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On generalized power methods of limitation

by

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Introduction

The subject of this paper is to discuss the properties of so-called generalized power methods, in particular power methods. A *generalized power method* $M(p, D)$ is determined by the function

$$p(w) = \sum_{n=0}^{\infty} p_n w^n$$

holomorphic in the circle $K_r = \{w: |w| < r\}$ satisfying $f(w) \neq 0$ for $r_0 \leqslant w = \mathrm{Re}(w) < r$ and by a domain $D \subset K_r$ such that $0 \in D$, $\langle r_0, r) \subset D$; the sequence $x = (t_n)$ is called *limitable to t by the method $M(p, D)$* if there exists a function $m(p, D; x, w)$ holomorphic in D such that

$$m(p, D; x, w) = \sum_{n=0}^{\infty} p_n t_n w^n$$

in a neighbourhood of 0 and

$$\lim_{w=\mathrm{Re}(w)\to r-} \frac{m(p, D; x, w)}{p(w)} = t.$$

In the particular case of $D = K_r$ the method $M(p, D) = M(p)$ is called a *power method*. For $p(w) = (1-w)^{-1}$ resp. e^w we obtain in this way the classical methods of Abel resp. Borel. The method $M(p, D)$ is *permanent* if, for instance, the function $p(w)$ satisfies the condition

$$(\beta) \qquad \begin{cases} p_n > 0 \quad \text{for} \quad n = 0, 1, \ldots, \\ \lim\limits_{w=\mathrm{Re}(w)\to r-} p(w) = +\infty, \end{cases}$$

which will be assumed very often.

Power methods have been investigated by several writers. Włodarski [11] proved in 1954 that every method $M(p)$ for $r < +\infty$ is perfect. The perfection of the Abel method was noticed earlier by K. Zeller. The investigation of methods $M(p)$ for $r = +\infty$ is more difficult and the question whether these methods are perfect has been answered only in the case of the Borel method; Ryll-Nardzewski [9] proved in 1962 that this method is perfect. Generalized power methods have not been yet investigated except the generalized Abel methods, i.e. methods $M(p, D)$, where $p(w) = (1-w)^{-1}$. These methods were distinguished long ago, e.g. in the investigation of the consistency of Nörlund methods [10].

In this paper the methods of functional analysis are applied. The definitions of spaces such as (B), (B_0), (F) are those of [1] and [5].

The main results are contained in § 3, in which the problem of perfection of generalized power methods is treated. First of all it is proved that a method $M(p, D)$ for $r < +\infty$ is perfect if and only if the domain D is simply connected. This result covers the theorem of Włodarski, mentioned above, concerning power methods. S. Mazur suggested to the present writer the investigation of the problem whether every power method $M(p)$ for $r = +\infty$ is perfect. Only partial results concerning this problem have been obtained. It is proved that the power method $M(p)$ (p is supposed to satisfy condition (β)) is perfect if and only if the function $p(w)$ has the following property (M):

(M) If $\omega(u)$ is a function continuous on the right in the interval $\langle 0, +\infty)$ and $\bigvee_0^{+\infty}(\omega) < +\infty$, then for $\varrho > 0$ the condition

$$\int_0^{+\infty} \frac{u^n}{p(u)} d\omega(u) = O(\varrho^n)$$

implies $\omega(u) = \text{const}$ for $u \geqslant \varrho$.

This result, in the writer's opinion, shows the difficulties connected with the solving the problem of perfection of methods $M(p)$ for $r = +\infty$. Let us remark that if the function $p(w)$ has the property (M), then it has the following property (LD):

(LD) The set of functions $u^n/p(u)$, $n = 0, 1, \ldots$, is linearly dense (in the sense of uniform convergence) in the space of functions $f(u)$ continuous in the interval $0 \leqslant u < +\infty$ and such that $\lim\limits_{n\to+\infty} f(u) = 0$.

The writer does not know the answer to the question whether every entire function $p(w) = \sum_{n=0}^{\infty} p_n w^n$, where $p_n > 0$, has the property (LD) or whether the property (LD) implies (M) (¹).

In this paper a simple condition for the method $M(p, D)$ to be perfect is given (Lemma 2). Elementary considerations enable us to deduce the theorem of Ryll-Nardzewski, mentioned above on the perfection of the Borel method.

(¹) After the author had sent the paper to press, he noticed that the answer to the former question was negative and gave a condition for the function $p(w)$ to have the property (LD) (see [2a]).

I wish to thank Professor S. Mazur for drawing my attention to these problems and for his helpful comments and suggestions made in the course of my writing the present paper.

§ 0. The terminology and the notation concerning certain notions of functional analysis and the theory of limitation

Definition 1. X is called a *sequence-space* if X is a linear space of complex-valued sequences $x = (t_n)$ with the usual definition of addition and multiplication by scalars.

Definition 2. Let X be a sequence-space of type (F). We say that X is a *sc-space* if $\varphi_n(x) = t_n$ are continuous functionals over X for $n = 0, 1, \ldots;$ $x = (t_n)$.

Definition 3. Let X be a sequence-space and let $\varphi(x)$ be an additive and homogeneous functional over X. Then the pair (X, φ) is called the *method of limitation* M. The set $X = M^*$ is called the *field* of the method M; the number $\varphi(x) = M(x)$ is called the *generalized limit* of the sequence $x \in M^*$ (corresponding to the method M).

Definition 4. A method M is called *permanent* if $T_c \subset M^*$ and $M(x) = \lim\limits_n t_n$ for every $x = (t_n) \in T_c$, where T_c denotes the set of all convergent sequences.

Definition 5. Let M be a method of limitation. If M^* is an (F)-sc-space such that $M(x)$ is a continuous functional over M^*, we shall say that M is an *(F)-method*. In particular, if M^* is a (B_0)-sc-space we shall say that M is a *(B_0)-method* and similarly for the other spaces (see [4] and [11]).

In this paper we use the following notation:

(1) T_c denotes the set of all convergent complex-valued sequences.

(2) c denotes the (B)-space of all convergent complex-valued sequences $x = (t_n)$ with the norm $\|x\| = \sup\limits_n |t_n|$.

(3) s denotes the (B_0)-space of all complex-valued sequences $x = (t_n)$ with the family of pseudonorms $\|x\|_n = |t_n|$; $n = 0, 1, \ldots$

(4) $C(\langle r_0, r))$ where $-\infty < r_0 < r \leqslant +\infty$ denotes the (B)-space of all continuous complex-valued functions $f(u)$ on the interval $r_0 \leqslant u < r$ with the finite limit $\lim\limits_{u\to r-} f(u)$. The norm in $C(\langle r_0, r\rangle)$ is given by the formula $\|f\| = \sup\limits_{r_0 \leqslant u < r} |f(u)|$.

(5) $H(D)$ denotes the (B_0)-space of all functions $f(w)$ holomorphic in domain D with the family of pseudonorms $\|f\|_Z = \sup_{w\in Z}|f(w)|$ where Z are compact subsets of D.

(6) H_0 denotes the (B_0)-space of all integer functions $f(w)$ with the finite limit $\lim_{w=\mathrm{Re}(w)\to+\infty} f(w)$. The pseudonorms in H_0 are given by the formulas $\|f\|_0 = \sup_{0\leqslant w=\mathrm{Re}(w)} |f(w)|$, $\|f\|_\varrho = \sup_{|w|\leqslant\varrho}|f(w)|$; $\varrho = 1, 2, \ldots$

(7) K_r denotes the circle $\{w\colon |w| < r\}$ in the complex plane (where $0 < r \leqslant +\infty$).

(8) $\mathcal{N}_r$ denotes the family of all domains D in the complex plane satisfying the following conditions:

1° $D \subset K_r$;

2° $0 \in D$;

3° there exists a number $r_0 < r$ such that $\{w\colon r_0 \leqslant w = \mathrm{Re}(w) < r\} \subset D$.

(9) $\mathcal{F}_r$ denotes the family of all functions

$$p(w) = \sum_{n=0}^{\infty} p_n w^n$$

satisfying the following conditions:

1'. the radius of convergence of the power series $\sum_{n=0}^{\infty} p_n w^n$ is equal to r;

2'. there exists a number $r_0 < r$ such that $p(w) \neq 0$ for $r_0 \leqslant w = \mathrm{Re}(w) < r$.

(10) For $p \in \mathcal{F}_r$ and $D \in \mathcal{N}_r$ we shall denote by $H(p, D)$ the (B_0)-space of all functions $f(w)$ holomorphic in D with the finite limit

$$\lim_{w=\mathrm{Re}(w)\to r-} \frac{f(w)}{p(w)}.$$

The pseudonorms in $H(p, D)$ are given by the formulas

$$\|f\|_0 = \sup_{r_0\leqslant w=\mathrm{Re}(w)<r} \left|\frac{f(w)}{p(w)}\right|, \qquad \|f\|_Z = \sup_{w\in Z}|f(w)|,$$

where Z are compact subsets of D.

(11) $\mathcal{M}_r$ denotes the family of all domains D in the complex plane such that $D \in \mathcal{N}_r$, $\{w\colon 0 \leqslant w = \mathrm{Re}(w) < r\} \subset D$.

§ 1. The definition and simple properties of the generalized power methods

Definition 6. By $M(p, D)$ we denote a method of limitation determined by a function $p \in \mathcal{F}_r$ and a domain $D \in \mathcal{N}_r$, defined as follows:

The field $M^*(p, D)$ of the method $M(p, D)$ is the set of all complex-valued sequences $x = (t_n)$ such that:

(a) the power series $\sum_{n=0}^{\infty} p_n t_n w^n$ has a positive radius of convergence;

(b) the function $\sum_{n=0}^{\infty} p_n t_n w^n$ is extendible to a function $m(p, D; x, w)$ holomorphic in the domain D;

(c) there exists a finite limit

$$\lim_{w=\mathrm{Re}(w)\to r-} \frac{m(p, D; x, w)}{p(w)}.$$

The generalized limit $M(p, D; x)$ of the sequence $x \in M^*(p, D)$ (corresponding to the method $M(p, D)$) is defined as the limit

$$M(p, D; x) = \lim_{w=\mathrm{Re}(w)\to r-} \frac{m(p, D; x, w)}{P(w)}.$$

In the particular case of $D = K_r$ we shall denote the method $M(p, D)$ by $M(p)$; $M(p)$ is called a *power method*. For example if $r = 1$ and $p(w) = (1-w)^{-1}$ or $r = +\infty$ and $p(w) = e^w$, is $M(p)$ the Abel method or the Borel method, respectively.

REMARK 1. *Let $p \in \mathcal{F}_r$ and $D_1, D_2 \in \mathcal{N}_r$. If $D_1 \subset D_2$, then $M^*(p, D_1) \supset M^*(p, D_2)$ and the methods $M(p, D_1)$, $M(p, D_2)$ are consistent: $M(p, D_1; x) = M(p, D_2; x)$ for $x \in M^*(p, D_2)$.*

In the space $M^*(p, D)$ we introduce the family of pseudonorms

$$(\mathrm{p}_1)\quad \begin{cases} \|x\|'_n = |t_n|, \quad \text{where} \quad n = 0, 1, \ldots;\ x = (t_n) \in M^*(p, D); \\ \|x\|_0 = \sup_{r_0\leqslant w=\mathrm{Re}(w)<r} \left|\dfrac{m(p, D; x, w)}{p(w)}\right|; \\ \|x\|_Z = \sup_{w\in Z}|m(p, D; x, w)|, \end{cases}$$

where Z takes on the values from the family of all compact subsets of D. (In fact the family of pseudonorms $\|x\|_Z$ is equivalent to a denumerable family of pseudonorms $\|x\|_{Z_k}$, $k = 1, 2, \ldots$).

It is easy to verify that $M^*(p, D)$ is the (B_0)-sc-space under the totality of pseudonorms (p_1).

Now we consider a mapping U of the space $M^*(p, D)$ into the product $s \times H(D) \times C(\langle r_0, r\rangle)$, defined by the formula

$$U(x) = \left(x, m(p, D; x, \cdot), \frac{m(p, D; x, \cdot)}{p(\cdot)}\right).$$

Of course U is a linear homeomorphism. According to the separability of the spaces s, $H(D)$, $C(\langle r_0, r\rangle)$ and the general form of linear functionals on those spaces, this implies that the space $M^*(p, D)$ is separable and the general form of linear functionals on $M^*(p, D)$ is given by the formula

$$(l_1)\quad \varphi(x) = \sum_{n=0}^{k} a_n t_n + \Phi(m(p, D; x, \cdot)) + \int_{r_0}^{r-} \frac{m(p, D; x, u)}{p(u)} d\omega(u) + aM(p, D; x),$$

where Φ is a linear functional on $H(D)$; $\omega(u)$ is a function continuous on the right with $\bigvee_{r_0}^{r-}(\omega) < +\infty$. The meaning of the symbol $\int_{r_0}^{r-}$ is

$$\int_{r_0}^{r-} = \lim_{s\to r-} \int_{r_0}^{s}.$$

Let us suppose now that the function $p \in \mathcal{F}_r$ fulfils the condition

$$(\alpha)\qquad n!\, p_n \equiv p^{(n)}(0) \neq 0 \quad \text{for} \quad n = 0, 1, \ldots$$

It is easy to establish that $M^*(p, D)$ is now the (B_0)-sc-space under the family of pseudonorms

$$(p_2)\qquad \begin{cases} \|x\|_0 = \sup\limits_{r_0\leqslant w=\mathrm{Re}(w)<r} \left|\dfrac{m(p, D; x, w)}{p(w)}\right|; \\ \|x\|_Z = \sup\limits_{w\in Z} |m(p, D; x, w)| \end{cases}$$

(Z has the same meaning as in the formula (p_1)).

Under the hypothesis (α) it is easy to establish that the space $M^*(p, D)$ is isomorphic to the space $H(p, D)$. The isomorphism between $M^*(p, D)$ and $H(p, D)$ is given by the formula

(i) $f(w) = m(p, D; x, w)$, where $w \in D;\, x \in M^*(p, D);\, f \in H(p, D)$.

The general form of linear functionals on $M^*(p, D)$ is given by the formula

$$(l_2)\quad \varphi(x) = \Phi(m(p, D; x, \cdot)) + \int_{r_0}^{r-} \frac{m(p, D; x, u)}{p(u)} d\omega(u) + aM(p, D; x),$$

where Φ, ω have the same meaning as in the formula (l_1).

In some considerations we shall suppose that

$$(\beta)\qquad \begin{cases} p_n > 0 \quad \text{for} \quad n = 0, 1, \ldots, \\ \lim\limits_{w=\mathrm{Re}(w)\to r-} p(w) = +\infty. \end{cases}$$

The fulfilment of (β) implies the permanence of the method $M(p, D)$.

From the above considerations we get the following

THEOREM 1. *Let $p \in \mathcal{F}_r$, $D \in \mathcal{N}_r$. Then*

(a) $M(p, D)$ is a separable (B_0)-method under the totality of pseudonorms (p_1); the general form of linear functionals on $M^*(p, D)$ is given by formula (l_1).

(b) If $p^{(n)}(0) \neq 0$ for $n = 0, 1, \ldots$, then $M(p, D)$ is a (B_0)-method under the totality of pseudonorms (p_2); the general form of linear functionals on $M^*(p, D)$ is given by formula (l_2); the spaces $M^*(p, D)$ and $H(p, D)$ are isomorphic and the isomorphism between $M^*(p, D)$ and $H(p, D)$ is given by formula (i).

§ 2. Rate of growth of the method $M(p, D)$

Definition 7. A sequence (θ_n) is called a *rate of growth* of the method M if $\theta_n > 0$ and $\sup\limits_n \theta_n |t_n| < +\infty$ for every $x = (t_n) \in M^*$.

Definition 8. A sequence (θ_n) is called a *strict rate of growth* of the method M if (θ_n) is a rate of growth of the method M and for every rate of growth (θ_n') of the method M there exists a constant C such that $\theta_n' \leqslant C\theta_n$ for $n = 0, 1, \ldots$ (see [4]).

REMARK 2. *Let $p \in \mathcal{F}_r$, $D \in \mathcal{N}_r$. If $p^{(n)}(0) \neq 0$ for $n = 0, 1, \ldots$, then the method $M(p, D)$ has a rate of growth.*

Proof. Let us write

$$\varrho_0 = \inf_{w\notin D} |w|$$

and let us fix $0 < \varrho < \varrho_0$. For every $x = (t_n) \in M^*(p, D)$ we have

$$\sum_{n=0}^{\infty} |p_n t_n| \varrho^n < +\infty.$$

Thus the sequence $\theta_n = |p_n| \varrho^n$ is the rate of growth of the method $M(p, D)$, q.e.d.

THEOREM 2. *Let $p \in \mathcal{N}_r$ and $p^{(n)}(0) \neq 0$ for $n = 0, 1, \ldots$ Then the power method $M(p)$ has no strict rate of growth.*

Proof. Let us suppose on the contrary that a sequence (θ_n) is a strict rate of growth of the power method $M(p)$. We consider the space $M^*(p)$ as the (B_0)-space under the family of pseudonorms

$$\|x\|_0 = \sup_{r_0\leqslant w=\mathrm{Re}(w)<r} \left|\frac{m(p; x, w)}{p(w)}\right|; \qquad \|x\|_\varrho = \sup_{|w|\leqslant\varrho} |m(p; x, w)|,$$

where $0 < \varrho < r$.

Since $\varphi_n(x) = \theta_n t_n$ are linear functionals on $M^*(p)$ and

$$\sup_n |\varphi_n(x)| = \sup_n \theta_n |t_n| < +\infty$$

for every $x = (t_n) \in M^*(p)$, there exist numbers C and $0 < \varrho_0 < r$ such that (see [5])

$$(*) \quad \theta_n |t_n| \leqslant C \max(\|x\|_0, \|x\|_{\varrho_0}) \quad \text{for} \quad n = 0, 1, \ldots; x = (t_n) \in M^*(p).$$

Since the sequence $(|p_n| \varrho^n)$ is the rate of growth of the method $M(p)$ for $0 < \varrho < r$, there exists a constant C_ϱ such that

$$(**) \qquad |p_n| \varrho^n \leqslant C_\varrho \theta_n; \quad n = 0, 1, \ldots; 0 < \varrho < r.$$

In virtue of inequalities (*), (**) we have for $0 < \varrho < \varrho_1 < r$ and $|w| \leqslant \varrho$:

$$|m(p; x, w)| = \Big| \sum_{n=0}^{\infty} p_n t_n w^n \Big| \leqslant \sum_{n=0}^{\infty} |p_n| |t_n| \varrho^n = \sum_{n=0}^{\infty} |p_n| \varrho_1^n |t_n| \left(\frac{\varrho}{\varrho_1}\right)^n$$
$$\leqslant C_{\varrho_1} \sum_{n=0}^{\infty} \theta_n |t_n| \left(\frac{\varrho}{\varrho_1}\right)^n \leqslant C C_{\varrho_1} \frac{\varrho_1}{\varrho_1 - \varrho} \max(\|x\|_0, \|x\|_{\varrho_0}).$$

Thus we have

$$(***) \qquad \|x\|_\varrho \leqslant C C_{\varrho_1} \frac{\varrho_1}{\varrho_1 - \varrho} \max(\|x\|_0, \|x\|_{\varrho_0}).$$

Inequality (***) shows that the conditions

$$x_k \in M^*(p), \quad \lim_k \|x_k\|_0 = \lim_k \|x_k\|_{\varrho_0} = 0$$

imply that

$$\lim_k \|x_k\|_\varrho = 0 \quad \text{for every } 0 < \varrho < r.$$

Thus in virtue of Theorem 1(b) the conditions

$$f_k \in H(p, K_r), \quad \lim_k \sup_{r_0 \leqslant w = \mathrm{Re}(w) < r} \left| \frac{f_k(w)}{p(w)} \right| = \lim_k \sup_{|w| \leqslant \varrho_0} |f_k(w)| = 0$$

imply that

$$\lim_k \sup_{|w| \leqslant \varrho} |f_k(w)| = 0$$

for every $0 < \varrho < r$, but it is easy to establish that it is impossible, q.e.d.

The next theorem concerns the rate of growth of the generalized Abel method.

In the case $p(w) = (1-w)^{-1}$, $D \in \mathcal{N}_1$ we denote the method $M(p, D)$ by $A(D)$. Let us write $\varrho_0 = \inf_{w \notin D} |w|$.

THEOREM 3. *A sequence (θ_n) $(\theta_n > 0)$ is a rate of growth of the generalized Abel method $A(D)$ if and only if there exist constants ϱ, C such that $0 < \varrho < \varrho_0$, $\theta_n \leqslant C\varrho^n$ for $n = 0, 1, \ldots$*

The sufficiency of the condition $\theta_n \leqslant C\varrho^n$ is trivial. To prove the necessity we need two simple remarks and a lemma.

For $x = (t_n)$ we shall write $x^{(k)} = (t_{n-k}) = (0, \ldots, 0, t_0, t_1, \ldots)$, where $t_n = 0$ for $n < 0$. Let us write

$$\|x\|_Z = \sup_{w \in Z} |A(D; x, w)|$$

for $Z \subset D$ and $x \in A^*(D)$, where $A(D; x, w) = (1-w) m(p, D; x, w)$; $p(w) = (1-w)^{-1}$.

REMARK 1. *If $x \in A^*(D)$, then $x^{(k)} \in A^*(D)$ for $k = 0, 1, \ldots$ and $\|x^{(k)}\|_Z \leqslant \|x\|_Z$.*

This is a consequence of the simple identity $A(D; x^{(k)}, w) = w^k A(D; x, w)$.

REMARK 2. *To prove Theorem 3 it is sufficient to prove it for a domain of the form $D_0 = \{w: |w| < 1\} - \{w_0\}$, where $|w_0| \leqslant 1$.*

Indeed, let w_0 be a point such that $|w_0| = \varrho_0$, $w_0 \notin D$ (if $\varrho_0 = 1$ we put $w_0 = -1$). Since $D \subset D_0$, we have $A^*(D) \supset A^*(D_0)$. Thus if (θ_n) is a rate of growth of the method $A(D)$, then (θ_n) is also a rate of growth of the method $A(D_0)$.

We consider the space $A^*(D)$ as the (B_0)-space under the totality of pseudonorms

$$\|x\|_0 = \sup_{r_0 \leqslant w = \mathrm{Re}(w) < 1} |A(D; x, w)|,$$

$$\|x\|_i = \sup_{\substack{|w - w_0| \geqslant 1/i \\ |w| \leqslant 1 - 1/i}} |A(D; x, w)| \quad \text{for} \quad i = i_0, i_0 + 1, \ldots,$$

where r_0, i_0 are fixed numbers (i_0 — a positive integer) such that the circle $\{w: |w - w_0| \leqslant 1/i_0\}$ lies inside the circle $\{w: |w| \leqslant 1 - 1/i_0\}$ and the interval $\{w: r_0 \leqslant w = \mathrm{Re}(w) < 1\}$ lies outside the circle $\{w: |w - w_0| \leqslant 1/i_0\}$.

LEMMA. *Let (ϑ_n) be a sequence such that $\vartheta_n \geqslant 0$ and the inequalities*

$$\vartheta_n |t_n| \leqslant a \max(\|x\|_0, \|x\|_l)$$

are satisfied for $n = 0, 1, \ldots$ and for every $x = (t_n) \in A^*(D)$, where a and l are fixed $(l \geqslant i_0)$. Then there exists a constant β such that

$$\vartheta_{lm} \leqslant \beta \left(\frac{\varrho_0}{\sqrt{2}}\right)^{lm} \quad \text{for} \quad m = 0, 1, \ldots$$

Proof. Let us consider the functions

$$f_m(w) = \frac{1 - w}{(1 - w/w_0)^{m+1}} \quad \text{for} \quad m = 0, 1, \ldots$$

Expanding them in a power series we get

$$f_m(w) = (1-w)\sum_{n=0}^{\infty}\binom{n+m}{m}\left(\frac{w}{w_0}\right)^n, \qquad |w| < |w_0| = \varrho_0.$$

Writing

$$x_m = (t_n^{(m)}), \quad \text{where} \quad t_n^{(m)} = \binom{n+m}{m}\frac{1}{w_0^n},$$

we get $x_m \in A^*(D)$ and $f_m(w) = A(D; x_m, w)$.

For $|w-w_0| \geqslant 1/l$ we have

$$|f_m(w)| = |1-w|\left|\frac{w_0}{w_0-w}\right|^{m+1} \leqslant 2|w_0|^{m+1}l^{m+1} = 2\varrho_0^{m+1}l^{m+1}$$

and we get the inequalities

$$\|x_m\|_0 \leqslant 2\varrho_0^{m+1}l^{m+1}, \qquad \|x_m\|_l \leqslant 2\varrho_0^{m+1}l^{m+1}.$$

According to the hypothesis of the Lemma we have $\vartheta_n|t_n^{(m)}| \leqslant a\max(\|x_m\|_0, \|x_m\|_l)$ and consequently

$$\vartheta_n\binom{n+m}{m}\frac{1}{\varrho_0^n} \leqslant 2a\varrho_0^{m+1}l^{m+1} \quad \text{for} \quad n, m = 0, 1, \ldots$$

Putting in the above inequality $n = lm$ we get

$$\vartheta_{lm}\binom{lm+m}{m} \leqslant 2al^{m+1}\varrho_0^{m+1+lm} \quad \text{for} \quad m = 0, 1, \ldots$$

We have

$$\binom{lm+m}{m} = \frac{(lm+m)(lm+m-1)\ldots(lm+1)}{m!} \geqslant \frac{l^m m^m}{m!}$$

and consequently

$$\vartheta_{lm}\frac{l^m m^m}{m!} \leqslant 2al^{m+1}\varrho_0^{lm+m+1},$$

and after simplification we get

$$\vartheta_{lm} \leqslant 2al\frac{m!}{m^m}\varrho_0^{lm} = 2al\frac{m!}{m^m}2^m\left(\frac{\varrho_0}{\sqrt[l]{2}}\right)^{lm}.$$

The radius of convergence of the power series

$$\sum_{m=0}^{\infty}\frac{m!}{m^m}z^m$$

is equal to e; thus

$$\sum_{m=0}^{\infty}\frac{m!}{m^m}2^m < +\infty$$

and consequently there exists a constant β such that

$$2al\frac{m!}{m^m}2^m \leqslant \beta \quad \text{for} \quad m = 0, 1, \ldots$$

Finally we get

$$\vartheta_{lm} \leqslant \beta\left(\frac{\varrho_0}{\sqrt[l]{2}}\right)^{lm} \quad \text{for} \quad (m = 0, 1, \ldots),$$

q.e.d.

Proof of Theorem 3. Let the sequence (θ_n) be a rate of growth of the method $A(D)$. As $\psi_n(x) = \theta_n t_n$ are linear functionals over $A^*(D)$ and

$$\sup_n|\psi_n(x)| = \sup_n \theta_n|t_n| < +\infty$$

for every $x = (t_n) \in A^*(D)$, there exist an index $l \geqslant i_0$ and a constant a such that (see [5])

$$\theta_n|t_n| \leqslant a\max(\|x\|_0, \|x\|_1), \qquad x = (t_n) \in A^*(D).$$

Putting $x^{(k)} = (t_{n-k})$ in place of x we get

$$\theta_n|t_{n-k}| \leqslant a\max(\|x^{(k)}\|_0, \|x^{(k)}\|_l),$$

and according to Remark 1 we get

$$\theta_n|t_{n-k}| \leqslant a\max(\|x\|_0, \|x\|_l), \qquad k, n = 0, 1, \ldots,$$

i.e.

$$\theta_{n+k}|t_n| \leqslant a\max(\|x\|_0, \|x\|_l), \qquad k, n = 0, 1, \ldots$$

In virtue on the Lemma there exist constants β_k for $k = 0, 1, \ldots$ such that

$$\theta_{lm+k} \leqslant \beta_k\varrho^{lm}, \quad \text{where} \quad \varrho = \frac{\varrho_0}{\sqrt[l]{2}} < \varrho_0.$$

Let us write $\beta = \max(\beta_0, \beta_1, \ldots, \beta_{l-1})$.

Now let n be an arbitrary non-negative integer. We may write $n = lm+k$, where $k = 0, 1, \ldots, l-1$; $m = 0, 1, \ldots$ Thus we have

$$\theta_n = \theta_{lm+k} \leqslant \beta\varrho^{lm} = \beta\varrho^{-k}\varrho^{lm+k} = \beta\varrho^{-k}\varrho^n \leqslant \beta\varrho^{1-l}\varrho^n = C\varrho^n,$$

where $C = \beta\varrho^{1-l}$, q.e.d.

COROLLARY. *A sequence* (θ_n) $(\theta_n > 0)$ *is a rate of growth of the Abel power method A if and only if there exist constants* ϱ *and* C *such that* $0 < \varrho < 1$, $\theta_n \leqslant C\varrho^n$ *for* $n = 0, 1, \ldots$

§ 3. Perfection of the methods $M(p, D)$

Definition 9. Let M be a permanent (B_0)-method. M is called a *perfect method at a point* $x_0 \in M^*$ if for every permanent (B_0)-method N such that $M^* \subset N^*$ the equality $M(x_0) = N(x_0)$ is satisfied. M is called a *perfect method* if it is perfect at every point $x_0 \in M^*$.

The following criterion plays an essential role in the considerations concerning the problem of perfection in the theory of limitation (see [3]):

LEMMA 1. *Let* M *be a permanent* (B_0)*-method. The method* M *is perfect in* $x_0 \in M^*$ *if and only if* x_0 *is a point of accumulation of the set* T_c *of the convergent sequences. In particular, the method* M *is perfect if and only if the set* T_c *of all convergent sequences is dense in the space* M^*.

Proof. 1° Suppose that $x_0 \notin \overline{T}_c$. Thus there exists on M^* a linear functional $\psi(x)$ such that $\psi(x) = 0$ for $x \in T_c$ and $\psi(x_0) = 1$. Let us consider a method N defined in following way: $N^* = M^*$, $N(x) = M(x) + \psi(x)$ for $x \in N^*$. Thus N is a permanent (B_0)-method but $N(x_0) = M(x_0) + \psi(x_0) = M(x_0) + 1 \neq M(x_0)$, whence M is not a perfect method at the point x_0.

2° Suppose that $x_0 \in \overline{T}_c$ and $M^* \subset N^*$, where N is a permanent (B_0)-method. Let us denote by $\|\cdot\|_M$, $\|\cdot\|_N$ the (F)-norms in the spaces M^* and N^*, respectively. Since

$$\lim_k \|x_k - x_0\|_M = 0,$$

where $x_k \in T_c$ for $k = 1, 2, \ldots$, we obtain

$$\lim_k \|x_k - x_0\|_N = 0$$

(it is a result of K. Zeller; see e.g. [11], p. 190).

Thus we have

$$N(x_0) = \lim_k N(x_k) = \lim_k M(x_k) = M(x_0),$$

q.e.d.

In this paragraph we shall consider the methods $M(p, D)$ under the hypothesis

$$(\beta) \qquad \begin{cases} p^{(n)}(0) > 0 & \text{for} \quad n = 0, 1, \ldots, \\ \lim\limits_{w = \mathrm{Re}(w) \to r-} p(w) = +\infty. \end{cases}$$

REMARK 3. *Let* $D \in \mathcal{N}_r$, $p \in \mathcal{F}_r$. *If the method* $M(p, D)$ *is perfect, then* D *is a simply connected domain.*

Proof. Suppose that D is not a simply connected domain. Hence there exist a point $w_0 \notin D$ and a simply connected domain G such that $w_0 \in G$ and $\Gamma = \overline{G} - G \subset D$. Let us consider the sequence

$$x_0 = \left(\frac{1}{p_n w_0^n}\right).$$

Since

$$m(p, D; x_0, w) = \sum_{n=0}^{\infty} p_n \frac{w^n}{p_n w_0^n} = \frac{w_0}{w - w_0}$$

and

$$\lim_{w = \mathrm{Re}(w) \to r-} \frac{m(p, D; x_0, w)}{p(w)} = 0,$$

we obtain $x_0 \in M^*(p, D)$.

Suppose that the method $M(p, D)$ is perfect. According to Lemma 1 there exists a sequence $x_k \in T_c$ such that

$$\lim_k x_k = x_0.$$

Consequently the sequence of the functions $m(p, D; x_k, w)$ is uniformly convergent on Γ to the function $w_0/(w - w_0)$, and consequently the sequence of the functions $m(p, D; x_k, w)$ is uniformly convergent on the domain G, but this is impossible since $m(p, D; x_k, w)$ are holomorphic functions for $|w| < r$, q.e.d.

LEMMA 2. *Let* $D \in \mathcal{N}_r$, $p \in \mathcal{F}_r$. *The method* $M(p, D)$ *is perfect if and only if the set of all functions of the form* $W(w) + tp(w)$, *where* $W(w)$ *denotes a polynomial and* t *— an arbitrary constant, is a dense subset of the space* $H(p, D)$.

Proof. 1° The sufficiency is a simple consequence of the equality

$$\sum_{n=0}^{k} a_n w^n + tp(w) = m(p, D; x, w),$$

where

$$x = \left(\frac{a_0}{p_0} + t, \frac{a_1}{p_1} + t, \ldots, \frac{a_k}{p_k} + t, t, t, \ldots\right),$$

since the spaces $M^*(p, D)$ and $H(p, D)$ are isomorphic.

2° Suppose that the method $M(p, D)$ is perfect. Hence the set of the functions of the form

$$\sum_{n=0}^{\infty} p_n t_n w^n,$$

where $(t_n)\epsilon T_c$, is dense in the space $H(p, D)$. Let

$$\lim_n t_n = t \quad \text{and} \quad f_m(w) = \sum_{n=0}^{m} p_n(t_n - t)w^n + tp(w).$$

We show that the sequence of the functions $f_m(w)$ tends to the function

$$f(w) = \sum_{n=0}^{\infty} p_n t_n w^n$$

(in the sense of convergence in $H(p, D)$). Of course $f_m(w) \to f(w)$ uniformly in the circle $|w| \leqslant \varrho < r$. It remains to prove that $f_m(u)/p(u) \to f(u)/p(u)$ uniformly on the interval $0 \leqslant u < r$.

Let $\varepsilon > 0$. We have $|t_n - t| < \varepsilon$ for $n > n_0$. Hence we get for $m \geqslant n_0$ and $0 \leqslant u < r$:

$$\left|\frac{f_m(u)}{p(u)} - \frac{f(u)}{p(u)}\right| = \frac{1}{p(u)}\left|\sum_{n=m+1}^{\infty} p_n(t_n - t)u^n\right| \leqslant \frac{1}{p(u)}\sum_{n=m+1}^{\infty} p_n|t_n - t|u^n$$
$$\leqslant \frac{\varepsilon}{p(u)}\sum_{n=m+1}^{\infty} p_n u^n \leqslant \varepsilon,$$

q.e.d.

LEMMA 3. *Let $g(\zeta)$ be a holomorphic function in the circle $|\zeta - \zeta_0| < \varrho < +\infty$ with the finite radial limit $\lim_{\zeta \to \zeta_1} g(\zeta)$, where ζ_1 denotes a fixed number such that $|\zeta_1 - \zeta_0| = \varrho$. Then for every $\varepsilon > 0$ and for every $0 < \varrho_1 < \varrho$ there exists a polynomial $W(\zeta)$ such that*

$$|W(\zeta) - g(\zeta)| < \varepsilon \quad \textit{for} \quad |\zeta - \zeta_0| \leqslant \varrho_1 \ \textit{and for} \ \zeta \epsilon [\zeta_0, \zeta_1).$$

Proof. According to the uniform continuity of the function $g(\zeta)$ on the set $Z = \{\zeta: |\zeta - \zeta_0| \leqslant \varrho_1\} \cup [\zeta_0, \zeta_1)$ there exists $0 < r < 1$ such that

$$(*) \qquad \big|g\big(\zeta_0 + r(\zeta - \zeta_0)\big) - g(\zeta)\big| < \frac{\varepsilon}{2} \quad \text{for} \quad \zeta \epsilon Z.$$

Since $g\big(\zeta_0 + r(\zeta - \zeta_0)\big)$ is a holomorphic function on the circle $|\zeta - \zeta_0| < \varrho/r > \varrho$, there exists a polynomial $W(\zeta)$ such that

$$(**) \qquad \big|W(\zeta) - g\big(\zeta_0 + r(\zeta - \zeta_0)\big)\big| < \frac{\varepsilon}{2} \quad \text{for} \quad |\zeta - \zeta_0| \leqslant \varrho.$$

In virtue of the inequalities (*) and (**) the proof of the Lemma is completed, q.e.d.

LEMMA 4. *The set of the functions of the form*

$$a_0 + \sum_{j=1}^{m} a_j e^{-\varrho_j w},$$

where $\varrho_j > 0$, is dense in the space H_0.

Proof. Given an arbitrary number $\varepsilon > 0$ and a square $K = \{w: |\mathrm{Re}(w)| \leqslant R, |\mathrm{Im}(w)| \leqslant R\}$. Let us fix a number $a > 0$ such that $aR < \pi/2$. Let us consider the mapping $\zeta = e^{-aw}$ which maps the square K onto the set $\Omega = \{\zeta: e^{-aR} \leqslant |\zeta| \leqslant e^{aR}, |\mathrm{Arg}\,\zeta| \leqslant aR\}$ and the interval $0 \leqslant w = \mathrm{Re}(w) < +\infty$ onto the interval $0 < \zeta = \mathrm{Re}(\zeta) \leqslant 1$. The inverse mapping is given by the formula

$$w = -\frac{1}{a}\mathrm{Log}\,\zeta.$$

Let $f\epsilon H_0$. We consider the function

$$g(\zeta) = f\left(-\frac{1}{a}\mathrm{Log}\,\zeta\right),$$

which is a holomorphic function on the open complex plane except $\zeta = \mathrm{Re}(\zeta) \leqslant 0$, with the finite limit

$$\lim_{\zeta = \mathrm{Re}(\zeta) \to +0} g(\zeta).$$

Let us write $S = \{\zeta: |\zeta - \zeta_0| \leqslant \varrho_1\}$, where ζ_0, ϱ_1 are fixed numbers such that $\zeta_0 = \mathrm{Re}(\zeta_0) > 0, \Omega \subset S, \varrho_1 < \zeta_0$. (Such numbers ζ_0, ϱ_1 exist, since $aR < \pi/2$). Putting, in Lemma 3, $\varrho = \zeta_0$ and $\zeta_1 = 0$ we find that there exists a polynomial $W(\zeta)$ such that

$$|W(\zeta) - g(\zeta)| < \varepsilon \quad \text{for} \quad |\zeta - \zeta_0| \leqslant \varrho_1 \ \text{and for} \ 0 < \zeta = \mathrm{Re}(\zeta) \leqslant 1.$$

After the substitution $\zeta = e^{-aw}$ we get

$$|W(e^{-aw}) - f(w)| < \varepsilon \quad \text{for} \quad w \epsilon K \ \text{and for} \ 0 \leqslant w = \mathrm{Re}(w) < +\infty.$$

Writing

$$W(\zeta) = \sum_{j=0}^{m} a_j \zeta^j \quad \text{and} \quad \varrho_j = aj$$

we have

$$W(e^{-aw}) = a_0 + \sum_{j=1}^{m} a_j e^{-\varrho_j w},$$

q.e.d.

LEMMA 5. *The set of functions of the form $e^{-w}W(w) + t$, where $W(w)$ denotes a polynomial and t — an arbitrary constant, is dense in the space H_0.*

Proof. In virtue of Lemma 4 it remains to prove that the function $e^{-\varrho w}$, where $\varrho > 0$, can be approximated (in the sense of H_0) by functions of the form $e^{-w}W(w)$, where $W(w)$ is a polynomial. At first we shall prove it for the same special cases with respect to ϱ.

We use the following notation:

$f_k(w) \rightrightarrows f(w)$ denotes uniform convergence on every compact subset of the open complex plane;

$f_k(w) \Rightarrow f(w)$ denotes uniform convergence on the interval $0 \leqslant w = \mathrm{Re}(w) < +\infty$;

$f_k(w) \to f(w)$ denotes convergence in the sense of the space H_0.

If $f_k, f \in H_0$, then $f_k(w) \to f(w)$ if and only if $f_k(w) \rightrightarrows f(w)$ and $f_k(w) \Rightarrow f(w)$.

1° Let $0 < \varrho < 2$. We have $e^{-\varrho w} = e^{-w}e^{aw}$, where $a = 1-\varrho$, $|a| < 1$. Writing

$$W_k(w) = \sum_{j=0}^{k-1} \frac{(aw)^j}{j!}$$

we get $e^{-w}W_k(w) \rightrightarrows e^{-\varrho w}$.

Let $\varepsilon > 0$. For $j \geqslant m$ we have $|a^j| < \varepsilon$; consequently we get for $k \geqslant m$ and $0 \leqslant w = \mathrm{Re}(w) < +\infty$:

$$|e^{-\varrho w} - e^{-w}W_k(w)| \leqslant e^{-w}\sum_{j=k}^{\infty} \frac{|a|^j}{j!} w^j \leqslant \varepsilon e^{-w} \sum_{j=k}^{\infty} \frac{w^j}{j!} < \varepsilon;$$

thus $e^{-w}W_k(w) \Rightarrow e^{-\varrho w}$ and consequently $e^{-w}W_k(w) \to e^{-\varrho w}$.

2° Let $\varrho = 2$. Writing

$$W_k(w) = \sum_{j=0}^{k-1} \frac{(-w)^j}{j!}$$

we obtain $e^{-w}W_k(w) \rightrightarrows e^{-2w}$.

For $0 \leqslant w = \mathrm{Re}(w)$ we have

$$e^{-2w} = e^{-w}W_k(w) + (-1)^k e^{-w}\frac{e^{-\theta w}}{k!}w^k, \quad \text{where} \quad 0 \leqslant \theta = \theta(k, w) \leqslant 1;$$

hence we obtain

$$|e^{-2w} - e^{-w}W_k(w)| = e^{-w}e^{-\theta w}\frac{w^k}{k!} \leqslant \frac{1}{k!}e^{-w}w^k \leqslant \frac{k^k e^{-k}}{k!}.$$

According to Stirling's formula $k! = k^k e^{-k}\sqrt{k}\, a_k$, $a_k \to \sqrt{2\pi}$, we get

$$\frac{k^k e^{-k}}{k!} = \frac{1}{a_k\sqrt{k}} \to 0.$$

Thus we have proved that $e^{-w}W_k(w) \Rightarrow e^{-2w}$ and consequently, $e^{-w}W_k(w) \to e^{-2w}$.

3° Now we prove that every function $e^{-3w/2}W_0(w)$, where $W_0(w)$ is a polynomial, can be approximated by functions of the form $e^{-w}W(w)$, where $W(w)$ is a polynomial.

It suffices to prove this for $W_0(w) = w^m$, $m = 0, 1, \ldots$ We have

$$e^{-3w/2}w^m = e^{-w}w^m e^{-w/2} = e^{-w}w^m \sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{w}{2}\right)^j.$$

Writing

$$W_k(w) = w^m \sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{w}{2}\right)^j$$

we obtain $e^{-w}W_k(w) \rightrightarrows e^{-3w/2}w^m$. For $0 \leqslant w = \mathrm{Re}(w) < +\infty$ we have

$$e^{-3w/2}w^m = e^{-w}W_k(w) + e^{-w}w^m \frac{1}{k!}\left(-\frac{1}{2}\right)^k e^{-\theta w/2}w^k, \quad \text{where} \quad 0 \leqslant \theta \leqslant 1.$$

Hence we get

$$|e^{-3w/2}w^m - e^{-w}W_k(w)| = \frac{1}{2^k k!}e^{-w}e^{-\theta w/2}w^{k+m} \leqslant \frac{1}{2^k k!}e^{-w}w^{k+m}$$

$$\leqslant \frac{1}{2^k k!}e^{-(k+m)}(k+m)^{k+m} = \frac{(k+m)!}{2^k k!} \cdot \frac{e^{-(k+m)}(k+m)^{k+m}}{(k+m)!}$$

$$\leqslant \frac{(k+m)^m}{2^k}e^{-1} \to 0 \quad \text{if} \quad k \to +\infty.$$

Thus we obtain $e^{-w}W_k(w) \Rightarrow e^{-3w/2}w^m$ and consequently $e^{-w}W_k(w) \to e^{-3w/2}$.

4° Now we show that every function $e^{-2w}W_0(w)$ can be approximated by functions of the form $e^{-3w/2}W(w)$, where $W_0(w)$ and $W(w)$ denote polynomials.

Indeed, in virtue of 2° there exists a sequence of polynomials $W_k(w)$ such that $e^{-w}W_k(w) \to e^{-2w}$. After substituting $w/2$ for w we establish $e^{-w/2}W_k(w/2) \to e^{-w}$, and multiplying by $e^{-w}W_0(w)$ we get

$$e^{-3w/2}W_0(w)W_k\left(\frac{w}{2}\right) \to e^{-2w}W_0(w).$$

5° In virtue of 3° and 4° we establish that every function $e^{-2w}W_0(w)$ can be approximated by functions of the form $e^{-w}W(w)$, where $W_0(w)$, $W(w)$ are polynomials.

6° Now the proof of Lemma 5 can be completed by the method of mathematical induction. Let $i < \varrho \leqslant i+1$, where $i = 0, 1, \ldots$ According to 1° the assertion of the Lemma is true for $i = 0$. Suppose that the assertion of the lemma is true for i and let $i+1 < \varrho \leqslant i+2$. In virtue of the inductive hypothesis and $i < \varrho - 1 \leqslant i+1$ we conclude that there exists a sequence of polynomials $P_k(w)$ such that $e^{-w}P_k(w) \to e^{-(\varrho-1)w}$. After multiplication by e^{-w} we obtain $e^{-2w}P_k(w) \to e^{-\varrho w}$. In virtue of 5° there exists a sequence of polynomials $W_k(w)$ such that $e^{-w}W_k(w) \to e^{-\varrho w}$, q.e.d.

Elementary Lemmas 2 and 5 enable us to prove the following theorem of Ryll-Nardzewski (see [9]):

THEOREM 4. *The Borel power method is perfect.*

Proof. Let $p(w) = e^w$. According to Lemma 2 we have to prove that the set of functions of the form $W(w) + te^w$ is dense in the space $H(p, K_{+\infty})$, where $W(w)$ is a polynomial.

Let $f \in H(p, K_{+\infty})$. Hence the function $e^{-w}f(w)$ is an element of the space H_0. According to Lemma 5 there exists a sequence of polynomials $W_k(w)$ and a sequence of numbers $t^{(k)}$ such that the sequence of functions $e^{-w}W_k(w) + t^{(k)}$ tends to $e^{-w}f(w)$ uniformly on every compact subset of the open complex plane and uniformly in the interval $0 \leqslant w = \mathrm{Re}(w) < +\infty$. Hence $W_k(w) + t^{(k)}p(w)$ tends to $f(w)$ uniformly on every compact subset of the open complex plane and $\big(W_k(w) + t^{(k)}p(w)\big)/p(w)$ tends to $f(w)/p(w)$ uniformly in the interval $0 \leqslant w = \mathrm{Re}(w) < +\infty$, but this means that the sequence of functions $W_k(w) + t^{(k)}p(w)$ tends to $f(w)$ in the sense of convergence in the space $H(p, K_{+\infty})$, q.e.d.

Definition 10. Let $\chi(u)$ be a positive continuous function in the interval $0 \leqslant u < r$ $(0 < r \leqslant +\infty)$ such that

$$\sup_{0 \leqslant u < r} \frac{u^n}{\chi(u)} < +\infty$$

for $n = 0, 1, \ldots$ (e.g. we may take $\chi(u) = p(u)$ where $p \in \mathscr{F}_r$ if hypothesis (β) is fulfilled). We shall say that $\chi(u)$ has the (M)-*property* if for every number $0 < \varrho < r$ and for every function $\omega(u)$ continuous on the right such that $\bigvee_0^{r-}(\omega) < +\infty$ the condition

$$\int_0^{r-} \frac{u^n}{\chi(u)} d\omega(u) = O(\varrho^n)$$

implies $\omega(u) = \text{const}$ for $\varrho \leqslant u < r$.

According to Mikusiński's theorem of bounded moments the function $\chi(u) \equiv 1$ has the (M)-property if $r < +\infty$ (see [7] and [8]).

LEMMA 6. *Let $p \in \mathscr{F}_r$, $D \in \mathscr{N}_r$ and let condition* (β) *be fulfilled. If D is a simply connected domain and the function $p(u)$ has the* (M)-*property, then the method $M(p, D)$ is perfect.*

Proof. Let $e = (1, 1, 1, \ldots)$, $e_m = (\underset{0}{0}, \ldots, 0, \underset{m}{1}, 0, \ldots)$ for $m = 0, 1, \ldots$

Let $\varphi(x)$ be a linear functional on $M^*(p, D)$ such that $\varphi(e) = \varphi(e_m) = 0$ for $m = 0, 1, \ldots$ To prove this lemma we show that $\varphi(x) = 0$ for every $x \in M^*(p, D)$.

In virtue of formula (1_2), § 1, we have

$$\varphi(x) = \Phi\big(m(p, D; x, \cdot)\big) + \int_{r_0}^{r-} \frac{m(p, D; x, u)}{p(u)} d\omega(u) + aM(p, D; x).$$

Putting $x = e_m$ we get

$$(*) \quad \int_{r_0}^{r-} \frac{u^m}{p(u)} d\omega(u) = -\Phi(f_m), \qquad \text{where} \quad f_m(w) = w^m,\ m = 0, 1, \ldots$$

Since Φ is a linear functional over $H(D)$, there exist a constant C and a compact subset Z of D such that

$$|\Phi(f)| \leqslant C \sup_{w \in Z} |f(w)| \quad \text{for} \quad f \in H(D).$$

Writing

$$\varrho = \sup_{w \in Z} |w|$$

we obtain

$$|\Phi(f_m)| \leqslant C\varrho^m, \quad m = 0, 1, \ldots$$

Thus in virtue of (*) we get

$$\int_{r_0}^{r-} \frac{u^m}{p(u)} d\omega(u) = O(\varrho^m).$$

According to the hypothesis of the Lemma, $\omega(u) = \text{const}$ for $\varrho \leqslant u < r$. The functional $\varphi(x)$ may now be written in the form

$$\varphi(x) = \Psi\big(m(p, D; x, \cdot)\big) + aM(p, D; x),$$

where

$$\Psi(f) = \Phi(f) + \int_{r_0}^{\varrho} \frac{f(u)}{p(u)} d\omega(u) \quad \text{for} \quad f \in H(D).$$

Equality (*) implies $\Psi(f_m) = 0$ for $m = 0, 1, \ldots$ and consequently $\Psi(W) = 0$ for every polynomial W.

According to the theorem of Runge we get $\psi(f) = 0$ for every function $f \in H(D)$ since Ψ is a linear functional over $H(D)$. Thus we have, $\varphi(x) = aM(p, D; x)$ for $x \in M^*(p, D)$. Putting $x = e$ we finally get $a = 0$, q.e.d.

LEMMA 7. *Let $\chi(u)$ be a positive continuous function on the interval $0 \leqslant u < r$ such that*

$$\sup_{0 \leqslant u < r} \frac{u^n}{\chi(u)} < +\infty$$

for $n = 0, 1, \ldots$, where $0 < r < +\infty$. Then $\chi(u)$ has the (M)*-property.*

Taking $\chi(u) \equiv 1$ we get Mikusiński's theorem of bounded moments (see [7] and [8]). The above lemma may be deduced from the theorem of Mikusiński but we shall give a new proof which is based on the following well-known approximation theorem of Mergelian (see [6]):

THEOREM OF MERGELIAN. *Let $g(w)$ be a function defined on a compact subset F of the open complex plane C and suppose that the following conditions are satisfied:*

1. *The function $g(w)$ is continuous on F.*
2. *The function $g(w)$ is holomorphic in* $\mathrm{Int}(F)$.
3. *The set $C - F$ is connected.*

Then for every $\varepsilon > 0$ there exists a polynomial $W(w)$ such that $|g(w) - W(w)| < \varepsilon$ for every $w \in F$.

Proof of Lemma 7. Suppose that

$$\int_0^{r-} \frac{u^n}{\chi(u)} d\omega(u) = O(\varrho^n),$$

where $0 < \varrho < r$ and the function $\omega(u)$ is such as in definition 10. Hence

$$\int_\varrho^{r-} \frac{u^n}{\chi(u)} d\omega(u) = O(\varrho^n).$$

Thus we have

$$\int_\varrho^{r-} \frac{u^n}{\chi(u)} d\omega(u) = \alpha_n \varrho^n,$$

where $|\alpha_n| \leqslant C$ for $n = 0, 1, \ldots$ Hence for an arbitrary polynomial

$$W(u) = \sum_{n=0}^{m} a_n u^n$$

we get

$$\int_\varrho^{r-} \frac{W(u)}{\chi(u)} d\omega(u) = \sum_{n=0}^{m} \alpha_n a_n \varrho^n.$$

Let us fix the numbers ϱ_1, ϱ_2 so that $\varrho < \varrho_1 < \varrho_2 < r$. Applying the inequalities of Cauchy

$$|a_n| \leqslant \frac{1}{\varrho_1^n} \sup_{|w| \leqslant \varrho_1} |W(w)|$$

we get

$$\left| \int_\varrho^{r-} \frac{W(u)}{\chi(u)} d\omega(u) \right| \leqslant \sum_{n=0}^{m} |\alpha_n| \, |a_n| \, \varrho^n \leqslant C \sum_{n=0}^{m} \left(\frac{\varrho}{\varrho_1}\right)^n \sup_{|w| \leqslant \varrho_1} |W(w)|$$

$$\leqslant C \frac{\varrho_1}{\varrho_1 - \varrho} \sup_{|w| \leqslant \varrho_1} |W(w)|.$$

Thus we have

$$(*) \qquad \left| \int_\varrho^{r-} \frac{W(u)}{\chi(u)} d\omega(u) \right| \leqslant C \frac{\varrho_1}{\varrho_1 - \varrho} \sup_{|w| \leqslant \varrho_1} |W(w)|.$$

Now let $f(u)$ be an arbitrary continuous function in the interval $\varrho_1 \leqslant u < r$ such that $f(\varrho_1) = 0 = f(u)$ for $\varrho_2 \leqslant u < r$. Let us consider a function $g(w)$ of the complex variable defined as follows:

$$g(w) = \begin{cases} 0 & \text{for} \quad |w| \leqslant \varrho_1, \\ \chi(u) f(u) & \text{for} \quad \varrho_1 \leqslant w = u < r. \end{cases}$$

Applying the theorem of Mergelian to the function $g(w)$ and the set $F = \{w: |w| \leqslant \varrho_1\} \cup \{w: \varrho_1 \leqslant w = \mathrm{Re}(w) \leqslant r\}$, we can choose a sequence of the polynomials $W_k(w)$ tending to $g(w)$ uniformly on F. It is easy to see that $W_k(u)/\chi(u)$ tends to $f(u)$ uniformly on the interval $\varrho_1 \leqslant u < r$ since the function $1/\chi(u)$ is bounded for $\varrho_1 \leqslant u < r$. In virtue of $(*)$ we have

$$\left| \int_\varrho^{r-} \frac{W_k(u)}{\chi(u)} d\omega(u) \right| \leqslant C \frac{\varrho_1}{\varrho_1 - \varrho} \sup_{|w| \leqslant \varrho_1} |W_k(w)|.$$

In the limit for $k \to +\infty$ we finally obtain

$$\int_{\varrho_1}^{\varrho_2} f(u) d\omega(u) = 0.$$

Since $f(u)$ is an arbitrary function, we deduce that $\omega(u) = \mathrm{const}$ for $\varrho_1 \leqslant u < \varrho_2$ and since ϱ_1, ϱ_2 are arbitrary numbers we deduce that $\omega(u) = \mathrm{const}$ for $\varrho \leqslant u < r$, q.e.d.

According to Lemmas 6, 7 and remark 3 we have the following theorem (see [11], Th. XI):

THEOREM 5. *Let $p \in \mathcal{F}_r$, $D \in \mathcal{N}_r$ and let hypothesis* (β) *be satisfied for $r < +\infty$. Then the method $M(p, D)$ is perfect if and only if D is a simply connected domain.*

LEMMA 8. *Let $f(u)$ be a continuous function for $u \geqslant R > 0$ such that*

$$f(R) = 0 = \lim_{u\to+\infty} f(u).$$

Then for every $\varepsilon > 0$ there exists a function $h(w)$ of the form

$$h(w) = \sum_{j=1}^{m} a_j e^{-\varrho_j w}$$

where $\varrho_j > 0$ such that

$$|h(u) - f(u)| < \varepsilon \quad \text{for} \quad u \geqslant R,$$

$$|h(w)| < \varepsilon \quad \text{for} \quad |\mathrm{Re}(w)| \leqslant R,\ |\mathrm{Im}(w)| \leqslant R.$$

Proof. We apply the same method as in the proof of Lemma 4. Let the symbols K, a, Ω have the same meaning as in the proof of Lemma 4. Let us consider a function $g(\zeta)$ defined as follows:

$$g(\zeta) = \begin{cases} 0 & \text{for} \quad \zeta \in \Omega, \\ f\left(-\frac{1}{a}\operatorname{Log}\zeta\right) & \text{for} \quad 0 < \zeta = \mathrm{Re}(\zeta) \leqslant e^{-aR}. \end{cases}$$

Applying the approximation theorem of Mergelian to the function $g(\zeta)$ and to the set $F = \Omega \cup \{\zeta: 0 \leqslant \zeta = \mathrm{Re}(\zeta) \leqslant e^{-aR}\}$, we can choose a polynomial

$$W(\zeta) = \sum_{j=1}^{m} a_j \zeta^j$$

such that $|W(\zeta) - g(\zeta)| < \varepsilon$ for $\zeta \in F$. After the substition $\zeta = e^{-aw}$ we get the assertion of Lemma 8, q.e.d.

LEMMA 9. *Let $p \in \mathscr{F}_r$ and hypothesis* (β) *be satisfied for $r = +\infty$, and let the power method $M(p)$ be perfect. If $f \in H_0$ and*

$$\lim_{w=\mathrm{Re}(w)\to+\infty} f(w) = 0,$$

then for every $\varepsilon > 0$ and for every compact subset Z of the open complex plane there exists such a polynomial $W(w)$ that

$$|W(w) - p(w)f(w)| < \varepsilon \quad \text{for} \quad w \in Z,$$

$$\left|\frac{W(w)}{p(w)} - f(w)\right| < \varepsilon \quad \text{for} \quad w = \mathrm{Re}(w) \geqslant 0.$$

Proof. Let

$$\lambda = \sup_{w\in Z} |p(w)|.$$

According to Lemma 2, since the function $p(w)\cdot f(w)$ belongs to the space $H(p, K_{+\infty})$, there exist a polynomial $W(w)$ and a number t such that

$$(*) \qquad |W(w) + tp(w) - p(w)f(w)| < \frac{\varepsilon}{1+\lambda} \quad \text{for} \quad w \in Z,$$

$$(**) \qquad \left|\frac{W(w)}{p(w)} + t - f(w)\right| < \frac{\varepsilon}{2(1+\lambda)} \quad \text{for} \quad w = \mathrm{Re}(w) \geqslant 0.$$

Taking the limit in (**) if $w = \mathrm{Re}(w) \to +\infty$, we obtain $|t| \leqslant \varepsilon/2(1+\lambda)$. Hence we get for $w \in Z$

$$|W(w) - p(w)f(w)| \leqslant |W(w) + tp(w) - p(w)f(w)| + |t|\,|p(w)| < \frac{\varepsilon}{1+\lambda} + \frac{\varepsilon\lambda}{2(1+\lambda)} < \frac{\varepsilon}{1+\lambda}(1+\lambda) = \varepsilon.$$

For $w = \mathrm{Re}(w) \geqslant 0$ we get

$$\left|\frac{W(w)}{p(w)} - f(w)\right| \leqslant \left|\frac{W(w)}{p(w)} + t - f(w)\right| + |t| < \frac{\varepsilon}{2(1+\lambda)} + \frac{\varepsilon}{2(1+\lambda)} < \varepsilon,$$

q.e.d.

LEMMA 10. *Let $p \in \mathscr{F}_r$ and hypothesis* (β) *be satisfied for $r = +\infty$; let $\chi(u)$ be a positive continuous function for $0 \leqslant u < +\infty$ such that*

$$\sup_{u\geqslant 0} \frac{u^n}{\chi(u)} < +\infty \quad \text{for} \quad n = 0, 1, \ldots$$

If the power method $M(p)$ is perfect and the inequality $\chi(u) \geqslant p(u)$ is satisfied for $u \geqslant 0$, then the function $\chi(u)$ has the (M)*-property.*

Proof. Suppose that

$$\int_0^{+\infty} \frac{u^n}{\chi(u)}\, d\omega(u) = O(\varrho^n), \quad 0 < \varrho < +\infty,$$

and fix the numbers ϱ_1, ϱ_2 so that $\varrho < \varrho_1 < \varrho_2 < +\infty$. Repeating partially the proof of Lemma 7 we get

$$(*) \qquad \left|\int_0^{+\infty} \frac{W(u)}{\chi(u)}\, d\omega(u)\right| \leqslant C\frac{\varrho_1}{\varrho_1 - \varrho} \sup_{|w|\leqslant\varrho_1} |W(w)|$$

for an arbitrary polynomial $W(w)$.

Now let $f(u)$ be an arbitrary continuous function in the interval $\varrho_1 \leqslant u < +\infty$ such that $f(\varrho_1) = 0 = f(u)$ for $\varrho_2 \leqslant u < +\infty$. Let

$$g(u) = \frac{f(u)\chi(u)}{p(u)} \quad \text{for} \quad \varrho_1 \leqslant u < +\infty.$$

Let $\varepsilon > 0$. According to Lemma 8 there exists a function

$$h(w) = \sum_{j=1}^{m} a_j e^{-r_j w}$$

such that

$$|h(u) - g(u)| < \varepsilon \quad \text{for} \quad u \geqslant \varrho_1,$$

$$|h(w)| < \varepsilon \quad \text{for} \quad |w| \leqslant \varrho_1.$$

According to Lemma 9 there exists a polynomial $W(w)$ such that

$$\left|\frac{W(u)}{p(u)} - h(u)\right| < \varepsilon \quad \text{for} \quad u \geqslant \varrho_1,$$

$$|W(w) - p(w)h(w)| < \varepsilon \quad \text{for} \quad |w| \leqslant \varrho_1.$$

Writing

$$\lambda = \sup_{|w| \leqslant \varrho_1} |p(w)|$$

we get for $|w| \leqslant \varrho_1$:

$$|W(w)| \leqslant |W(w) - p(w)h(w)| + |p(w)h(w)| < \varepsilon + \lambda\varepsilon.$$

For $u \geqslant \varrho_1$ we get

$$\frac{W(u)}{\chi(u)} - f(u) = \frac{p(u)}{\chi(u)}\left(\frac{W(u)}{p(u)} - g(u)\right),$$

whence

$$\left|\frac{W(u)}{\chi(u)} - f(u)\right| \leqslant \left|\frac{W(u)}{p(u)} - g(u)\right| \leqslant \left|\frac{W(u)}{p(u)} - h(u)\right| + |h(u) - g(u)| < 2\varepsilon.$$

Thus we have proved that for every $\varepsilon > 0$ there exists a polynomial $W(w)$ such that

$$|W(w)| < (1+\lambda)\varepsilon \quad \text{for} \quad |w| \leqslant \varrho_1,$$

$$\left|\frac{W(u)}{\chi(u)} - f(u)\right| < 2\varepsilon \quad \text{for} \quad u \geqslant \varrho_1.$$

Taking $\varepsilon = 1/k$ $(k = 1, 2, \ldots)$ we get a sequence of polynomials $W_k(w)$ tending to zero uniformly in the circle $|w| \leqslant \varrho_1$ such that $W_k(u)/\chi(u)$ tends to $f(u)$ uniformly on the interval $\varrho_1 \leqslant u < +\infty$. In virtue of $(*)$ we have

$$\left|\int_{\varrho}^{+\infty} \frac{W_k(u)}{\chi(u)} d\omega(u)\right| \leqslant C \frac{\varrho_1}{\varrho_1 - \varrho} \sup_{|w| \leqslant \varrho_1} |W_k(w)|.$$

Taking the limit if $k \to +\infty$, we get

$$\int_{\varrho_1}^{\varrho_2} f(u) d\omega(u) = 0.$$

Hence we finally deduce that $\omega(u) = \text{const}$ for $u \geqslant \varrho$, q.e.d.

As a consequence of Lemmas 6 and 10 and Theorem 4 we get the following

THEOREM 6. *Let $p, q \in \mathcal{F}_r$, $D \in \mathcal{N}_r$ and let the functions p, q fulfil condition* (β) *for $r = +\infty$. Then*

(a) *The power method $M(p)$ is perfect if and only if the function $p(u)$ has the* (M)*-property.*

(b) *If the power method $M(p)$ is perfect, D is a simply connected domain and the inequality $q(u) \geqslant p(u)$ is satisfied for $u \geqslant 0$, then the method $M(q, D)$ is perfect; in particular, if $q(u) \geqslant e^u$ for $u \geqslant 0$, the method $M(q, D)$ is perfect.*

Note. Let us denote by $\hat{M}^*(p)$ $(p \in \mathcal{F}_r;\ r \leqslant +\infty)$ the set of all real sequences $x \in M^*(p)$ and put $\hat{M}(p; x) = M(p; x)$ for $x \in \hat{M}^*(p)$. Thus $\hat{M}(p)$ is a "real" power method. It is easy to see that the "real" power method $\hat{M}(p)$ is perfect if and only if the method $M(p)$ is perfect.

§ 4. Methods "extracted" from methods $M(p, D)$

Definition 11. Let $D \in \mathcal{N}_r$, $p \in \mathcal{F}_r$ and let $\hat{u} = (u_k)$ be a sequence of real numbers such that $u_k \in D$, $\lim_k u_k = r$.

By $M(p, D, \hat{u})$ we denote a method defined as follows. The *field* $M^*(p, D, \hat{u})$ of the method $M(p, D, \hat{u})$ is the set of all complex-valued sequences $x = (t_n)$ such that conditions (a), (b) from definition 6, § 1, are satisfied and

(c′) there exists a finite limit

$$\lim_k \frac{m(p, D; x, u_k)}{p(u_k)}.$$

The *generalized limit* $M(p, D, u; x)$ of the sequence $x \in M^*(p, D, \hat{u})$ (corresponding to the method $M(p, D, \hat{u})$) is defined as the limit

$$\dot{M}(p, D, \hat{u}; x) = \lim_k \frac{m(p, D; x, u_k)}{p(u_k)}.$$

REMARK 4. *The methods $M(p, D, \hat{u})$, $M(p, D)$ are consistent and $M^*(p, D) \subset M^*(p, D, \hat{u})$.*

In the space $M^*(p, D, \hat{u})$ we now introduce the family of pseudo-norms

$$(\mathrm{p}_3) \qquad \begin{cases} \|x\|_0 = \sup_k \left|\dfrac{m(p, D; x, u_k)}{p(u_k)}\right|; \\ \|x\|_Z = \sup_{w \in Z} |m(p, D; x, w)|, \quad Z \text{ — compact subsets of } D. \end{cases}$$

It is easy to verify that under the hypothesis $p^{(n)}(0) \neq 0$ $(n = 0, 1, \ldots)$ the space $M(p, D, \hat{u})$ is a (B_0)-sc-space under the family of pseudonorms (p_3). Considering the mapping U of the space $M^*(p, D, \hat{u})$ into the product $H(D) \times c$,

$$U(x) = \left(m(p, D; x, \cdot), \left(\frac{m(p, D; x, u_k)}{p(u_k)}\right)\right) \epsilon H(D) \times c,$$

we deduce that the space $M^*(p, D, \hat{u})$ is separable and the general form of linear functionals over $M^*(p, D, \hat{u})$ is given by the formula

$$(1_3) \quad \varphi(x) = \Phi\big(m(p, D; x, \cdot)\big) + \sum_{k=0}^{\infty} a_k \frac{m(p, D; x, u_k)}{p(u_k)} + aM(p, D, \hat{u}; x),$$

where

$$\sum_{k=0}^{\infty} |a_k| < +\infty;$$

Φ is a linear functional over $H(D)$.

Thus we have the following

LEMMA 11. *Let $D \epsilon \mathcal{N}_r$, $p \epsilon \mathcal{F}_r$, $p^{(n)}(0) \neq 0$ for $n = 0, 1, \ldots$ and let $\hat{u} = (u_k)$ be a sequence such as in definition* 11. *Then $M(p, D, \hat{u})$ is a separable (B_0)-method under the totality of pseudonorms* (p_3); *the general form of linear functionals over $M^*(p, D, \hat{u})$ is given by formula* (1_3).

THEOREM 7. *Let the hypotheses of Lemma* 11 *be satisfied, where the function p fulfils conditions* (β), § 3. *If the power method $M(p)$ is perfect and D is a simply connected domain, then the method $M(p, D, \hat{u})$ is also perfect* (see [11], Th. XII).

Proof. To prove the above theorem it suffices to consider the case of the increasing sequence $\hat{u} = (u_k)$.

Let $e = (1, 1, 1, \ldots)$, $e_m = (\underset{0}{0}, \ldots, 0, \underset{m}{1}, 0, 0, \ldots)$ for $m = 0, 1, \ldots$ Let $\varphi(x)$ be a linear functional on $M^*(p, D, \hat{u})$ such that $\varphi(e) = \varphi(e_m) = 0$ for $m = 0, 1, \ldots$ We have to show that $\varphi(x) \equiv 0$. Putting $x = e_m$ in formula (1_3), we have

$$(*) \qquad \sum_{k=0}^{\infty} a_k \frac{u_k^m}{p(u_k)} = -\Phi(f_m) \quad \text{where} \quad f_m(w) = w^m;\ m = 0, 1, \ldots$$

As in the proof of Lemma 6 we deduce that there exist constants C, $0 < \varrho < r$ such that

$$|\Phi(f_m)| \leqslant C\varrho^m \quad \text{for} \quad m = 0, 1, \ldots$$

In virtue of $(*)$ we get

$$(**) \qquad \sum_{k=0}^{\infty} a_k \frac{u_k^m}{p(u_k)} = O(\varrho^m).$$

Let $\omega(u)$ be a function defined as follows:

$$\omega(u) = \begin{cases} 0 & \text{for} \quad 0 \leqslant u < u_0, \\ a_0 + a_1 + \ldots + a_k & \text{for} \quad u_k \leqslant u < u_{k+1};\ k = 0, 1, \ldots \end{cases}$$

Thus $\omega(u)$ is continuous on the right and such that

$$\omega(u_k) - \omega(u_k-) = a_k,$$

$$\bigvee_0^{r-}(\omega) = \sum_{k=0}^{\infty} |a_k| < +\infty.$$

We have

$$\sum_{k=0}^{\infty} a_k \frac{u_k^n}{p(u_k)} = \int_0^{r-} \frac{u^n}{p(u)}\, d\omega(u).$$

In virtue of $(**)$ we obtain

$$\int_0^{r-} \frac{u^n}{p(u)}\, d\omega(u) = O(\varrho^n).$$

According to Theorem 6 (a) or Lemma 7 we establish that $\omega(u) = \text{const}$ for $\varrho \leqslant u < r$ and consequently $a_k = 0$ for $k > k_0$. Thus $\varphi(x)$ may be written in the form

$$\varphi(x) = \Phi\big(m(p, D; x, \cdot)\big) + \sum_{k=0}^{k_0} a_k \frac{m(p, D; x, u_k)}{p(u_k)} + aM(p, D, \hat{u}; x).$$

$$\varphi(x) = \Psi\big(m(p, D; x, \cdot)\big) + aM(p, D, \hat{u}; x),$$

where

$$\Psi(f) = \Phi(f) + \sum_{k=0}^{k_0} a_k \frac{f(u_k)}{p(u_k)} \quad \text{for} \quad f \epsilon H(D).$$

In virtue of $(*)$ we have $\Psi(f_m) = 0$ for $m = 0, 1, \ldots$ and $\Psi(W) = 0$ for every polynomial W. According to theorem of Runge we get $\Psi(f) = 0$ for every $f \epsilon H(D)$, since Ψ is a linear functional on $H(D)$. Thus we have $\varphi(x) = aM(p, D, \hat{u}: x)$. Putting $x = e$ we get $a = 0$, q.e.d.

§ 5. Relations between the generalized power methods $M(p, D)$ and the Toeplitz methods

THEOREM 8. *Let $D \epsilon \mathcal{N}_r$, $p \epsilon \mathcal{F}_r$ and let $\hat{u} = (u_k)$ be a sequence such as in definition* 11. *There exists a sequence of row-finite matrix methods*

$C_0, C_1, C_2, \ldots$ *such that*

$$M(p, D, \hat{u}; x) = \lim_k C_k(x)$$

for every $x \in M^*(p, D, \hat{u})$.

Proof. Let D_0 be such a domain that $D \in \mathscr{N}_r$, $D_0 \subset D$, which can be conformally mapped onto the circle $\{\zeta: |\zeta| < 1\}$. The existence of such a domain D_0 is the consequence of Riemann's theorem. Let $w = h(\zeta)$ be a conformal mapping of the circle $\{\zeta: |\zeta| < 1\}$ onto the domain D_0 such that $h(0) = 0$. It can be shown by easy induction that for $i = 0, 1, \ldots$ there exists such a system of functions $g_{i,0}(\zeta), g_{i,1}(\zeta), \ldots, g_{i,i}(\zeta)$ holomorphic for $|\zeta| < 1$ that the formula

$$\frac{d^i}{d\zeta^i} f\big(h(\zeta)\big) = \sum_{n=0}^{i} g_{i,n}(\zeta) f^{(n)}\big(h(\zeta)\big)$$

is satisfied for $|\zeta| < 1$ and for every function $f(w)$ holomorphic in D_0.

Writing $a_{i,n} = g_{i,n}(0)$ we obtain

$$f\big(h(\zeta)\big) = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{d^i}{d\zeta^i} f\big(h(\zeta)\big)\right)_{\zeta=0} \zeta^i = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\sum_{n=0}^{i} a_{i,n} f^{(n)}(0)\right)\zeta^i$$

for $|\zeta| < 1$.

Let $\zeta_k = h^{-1}(u_k)$ for $k = 0, 1, \ldots$ and let us put $f(w) = m(p, D; x, w)$. We have $f^{(n)}(0) = n!\, p_n t_n$ and consequently

$$M(p, D, \hat{u}; x) = \lim_k \frac{m(p, D; x, u_k)}{p(u_k)} = \lim_k \frac{f(u_k)}{p(u_k)} = \lim_k \frac{f\big(h(\zeta_k)\big)}{p(u_k)}$$

$$= \lim_k \frac{1}{p(u_k)} \sum_{i=0}^{\infty} \frac{1}{i!}\left(\sum_{n=0}^{i} a_{i,n} n!\, p_n t_n\right)\zeta_k^i$$

$$= \lim_k \frac{1}{p(u_k)} \left[\lim_m \sum_{i=0}^{m} \frac{1}{i!}\left(\sum_{n=0}^{i} a_{i,n} n!\, p_n t_n\right)\zeta_k^i\right]$$

$$= \lim_k \frac{1}{p(u_k)} \left[\lim_m \sum_{n=0}^{m}\left(\sum_{i=n}^{m} a_{i,n} \zeta_k^i \frac{1}{i!}\right) n!\, p_n t_n\right].$$

Writing

$$c_{k,m,n} = \frac{p_n n!}{p(u_k)} \sum_{i=n}^{m} a_{i,n} \zeta_k^i \frac{1}{i!}$$

for $n = 0, 1, \ldots, m$, $m = 0, 1, \ldots$ and $k = 0, 1, \ldots$, we finally obtain

$$M(p, D, \hat{u}; x) = \lim_k \left(\lim_m \sum_{n=0}^{m} c_{k,m,n} t_n\right) = \lim_k C_k(x),$$

where $C_k = (c_{k,m,n})$, q.e.d.

COROLLARY. *Let $D \in \mathscr{N}_r$, $p \in \mathscr{F}_r$. There exists a sequence of row-finite matrix methods $C_0, C_1, C_2, \ldots$ such that*

$$M(p, D; x) = \lim_k C_k(x)$$

for every $x \in M^*(p, D)$.

Definition 12. We shall say that a method M has the (Z)-*property* if there exists no permanent row-finite matrix method C such that $M^* \subset C^*$.

Zeller [12] has proved that the Abel power method has the (Z)-property. Zeller's theorem may be generalized as follows:

THEOREM 9. *Every perfect power method $M(p)$ has the (Z)-property, where $p \in \mathscr{F}_r$ and the function p satisfies hypothesis* (β), § 3.

In particular, if $r < +\infty$, then the power method $M(p)$ has the (Z)-property; the Borel power method has the (Z)-property.

The proof of this theorem may be obtained by a small modification of the proof of Zeller [12] with the use of Lemma 7 (if $r < +\infty$) or of Theorem 6 (a) (if $r = +\infty$).

THEOREM 10. *Let $D \in \mathscr{N}_r$ and let p be a function such as in Theorem 9. If the power method $M(p)$ has the (Z)-property and $D \neq K_r$, then there exists no permanent matrix method C such that $M^*(p, D) \subset C^*$.*

Proof. Suppose that such a method $C = (c_{m,n})$ exists. Thus we have $M^*(p) \subset M^*(p, D) \subset C^*$. Let w_0 be such a complex number that $|w_0| < r$, $w_0 \notin D$. Writing $x_0 = (1/p_n w_0^n)$, we easily establish that $x_0 \in M^*(p, D)$ and $x_0 \in C^*$, and consequently the series

$$\sum_{n=0}^{\infty} \frac{c_{m,n}}{p_n w_0^n}$$

is convergent for $m = 0, 1, \ldots$ Consequently there exist indices n_m where $n_m \geqslant m$ such that

$$\frac{|c_{m,n}|}{p_n |w_0^n|} \leqslant 1 \quad \text{for} \quad n \geqslant n_m;\ m = 0, 1, \ldots$$

Let us consider the row-finite matrix method $\bar{C}$ where

$$\bar{C}(x) = \lim_m \sum_{n=0}^{n_m} c_{m,n} t_n, \quad x = (t_n).$$

We show that $M^*(p) \subset \bar{C}^*$ and $\bar{C}(x) = C(x)$ for $x \in M^*(p)$, but this leads to a contradiction because the method $M(p)$ has the (Z)-property. Let $|w_0| < \varrho < r$ and $x = (t_n) \in M^*(p)$. Hence

$$\lim_n p_n t_n \varrho^n = 0$$

and consequently $p_n|t_n|\varrho^n \leqslant \lambda$ for $n = 0, 1, \ldots$ We have

$$\bar{C}_m(x) = C_m(x) - \sum_{n=n_m+1}^{\infty} c_{m,n} t_n,$$

whence

$$|\bar{C}_m(x) - C_m(x)| \leqslant \sum_{n=n_m+1}^{\infty} |c_{m,n}|\,|t_n|$$

$$\leqslant \lambda \sum_{n=m}^{\infty} \frac{1}{p_n \varrho^n} p_n |w_0|^n = \lambda \sum_{n=m}^{\infty} \left(\frac{|w_0|}{\varrho}\right)^n \to 0$$

if $m \to +\infty$. This shows that $x \in \bar{C}^*$ and $\bar{C}(x) = C(x)$, q.e.d.

§ 6. The $U(p)$ methods

REMARK 5. *Let $p \in \mathcal{F}_r$. If $D_1, D_2 \in \mathcal{M}_r$, then the methods $M(p, D_1)$, $M(p, D_2)$ are consistent.*

Proof. Let $x \in M^*(p, D_1) \cap M^*(p, D_2)$. We have $m(p, D_1; x, w) = m(p, D_2; x, w)$ for $0 \leqslant w = \mathrm{Re}(w) < r$, since $m(p, D_1; x, w)$ and $m(p, D_2; x, w)$ are analytic functions of the real variable on the interval $0 < w = \mathrm{Re}(w) < r$ and the above identity is satisfied in a neighbourhood of 0, q.e.d.

REMARK 6. *If $D_1, D_2 \in \mathcal{M}_r$, then there exists a $D_3 \in \mathcal{M}_r$ such that $D_3 \subset D_1 \cap D_2$, and consequently $M^*(p, D_1) \cup M^*(p, D_2) \subset M^*(p, D_3)$ where $p \in \mathcal{F}_r$.*

Proof. For every $u \in \langle 0, r)$ there exists an open circle $S_u = \{w: |w-u| < \varrho_u\} \subset D_1 \cap D_2$. It suffices to take $D_3 = \bigcup_{0 \leqslant u < r} S_u$, q.e.d.

Owing to Remarks 5 and 6 we can define the following method of limitation $U(p)$:

Definition 13. Let $p \in \mathcal{F}_r$. The *field of the method* $U(p)$ is defined as the set

$$U^*(p) = \bigcup_{D \in \mathcal{M}_r} M^*(p, D).$$

The *generalized limit* $U(p; x)$ of the sequence $x \in U^*(p)$ (corresponding to the method $U(p)$) is defined by the formula $U(p; x) = M(p, D; x)$, where $x \in M^*(p, D)$; $D \in \mathcal{M}_r$.

THEOREM 11. *Let $p \in \mathcal{F}_r$ and conditions* (β), § 3, *be fulfilled. The method $U(p)$ is a permanent method of limitation in the sense of definition* 3, *but $U(p)$ is not an (F)-method* (definition 5).

Proof. Suppose that $U^*(p)$ is an (F)-sc-space under a norm $\|\cdot\|$. Let us consider the sets $A_k \subset U^*(p)$, where

$$A_k = \left\{x = (t_n) \in U^*(p): |t_n| \leqslant \frac{k^n}{p_n} \text{ for } n = 1, 2, \ldots\right\}; \quad k = 1, 2, \ldots$$

The sets A_k are closed, since $\varphi_n(x) = t_n$ are continuous functionals on $U^*(p)$.

We show that

$$U^*(p) = \bigcup_{k=1}^{\infty} A_k.$$

Indeed, let $x \in U^*(p)$, whence $x \in M^*(p, D)$ where $D \in \mathcal{M}_r$. Thus the power series

$$\sum_{n=0}^{\infty} p_n t_n w^n$$

has a positive radius of convergence:

$$\overline{\lim_n} \sqrt[n]{p_n |t_n|} < +\infty.$$

Hence there exists a k such that $\sqrt[n]{p_n|t_n|} \leqslant k$ for $n = 1, 2, \ldots$ and consequently $x \in A_k$.

According to a theorem of Baire there exist a natural number k and a ball $K = \{x: \|x - x_0\| < \varrho\} \subset A_k$; $x_0 = (t_n^{(0)})$. Putting $y = (s_n)$, where $s_n = (-k-1)^n / p_n$, we have

$$\sum_{n=0}^{\infty} p_n s_n w^n = \frac{1}{1 + (k+1)w}$$

and consequently $y \in M^*(p, D)$, where

$$D = \left\{|w| < r: w \neq -\frac{1}{k+1}\right\} \in \mathcal{M}_r;$$

hence $y \in U^*(p)$. For sufficiently great integer m we have

$$\frac{1}{m} y + x_0 \in K$$

and consequently

$$\left|\frac{1}{m} s_n + t_n^{(0)}\right| \leqslant \frac{k^n}{p_n} \quad \text{for} \quad n = 1, 2, \ldots$$

Hence

$$\frac{1}{m}|s_n| \leqslant \frac{k^n}{p_n} + |t_n^{(0)}| \leqslant 2\frac{k^n}{p_n}; \quad \frac{1}{m} \leqslant 2\left(\frac{k}{k+1}\right)^n.$$

Taking the limit for $n \to +\infty$, we obtain a contradiction, q.e.d.

Remark 7. *Let p be such a function as in Theorem 11. A sequence θ_n ($\theta_n > 0$) is a rate of growth of the method $U(p)$ if and only if*

$$\lim_n \sqrt[n]{\frac{\theta_n}{p_n}} = 0.$$

Proof. 1° Let

$$\lim_n \sqrt[n]{\frac{\theta_n}{p_n}} = 0$$

and let $x = (t_n) \in U^*(p)$. We have

$$\lim_n p_n t_n \varrho^n = 0$$

for certain $\varrho > 0$, since the power series

$$\sum_{n=0}^{\infty} p_n t_n w^n$$

has a positive radius of convergence. For $n \geqslant n_0$ we have $\sqrt[n]{\theta_n/p_n} \leqslant \varrho$; $\theta_n \leqslant p_n \varrho^n$, hence $\theta_n |t_n| \leqslant p_n |t_n| \varrho^n \to 0$ if $n \to +\infty$.

2° Let (θ_n) be a rate of growth of the method $U(p)$ and let $\varepsilon > 0$. It is easy to see that

$$\left(\frac{1}{p_n(-\varepsilon)^n}\right) \in U^*(p)$$

and consequently the sequence $\left(\theta_n/p_n(-\varepsilon)^n\right)$ is bounded: $\theta_n/p_n\varepsilon^n \leqslant C$. Hence

$$\sqrt[n]{\frac{\theta_n}{p_n}} \leqslant \varepsilon \sqrt[n]{C}, \qquad \overline{\lim_n} \sqrt[n]{\frac{\theta_n}{p_n}} \leqslant \varepsilon$$

but this means that

$$\lim_n \sqrt[n]{\frac{\theta_n}{p_n}} = 0,$$

q.e.d.

Corollary. *If the hypothesis of Remark 7 is satisfied, then the method $U(p)$ has no strict rate of growth.*

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