On generalized power methods of limitation

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Introduction

The subject of this paper is to discuss the properties of so-called generalized power methods, in particular power methods. A generalized power method \( M(p, D) \) is determined by the function

\[
p(w) = \sum_{n=0}^{\infty} p_n w^n
\]

holomorphic in the circle \( K = \{ w : |w| < r \} \) satisfying \( f(w) \neq 0 \) for \( r < w = \text{Re}(w) < r \) and by a domain \( D \subset K \), such that \( 0 < D_1 \subset D \); the sequence \( \pi = (\pi_n) \) is called limitable to \( t \) by the method \( M(p, D) \) if there exists a function \( m(p, D; \pi, w) \) holomorphic in \( D \) such that

\[
m(p, D; \pi, w) = \sum_{n=0}^{\infty} \pi_n w^n
\]
in a neighbourhood of 0 and

\[
\lim_{w=\text{Re}(w)x} \frac{m(p, D; \pi, w)}{p(w)} = t.
\]

In the particular case of \( D = K \), the method \( M(p, D) = M(p) \) is called a power method. For \( p(w) = (1 - w)^{-1} \) resp. \( e^w \) we obtain in this way the classical methods of Abel resp. Borel. The method \( M(p, D) \) is permanent if, for instance, the function \( p(w) \) satisfies the condition

\[
\begin{cases}
p_n > 0 & \text{for } n = 0, 1, \ldots, \\
\lim_{w=\text{ln}e^w} p(w) = +\infty,
\end{cases}
\]

which will be assumed very often.

Power methods have been investigated by several writers. Wodarski [11] proved in 1904 that every method \( M(p) \) for \( r < +\infty \) is perfect. The perfection of the Abel method was noticed earlier by K. Zeller. The investigation of methods \( M(p) \) for \( r = +\infty \) is more difficult...
and the question whether these methods are perfect has been answered only in the case of the Borel method; Ryll-Nardzewski [9] proved in 1962 that this method is perfect. Generalized power methods have not been yet investigated except the generalized Abel methods, i.e., methods $M(p, D)$, where $p(w) = (1 - w)^\tau$. These methods were distinguished long ago, e.g., in the investigation of the consistency of Nörlund methods [10].

In this paper the methods of functional analysis are applied. The definitions of spaces such as $(B)$, $(B_0)$, $(F)$ are those of [1] and [5].

The main results are contained in § 3, in which the problem of perfection of generalized power methods is treated. First of all it is proved that a method $M(p, D)$ for $\tau < +\infty$ is perfect if and only if the domain $D$ is simply connected. This result covers the theorem of Wlodarski, mentioned above, concerning power methods. S. Mazur suggested to the present writer the investigation of the problem whether every power method $M(p)$ for $\tau = +\infty$ is perfect. Only partial results concerning this problem have been obtained. It is proved that the power method $M(p)$ ($p$ is supposed to satisfy condition (\text{\text{\text{\text{\text{\text{\text{\text{\text{\text{\text{\text{\text{(M)}}}}}}}}}}}}$) is perfect if and only if the function $p(w)$ has the following property (M):

\begin{equation}
(0, +\infty) \quad \int_0^{+\infty} \frac{u^n}{p(u)} \, du = O(u^\sigma) \quad \text{for} \quad q > 0
\end{equation}

implies $w(w) = \text{const}$ for $u \geq q$.

This result, in the writer's opinion, shows the difficulties connected with the solving the problem of perfection of methods $M(p)$ for $\tau = +\infty$. Let us remark that if the function $p(w)$ has the property (M), then it has the following property (LD):

\begin{equation}
\text{LD: The set of functions } w^n/p(u), n = 0, 1, \ldots, \text{ is linearly dense (in the sense of uniform convergence) in the space of functions } f(u) \text{ continuous in the interval } 0 \leq u < +\infty \text{ and such that } \lim f(u) = 0.
\end{equation}

The writer does not know the answer to the question whether every entire function $p(w) = \sum_{n=0}^{\infty} p_n w^n$, where $p_n > 0$, has the property (LD) or whether the property (LD) implies (M)\(^{(1)}\).

In this paper a simple condition for the method $M(p, D)$ to be

\begin{equation}
\text{perect is given (Lemma 2). Elementary considerations enable us to deduce the theorem of Ryll-Nardzewski, mentioned above on the perfection of the Borel method.}
\end{equation}

I wish to thank Professor S. Mazur for drawing my attention to these problems and for his helpful comments and suggestions made in the course of my writing the present paper.

\hspace{1cm}§ 0. The terminology and the notation concerning certain notions of functional analysis and the theory of limitation

\begin{definition}
\textbf{Definition 1.} $X$ is called a sequence-space if $X$ is a linear space of complex-valued sequences $x = (x_n)$ with the usual definition of addition and multiplication by scalars.
\end{definition}

\begin{definition}
\textbf{Definition 2.} Let $X$ be a sequence-space of type (F). We say that $X$ is a sc-space if $s_n(x) = t_n$ are continuous functionals over $X$ for $n = 0, 1, \ldots ; x = (x_n)$.
\end{definition}

\begin{definition}
\textbf{Definition 3.} Let $X$ be a sequence-space and let $\varphi(x)$ be an additive and homogeneous functional over $X$. Then the pair $(X, \varphi)$ is called the method of limitation $M$. The set $X = M^\circ$ is called the field of the method $M$; the number $\varphi(x) = M(x)$ is called the generalized limit of the sequence $x \in X$ (corresponding to the method $M$).
\end{definition}

\begin{definition}
\textbf{Definition 4.} A method $M$ is called \textit{permanent} if $T_x \subset M^\circ$ and $\varphi(x) = \lim F_x$ for every $x = (x_n)$, where $T_x$ denotes the set of all convergent sequences.
\end{definition}

\begin{definition}
\textbf{Definition 5.} Let $M$ be a method of limitation. If $M^\circ$ is an $(F)$-sequence space such that $M(x)$ is a continuous functional over $M^\circ$, we shall say that $M$ is an $(F)$-method. In particular, if $M^\circ$ is a $(B_0)$-sequence space, then the method $M$ is an $(F)$-method and similarly for the other spaces (see [4] and [11]).
\end{definition}

In this paper we use the following notation:

\begin{itemize}
\item \textbf{1.} $T_x$ denotes the set of all convergent complex-valued sequences.
\item \textbf{2.} $e$ denotes the (B)-space of all convergent complex-valued sequences $x = (x_n)$ with the norm $\|x\| = \sup |x_n|$.
\item \textbf{3.} $e$ denotes the (B)-space of all convergent complex-valued sequences $x = (x_n)$ with the family of pseudonorms $\|x\| = |x_n|; n = 0, 1, \ldots$
\item \textbf{4.} $C(r) = C(r, r)$ where $-\infty < r < t < +\infty$ denotes the $(B)$-space of all complex-valued functions $f(u)$ on the interval $r < u < r$ with the finite limit $\lim f(u)$. The norm in $C(r, r)$ is given by the formula $\|f\| = \sup_{u < r} |f(u)|$.
\end{itemize}

\footnote{After the author had sent the paper to press, he noticed that the answer to the former question was negative and gave a condition for the function $p(w)$ to have the property (LD) (see [2a]).}
(5) \( H(D) \) denotes the \((B_\infty)\)-space of all functions \( f(w) \) holomorphic in domain \( D \) with the family of pseudonorms \( \| f \|_w = \sup_{w \in D} |f(w)| \) where \( Z \) are compact subsets of \( D \).

(6) \( H_0 \) denotes the \((B_\infty)\)-space of all integer functions \( f(w) \) with the finite limit \( \lim_{w \to \infty} f(w) \). The pseudonorms in \( H_0 \) are given by the formulas \( \| f \|_w = \sup_{w \in D} |f(w)|, \| f \|_w = \sup_{w \in D} |f(w)|; e = 1, 2, \ldots \).

(7) \( K_r \) denotes the circle \( \{ w : |w| < r \} \) in the complex plane (where \( 0 < r < +\infty \)).

(8) \( \mathcal{N} \) denotes the family of all domains \( D \) in the complex plane satisfying the following conditions:

1. \( D \subset K_r \);
2. \( 0 \in D \);
3. there exists a number \( r_0 < r \) such that \( \{ w : r_0 < w = \text{Re}(w) < r \} \subset D \).

(9) \( \mathfrak{S} \) denotes the family of all functions \( p(w) = \sum_{n=0}^{\infty} p_n w^n \) satisfying the following conditions:

1. the radius of convergence of the power series \( \sum_{n=0}^{\infty} p_n w^n \) is equal to \( r \);
2. there exists a number \( r < r \) such that \( p(w) \neq 0 \) for \( r < \text{Re}(w) < r \).

(10) For \( p \in \mathfrak{S} \), and \( D \in \mathcal{N} \), we shall denote by \( H(p, D) \) the \((B_\infty)\)-space of all functions \( f(w) \) holomorphic in \( D \) with the finite limit

\[ \lim_{w \to \infty} \frac{f(w)}{p(w)} \]

The pseudonorms in \( H(p, D) \) are given by the formulas

\[ \| f \|_w = \sup_{w \in D} \left| \frac{f(w)}{p(w)} \right|, \| f \|_w = \sup_{w \in D} |f(w)|, \]

where \( Z \) are compact subsets of \( D \).

(11) \( \mathcal{M} \) denotes the family of all domains \( D \) in the complex plane such that \( D \in \mathcal{N} \), \( \{ w : 0 \leq w = \text{Re}(w) < r \} \subset D \).

§ 1. The definition and simple properties of the generalized power methods

Definition 6. By \( M(p, D) \) we denote a method of limitation determined by a function \( p \in \mathfrak{S} \) and a domain \( D \in \mathcal{N} \), defined as follows:

The field \( M^*(p, D) \) of the method \( M(p, D) \) is the set of all complex-valued sequences \( x = (x_n) \) such that:

(a) the power series \( \sum_{n=0}^{\infty} p_n x_n w^n \) has a positive radius of convergence;
(b) the function \( \sum_{n=0}^{\infty} p_n x_n w^n \) is extendible to a function \( m(p, D; x, w) \) holomorphic in the domain \( D \);
(c) there exists a finite limit

\[ \lim_{w \to \infty} \frac{m(p, D; x, w)}{p(w)} \]

The generalized limit \( M(p, D; x) \) of the sequence \( x \in M^*(p, D) \) (corresponding to the method \( M(p, D) \)) is defined as the limit

\[ M(p, D; x) = \lim_{w \to \infty} \frac{m(p, D; x, w)}{p(w)} \]

In the particular case of \( D = K \), we shall denote the method \( M(p, D) \) by \( M(p) \); \( M(p) \) is called a power method. For example if \( r = 1 \) and \( p(w) = (1-|w|)^{-1} \) or \( r = +\infty \) and \( p(w) = e^w \), is \( M(p) \) the Abel method or the Borel method, respectively.

Remark 1. Let \( p \in \mathfrak{S} \), and \( D_1, D_2 \in \mathcal{N} \). If \( D_1 \subset D_2 \), then \( M^*(p, D_1) \supset M^*(p, D_2) \) and the methods \( M(p, D_1) \) and \( M(p, D_2) \) are consistent: \( M(p, D_1; x) = M(p, D_2; x) \) for \( x \in M^*(p, D_1) \).

In the space \( M^*(p, D) \) we introduce the family of pseudonorms

\[ \| x \|_n = |x_n|, \| x \|_n = \sup_{n=0}^{\infty} |x_n| ; \]

where \( Z \) takes on the values from the family of all compact subsets of \( D \). (In fact the family of pseudonorms \( \| x \|_n \) is equivalent to a denumerable family of pseudonorms \( \| x \|_k, k = 1, 2, \ldots \)).

It is easy to verify that \( M^*(p, D) \) is the \((B_\infty)\)-space under the totality of pseudonorms \( (\| x \|_n) \).

Now we consider a mapping \( U \) of the space \( M^*(p, D) \) into the product \( s \times H(D) \times C(\gamma, \rho) \), defined by the formula

\[ U(x) = \left( x, m(p, D; x, \cdot), \frac{m(p, D; x, \cdot)}{p(\cdot)} \right) \]

Of course \( U \) is a linear homeomorphism. According to the separability of the spaces \( s, H(D), C(\gamma, \rho) \) and the general form of linear
functional on those spaces, this implies that the space $M^*(p, D)$ is separable and the general form of linear functionals on $M^*(p, D)$ is given by the formula

$$
(l_1) \quad \nu(x) = \sum_{n=1}^k a_n \omega_n + \Phi \left( m(p, D; x, \cdot) \right) + \int_{0}^{r} \frac{m(p, D; x, u) \omega(u)}{p(u)} + a M(p, D; x),
$$

where $\Phi$ is a linear functional on $H(D); \omega(u)$ is a function continuous on the right with $\int_{0}^{r} (u) < +\infty$. The meaning of the symbol $\int_{0}^{r}$ is

$$
\int_{0}^{r} = \lim_{\epsilon \to 0^+} \int_{0}^{r + \epsilon}.
$$

Let us suppose now that the function $p \in \mathcal{F}$ fulfills the condition

$$(a) \quad m(p_n, D; x, w) \neq 0 \quad \text{for} \quad n = 0, 1, \ldots
$$

It is easy to establish that $M^*(p, D)$ is now the $(B_1)$-space under the family of pseudonorms

$$(p_1) \quad \|x\|_0 = \sup_{\mu \in \mathcal{M}} \frac{m(p, D; x, w)}{p(w)}.
$$

(Z has the same meaning as in the formula $(p_1)$).

Under the hypothesis $(a)$ it is easy to establish that the space $M^*(p, D)$ is isomorphic to the space $H(p, D)$. The isomorphism between $M^*(p, D)$ and $H(p, D)$ is given by the formula

$$(i) \quad f(u) = m(p, D; x, w), \quad \text{where} \quad w \in \mathcal{E} \subset \mathcal{M}^*(p, D); f \in \mathcal{H}(p, D).
$$

The general form of linear functionals on $M^*(p, D)$ is given by the formula

$$
(l_2) \quad \nu(x) = \Phi \left( m(p, D; x, \cdot) \right) + \int_{0}^{r} \frac{m(p, D; x, u) \omega(u)}{p(u)} + a M(p, D; x),
$$

where $\Phi, \omega$ have the same meaning as in the formula $(l_1)$.

In some considerations we shall suppose that

$$(\beta) \quad \nu_n > 0 \quad \text{for} \quad n = 0, 1, \ldots, \quad \lim_{w \to 0^+} \nu_n(w) = +\infty.
$$

The fulfillment of $(\beta)$ implies the permanence of the method $M(p, D)$.

From the above considerations we get the following

**Theorem 1.** Let $p \in \mathcal{F}$, $D \in \mathcal{M}$. Then

$$(a) \quad M(p, D) \text{ is a separable (B_1)-method under the totality of pseudonorms (p_1), the general form of linear functionals on } M^*(p, D) \text{ is given by formula (l_1).}
$$

(b) If $p(0) \neq 0$ for $n = 0, 1, \ldots$, then $M(p, D)$ is a (B_1)-method under the totality of pseudonorms (p_1); the general form of linear functionals on $M^*(p, D)$ is given by formula (l_2); the spaces $M^*(p, D)$ and $H(p, D)$ are isomorphic and the isomorphism between $M^*(p, D)$ and $H(p, D)$ is given by formula (i).

**§ 2. Rate of growth of the method M(p, D)**

**Definition 7.** A sequence $(\theta_n)$ is called a rate of growth of the method $M$ if $\theta_n > 0$ and $\sup_{n} \theta_n |t_n| < +\infty$ for every $x = (t_n) \in M^*$.

**Definition 8.** A sequence $(\theta_n)$ is called a strict rate of growth of the method $M$ if $\theta_n$ is a rate of growth of the method $M$ and for every rate of growth $(\theta_n)$ of the method $M$ there exists a constant $C$ such that $\theta_n \leq C \theta_n$ for $n = 0, 1, \ldots$ (see [4]).

**Remark.** Let $p \in \mathcal{F}$, $D \in \mathcal{M}$. If $p(0) \neq 0$ for $n = 0, 1, \ldots$, then the method $M(p, D)$ has a rate of growth.

**Proof.** Let us write

$$
\theta_0 = \inf_{w \in \mathcal{D}} |w|
$$

and let us fix $0 < \epsilon < \theta_0$. For every $x = (t_n) \in M^*(p, D)$ we have

$$
\sum_{n=0}^{\infty} |t_n| \epsilon < +\infty.
$$

Thus the sequence $\theta_n = |p_n| \epsilon^\theta$ is the rate of growth of the method $M(p, D)$, q.e.d.

**Theorem 2.** Let $p \in \mathcal{F}$, and $p(0) \neq 0$ for $n = 0, 1, \ldots$. Then the power method $M(p)$ has no strict rate of growth.

**Proof.** Let us suppose on the contrary that a sequence $(\theta_n)$ is a strict rate of growth of the power method $M(p)$. We consider the space $M^*(p)$ as the $(B_1)$-space under the family of pseudonorms

$$
\|x\|_0 = \sup_{\mu \in \mathcal{M}} \frac{m(p; x, w)}{p(w)}, \quad \|x\|_0 = \sup_{\mu \in \mathcal{M}} \frac{m(p; x, w)}{p(w)},
$$

where $0 < \epsilon < \epsilon'$. Since $\nu_k(x) = \theta_n t_n$ are linear functionals on $M^*(p)$ and

$$
\sup_{n} |\nu_k(x)| = \sup_{n} \theta_n |t_n| < +\infty.
$$
for every \( x = \langle t_n \rangle \in M^*(p) \), there exist numbers \( C \) and \( 0 < \theta_n < r \) such that (see [5])

\[ \theta_n |\alpha| \leq C|\max(|\alpha|, |\|\alpha|\|)| \quad \text{for} \quad n = 0, 1, \ldots; \quad x = \langle t_n \rangle \in M^*(p). \]

Since the sequence \((|p_n| e^n)\) is the rate of growth of the method \( M(p) \) for \( 0 < \theta < r \), there exists a constant \( C_2 \) such that

\[ |p_n| e^n \leq C_2 |\|x|\| \quad \text{for} \quad n = 0, 1, \ldots, \quad 0 < \theta < r. \]

In virtue of inequalities (\( \ast \), \( \ast \ast \)) we have for \( 0 < \theta < \theta_1 < r \) and \( |\|x|\| \leq e \):

\[
|\|x|\|_{p, x, w} = \left| \sum_{n=0}^{\infty} p_n t_n w^n \right| \leq \sum_{n=0}^{\infty} |p_n| t_n |w| = \sum_{n=0}^{\infty} |p_n| t_n |w| \left( \frac{e}{\theta_1} \right)^n \\
\leq C_2 \sum_{n=0}^{\infty} \theta_n |\alpha| \left( \frac{\theta_1}{\theta} \right)^n \leq C C_2 \frac{\theta_1}{\theta_1 - \theta} \max(|\|x|\|, |\|w|\|). 
\]

Thus we have

\[ \theta_n |\|x|\| \leq C C_2 \frac{\theta_1}{\theta_1 - \theta} \max(|\|x|\|, |\|w|\|). \]

Inequality \( \ast \ast \ast \) shows the conditions

\[ \exists \alpha \in M^*(p), \quad \lim_k |\|x_k|\|_k = \lim_k |\|p_k|\|_k = 0 \]

imply that

\[ \lim_k |\|x_k|\|_k = 0 \quad \text{for every} \quad 0 < \theta < r. \]

Thus in virtue of Theorem 1(b) the conditions

\[ f_{x, \varrho} = H(p, \varrho), \quad \lim_{\varrho \to \rho} \sup_{k} |\|f_{x, \varrho}(w)|| = 0 \]

imply that

\[ \lim_{\varrho \to \rho} \sup_{k} |\|f_{x, \varrho}(w)|| = 0 \]

for every \( 0 < \varrho < r \), but it is easy to establish that it is impossible, q.e.d.

The next theorem concerns the rate of growth of the generalized Abel method.

In the case \( p(w) = (1 - w)^{-1}, D \subset \mathbb{N}, \) we denote the method \( M(p, D) \) by \( A(D) \). Let us write \( \theta_0 = \inf |\|x|\|. \)

**Theorem 3.** A sequence \( (\theta_n) \) (\( \theta_n > 0 \)) is a rate of growth of the generalized Abel method \( A(D) \) if and only if there exist constants \( q, C \) such that

\[ 0 < \theta < \theta_0, \quad \theta_n \leq C q^n \quad \text{for} \quad n = 0, 1, \ldots \]

The sufficiency of the condition \( \theta_n \leq C q^n \) is trivial. To prove the necessity we need two simple remarks and a lemma.

For \( x = \langle t_n \rangle \) we shall write \( x^n = \langle t_n, t_1, \ldots \rangle \), \( x^n = \langle t_n \rangle \rightarrow (0, \ldots, 0, t_n, t_1, \ldots) \), where \( t_n = 0 \) for \( n < 0 \). Let us write

\[ [x]_Z = \sup_{w \in Z} |A(D; x, w)| \]

for \( Z \subset D \) and \( x \in \mathbb{A}^*(D) \), where \( A(D; x, w) = (1 - w) m(p, D; x, w); p(w) = (1 - w)^{-1}. \)

**Remark 1.** If \( x \in \mathbb{A}^*(D) \), then \( \varrho^M \in \mathbb{A}^*(D) \) for \( k = 0, 1, \ldots \) and

\[ [\varrho^M]_Z \leq [x]_Z. \]

This is a consequence of the simple identity \( A(D; \varrho^M, w) = \varrho A(D; x, w). \)

**Remark 2.** To prove Theorem 3 it is sufficient to prove it for a domain of the form \( D_k = (w; |w| < 1)|\alpha|, \) where \( w = 1 \).

Indeed, let \( w_0 \) be a point such that \( |w_0| = \theta_0, w_0 \in D \) (if \( \theta_0 = 1 \) we put \( w_0 = -1 \)). Since \( D \subset D_k \), we have \( \mathbb{A}^*(D_k) \subset \mathbb{A}^*(D_k) \). Thus if \( \theta_n \) is a rate of growth of the method \( A(D) \), then \( \theta_n \) is also a rate of growth of the method \( A(D_k) \).

We consider the space \( \mathbb{A}^*(D) \) as the \( (B_k) \)-space under the totality of pseudonorms

\[ [x]_k = \sup_{0 \leq |\|x|\|, |\|w|\| \leq 1} |A(D; x, w)|, \]

\[ [x]_k = \sup_{0 \leq |\|x|\|, |\|w|\| \leq 1} |A(D; x, w)| \]

for \( i = i_0, i_0 + 1, \ldots \), where \( r_i, i_0 \) are fixed numbers \( (i_0 > 0) \) such that the circle \( |w| = 1/2 \) lies inside the circle \( |w| = 1 - 1/2 \) and the interval \( |w| = r_i \) lies outside the circle \( |w| = 1 - 1/2 \).

**Lemma.** Let \( \theta_n \) be a sequence such that \( \theta_n \rightarrow 0 \) and the inequalities

\[ \theta_n |\|x|\| \leq C \max(|\|x|\|, |\|w|\|) \]

are satisfied for \( n = 0, 1, \ldots \) and for every \( x = \langle t_n \rangle \in \mathbb{A}^*(D) \), where \( a \) and \( l \) are fixed \( (l > i_0) \). Then there exists a constant \( \beta \) such that

\[ \theta_{mn} \leq \beta \left( \theta_n \right)^{l/m} \quad \text{for} \quad m = 0, 1, \ldots \]

**Proof.** Let us consider the functions

\[ f_m(w) = \frac{1 - w}{(1 - w/m)^{m-1}} \quad \text{for} \quad m = 0, 1, \ldots \]
Expanding them in a power series we get

\[ f_m(w) = (1 - w) \sum_{n=1}^\infty \left( \frac{n + m}{m} \right) \frac{1}{w_0^n}, \quad |w| < |w_0| = \phi_0. \]

Writing

\[ x_m = (\phi_0)^n, \quad \text{where} \quad \phi_0 = \left( \frac{n + m}{m} \right)^{1/m} \frac{1}{w_0}, \]

we get \( x_m \in A^*(D) \) and \( f_m(w) = A(D; x_m, w) \).

For \( |w - w_0| > 1/\phi \) we have

\[ |f_m(w)| = |1 - w| \frac{x_m}{|w_0 - w|} \leq 2|x_m|^{m+1} = 2\phi_0^{m+1}. \]

and we get the inequalities

\[ ||x_m|| < 2\phi_0^{m+1}, \quad ||x_m|| > 2\phi_0^{m+1}. \]

According to the hypothesis of the Lemma we have \( \phi_h ||x_m|| < \max(||x_m||, ||x_m||) \) and consequently

\[ \phi_h \left( \frac{n + m}{m} \right)^{1/m} \leq 2\phi_0^{m+1}, \quad \text{for} \quad n, m = 0, 1, \ldots \]

Putting in the above inequality \( n = im \) we get

\[ \phi_{im} \left( \frac{m + m}{m} \right)^{1/m} \leq 2\phi_0^{m+1}. \]

We have

\[ \left( \frac{m + m}{m} \right) = \frac{(m + m)(m + m - 1) \ldots (m + 1)}{m^m} > \frac{m^m}{m!} \]

and consequently

\[ \phi_{im} \left( \frac{m^m}{m!} \right)^{1/m} \leq 2\phi_0^{m+1}. \]

and after simplification we get

\[ \phi_{im} \leq 2\phi_0^{m+1} \phi_0^{im} = 2\phi_0^{m+1} \left( \frac{1}{\sqrt{2}} \right)^{im} \phi_0. \]

The radius of convergence of the power series

\[ \sum_{n=0}^\infty \frac{m!}{m^m} \phi_0^n \]

is equal to \( \phi \); thus

\[ \sum_{n=0}^\infty \frac{m!}{m^m} \phi_0^n < \infty \]

and consequently there exists a constant \( \beta \) such that

\[ 2\phi_0^{m+1} \phi_0^{im} \leq \beta \quad \text{for} \quad m = 0, 1, \ldots \]

Finally we get

\[ \phi_{im} \leq \beta \left( \frac{\phi_0}{\sqrt{2}} \right)^{im} \quad \text{for} \quad m = 0, 1, \ldots, \]

q.e.d.

Proof of Theorem 3. Let the sequence \( (\theta_n) \) be a rate of growth of the function \( A(D) \). As \( \theta_n(x) = \theta_n x \) are linear functionals over \( A^*(D) \) and

\[ \sup_n \{ \theta_n(x) \} = \sup_n \theta_n ||x|| < \infty \]

for every \( x = (\xi_k) \in A^*(D) \), there exist an index \( l > i_n \) and a constant \( a \) such that (see [5])

\[ \theta_l ||x|| \leq \max(||x||, ||x||), \quad x = (\xi_k) \in A^*(D). \]

Putting \( x^{(k)} = (\xi_{n-k}) \) in place of \( x \) we get

\[ \theta_l ||x^{(k)}|| \leq \max(||x^{(k)}||, ||x^{(k)}||), \]

and according to Remark 1 we get

\[ \theta_l ||x^{(k)}|| \leq \max(||x^{(k)}||, ||x^{(k)}||), \quad k, n = 0, 1, \ldots, \]

i.e.

\[ \theta_{l+k} ||x|| \leq \max(||x||, ||x||), \quad k, n = 0, 1, \ldots \]

In virtue of the Lemma there exist constants \( \beta_k \) for \( k = 0, 1, \ldots \) such that

\[ \theta_{lm+k} \leq \beta_k \phi_0^{im}, \quad \text{where} \quad \phi = \frac{\phi_0}{\sqrt{2}}. \]

Let us write \( \beta = \max(\beta_0, \beta_1, \ldots, \beta_{l-1}). \)

Now let \( n \) be an arbitrary non-negative integer. We may write

\[ n = im + k, \quad \text{where} \quad k = 0, 1, \ldots, l-1; \quad m = 0, 1, \ldots \]

Thus we have

\[ \theta_n = \theta_{lm+k} \leq \beta \phi_0^{im} = \beta \phi_0^{im} \phi_0^{im} = \beta \phi_0^{im} \phi_0^{im} = \phi_0^{im}. \]

where \( C = \beta \phi_0^{im}, \) q.e.d.
Corollary. A sequence \((\theta_n) (\theta_n > 0)\) is a rate of growth of the Abel power method \(A\) if and only if there exist constants \(q\) and \(C\) such that \(0 < q < 1, \theta_n \leq C \theta^n\) for \(n = 0, 1, \ldots\)

§ 3. Perfection of the methods \(M(p, D)\)

Definition 9. Let \(M\) be a permanent \((R_b)\)-method. \(M\) is called a perfect method at a point \(x_0 \in M^*\) if for every permanent \((R_b)\)-method \(N\) such that \(M^* \subseteq N^*\) the equality \(M(x_0) = N(x_0)\) is satisfied. \(M\) is called a perfect method if it is perfect at every point \(x_0 \in M^*\).

The following criterion plays an essential role in the considerations concerning the problem of perfection in the theory of limitation (see [3]):

Lemma 1. Let \(M\) be a permanent \((R_b)\)-method. The method \(M\) is perfect in \(x_0 \in M^*\) if and only if \(x_0\) is a point of accumulation of the set \(T_\alpha\) of all convergent sequences. In particular, the method \(M\) is perfect if and only if the set \(T_\alpha\) of all convergent sequences is dense in the space \(M^*\).

Proof. 1° Suppose that \(x_0 \in T_\alpha\). Thus there exists on \(M^*\) a linear functional \(\varphi(x)\) such that \(\varphi(x) = 0\) for \(x \notin T_\alpha\), and \(\varphi(x_0) = 1\). Let us consider a method \(N\) defined in the following way: \(N^* = M^*\), \(N(x) = M(x) + \varphi(x)\) for \(x \in M^*\). Thus \(N\) is a permanent \((R_b)\)-method, but the sequence \(M(x_0) + \varphi(x_0) = M(x_0) + 1 \neq M(x_0)\), whence \(M\) is not a perfect method at the point \(x_0\).

2° Suppose that \(x_0 \notin T_\alpha\) and \(M^* \subseteq N^*\), where \(N\) is a permanent \((R_b)\)-method. Let us denote by \(\|\cdot\|_M, \|\cdot\|_N\) the \((\mathcal{F})\)-norms in the spaces \(M^*\) and \(N^*\), respectively. Since

\[
\lim_{k \to \infty} \|x_k - x_0\|_M = 0,
\]

where \(x_k \in T_\alpha\) for \(k = 1, 2, \ldots\), we obtain

\[
\lim_{k \to \infty} \|x_k - x_0\|_N = 0
\]

(it is a result of K. Zeller; see e.g. [11], p. 190).

Thus we have

\[N(x_0) = \lim_{k} N(x_k) = \lim_{k} M(x_k) = M(x_0),\]

q.e.d.

In this paragraph we shall consider the methods \(M(p, D)\) under the hypothesis

\[
\begin{align*}
p^m(0) &> 0 \quad \text{for} \quad m = 0, 1, \ldots, \\
\lim_{w \to \infty} p(w) &\to +\infty.
\end{align*}
\]

Remark 3. Let \(D \in \mathcal{A}_r, p \in \mathcal{F}_r\). If the method \(M(p, D)\) is perfect, then \(D\) is a simply connected domain.

Proof. Suppose that \(D\) is not a simply connected domain. Hence there exist a point \(w_0 \in D\) and a simply connected domain \(G\) such that \(w_0 \notin G\) and \(I = \overline{G} - G \subset D\). Let us consider the sequence

\[
x_0 = \left\{ \frac{1}{p(w)} \right\},
\]

Since

\[
m(p, D; x_0, w) = \lim_{w \to \infty} \frac{\sum_{k=0}^{\infty} \frac{p_k}{p_k w_0^k}}{w - w_0}
\]

and

\[
\lim_{w \to \infty} \frac{m(p, D; x_0, w)}{p(w)} = 0,
\]

we obtain

\[
x_0 \in M^*(p, D),
\]

Suppose that the method \(M(p, D)\) is perfect. According to Lemma 1 there exists a sequence \(x_k \in T_\alpha\) such that

\[
\lim_{k} x_k = x_0.
\]

Consequently the sequence of the functions \(m(p, D; x_k, w)\) is uniformly convergent on \(D\) to the function \(m_\alpha(w - w_0)\), and consequently the sequence of the functions \(m(p, D; x_0, w)\) is uniformly convergent on the domain \(G\), but this is impossible since \(m(p, D; x_0, w)\) are holomorphic functions for \(|w| < r\), q.e.d.

Lemma 2. Let \(D \in \mathcal{A}_r, p \in \mathcal{F}_r\). The method \(M(p, D)\) is perfect if and only if the set of all functions of the form \(W(w) + tp(w)\), where \(W(w)\) denotes a polynomial and \(t\) — an arbitrary constant, is a dense subset of the space \(H(p, D)\).

Proof. 1° The sufficiency is a simple consequence of the equality

\[
\sum_{k=0}^{\infty} a_k w^k + tp(w) = m(p, D; x, w),
\]

where

\[
x = \left( \frac{a_0}{p_0}, \frac{a_1}{p_1} + t, \frac{a_2}{p_2} + t, \ldots, \frac{a_k}{p_k} + t, \ldots \right),
\]

since the spaces \(M^*(p, D)\) and \(H(p, D)\) are isomorphic.

2° Suppose that the method \(M(p, D)\) is perfect. Hence the set of the functions of the form

\[
\sum_{k=0}^{\infty} a_k w^k,
\]

where

\[
a_k = \left( \frac{a_0}{p_0}, \frac{a_1}{p_1}, \ldots, \frac{a_k}{p_k}, \ldots \right),
\]
where \((t_n) \ast T_x\), is dense in the space \(H(p, D)\). Let

\[
\lim_{n \to \infty} t_n = t \quad \text{and} \quad f_n(w) = \sum_{n=0}^{\infty} p_n(t_n - t)w^n + tp(w).
\]

We show that the sequence of the functions \(f_n(w)\) tends to the function

\[
f(w) = \sum_{n=0}^{\infty} p_n t_n w^n
\]

(in the sense of convergence in \(H(p, D)\)). Of course \(f_n(w) \to f(w)\) uniformly in the circle \(|w| < \rho < r\). It remains to prove that \(f_n(w)/p(w) \to f(w)/p(w)\) uniformly on the interval \(0 \leq u < r\).

Let \(e > 0\). We have \(|t_n - t| < e\) for \(n > n_e\). Hence we get for \(m \geq n_e\) and \(0 \leq u < r\):

\[
\left| \frac{f_n(w)}{p(u)} - \frac{f(u)}{p(u)} \right| = \left| \frac{1}{p(u)} \sum_{n=0}^{m} p_n(t_n - t)u^n \right| \leq \frac{1}{p(u)} \sum_{n=0}^{m} p_n |t_n - t| u^n \\
\leq \frac{e}{p(u)} \sum_{n=0}^{m} p_n u^n < e,
\]

q.e.d.

**Lemma 3.** Let \(g(z)\) be a holomorphic function in the circle \(|z - z_0| < q < +\infty\) with the finite radial limit \(\lim_{z \to z_0^{\pm}} g(z)\), where \(z_0\) denotes a fixed number such that \(|z - z_0| = q\). Then for every \(e > 0\) and for every \(0 < e < q\) there exists a polynomial \(W(z)\) such that

\[
|W(z) - g(z)| < e \quad \text{for} \quad |z - z_0| \leq e_1 \quad \text{and for} \quad z \in [z_0 - e_1, z_0 + e_1].
\]

Proof. According to the uniform continuity of the function \(g(z)\) on the set \(z = \{z: |z - z_0| \leq e_1\} \cup [z_0 - e_1, z_0 + e_1]\) there exists \(0 < \rho < 1\) such that

\[
|g(z_0 + \rho(z - z_0)) - g(z)| < \frac{e}{2} \quad \text{for} \quad z \in Z.
\]

Since \(g(z_0 + \rho(z - z_0))\) is a holomorphic function on the circle \(|z - z_0| < q\rho > 0\), there exists a polynomial \(W(z)\) such that

\[
|W(z) - g(z_0 + \rho(z - z_0))| < \frac{e}{2} \quad \text{for} \quad |z - z_0| \leq q.
\]

In virtue of the inequalities (**) the proof of the Lemma is completed, q.e.d.

**Lemma 4.** The set of the functions of the form

\[
a_0 + \sum_{n=1}^{m} a_n e^{-a_n u},
\]

where \(a > 0\), is dense in the space \(H_a\).

Proof. Given an arbitrary number \(e > 0\) and a square \(K = \{w: |\text{Re}(w)| \leq R, |\text{Im}(w)| \leq R\}\). Let us fix a number \(u > 0\) such that \(aR < \pi/2\).

Let us consider the mapping \(\zeta = e^{-u}\) which maps the square \(K\) onto the set \(Q = \{\zeta: e^{-2R} \leq |\text{Re}(\zeta)| \leq e^{-2R}, |\text{Arg}(\zeta)| \leq \pi/2\}\) and the interval \(0 \leq w = \text{Re}(w) < +\infty\) onto the interval \(0 < \zeta = \text{Re}(\zeta) \leq 1\). The inverse mapping is given by the formula

\[
w = \frac{1}{a} \log \zeta.
\]

Let \(f \in H_a\). We consider the function

\[
g(\zeta) = f\left(\frac{1}{a} \log \zeta\right),
\]

which is a holomorphic function on the open complex plane except \(\zeta = \text{Re}(\zeta) \leq 0\), with the finite limit

\[
\lim_{|\text{Re}(\zeta)| \to 0} g(\zeta).
\]

Let us write \(S = \{\zeta: |\text{Re}(\zeta)| \leq \zeta_0, \text{Re}(\zeta) > 0\}\), where \(\zeta_0, \zeta_1\) are fixed numbers such that \(\zeta_0 = \text{Re}(\zeta_0) > 0, D \subset S, \zeta_1 < \zeta_2\). (Such numbers \(\zeta_0, \zeta_1\) exist, since \(aR < \pi/2\)). Putting, in Lemma 3, \(e = \zeta_0\) and \(\zeta = 0\), in we find that there exists a polynomial \(W(\zeta)\) such that

\[
|W(\zeta) - g(\zeta)| < e \quad \text{for} \quad \zeta_0 
\]

After the substitution \(\zeta = e^{-u}\) we get

\[
|W(e^{-u}) - f(w)| < e \quad \text{for} \quad w \in K \quad \text{and for} \quad 0 \leq w = \text{Re}(w) < +\infty.
\]

Writing

\[
W(\zeta) = \sum_{n=1}^{m} a_n \zeta^n \quad \text{and} \quad \zeta = \text{Re}(\zeta),
\]

we have

\[
W(e^{-u}) = a_0 + \sum_{n=1}^{m} a_n e^{-a_n u},
\]

q.e.d.

**Lemma 5.** The set of functions of the form \(e^{-u}W(w) + t\), where \(W(w)\) denotes a polynomial and \(t\) — an arbitrary constant, is dense in the space \(H_a\).
Proof. In virtue of Lemma 4 it remains to prove that the function $e^{-w}$, where $\varphi \geq 0$, can be approximated (in the sense of $H_1$) by functions of the form $e^{-w}W(\omega)$, where $W(\omega)$ is a polynomial. At first we shall prove it for the same special cases with respect to $\varphi$.

We use the following notation:

- $f_\omega(\omega) = f(\omega)$ denotes uniform convergence on every compact subset of the open complex plane;
- $f_\omega(\omega) \rightarrow f(\omega)$ denotes uniform convergence on the interval $0 < \omega = \operatorname{Re}(\omega) < +\infty$;
- $f_\omega(\omega) \rightarrow f(\omega)$ denotes convergence in the sense of the space $H_1$.

If $f_\omega, f \in H_1$, then $f_\omega(\omega) \rightarrow f(\omega)$ if and only if $f_\omega(\omega) \Rightarrow f(\omega)$.

1. Let $0 < \varepsilon < 2$. We have $e^{-w} = e^{-w}e^{\varphi}$, where $\varphi = 1 - \varphi$, $|\varphi| < 1$.

Writing

$$W_\omega(\omega) = \sum_{\omega=0}^{\infty} \frac{1}{\omega!} \left(\frac{\omega}{\omega}\right)^m$$

we get $e^{-w}W_\omega(\omega) \Rightarrow e^{-w}$.

Let $\varepsilon > 0$. For $j > m$ we have $|\omega|^j < \varepsilon$; consequently we get for $k \geq m$ and $0 < \omega = \operatorname{Re}(\omega) < +\infty$:

$$|e^{-w} - e^{-w}W_\omega(\omega)| \leq e^{-w} \sum_{\omega=0}^{\infty} \frac{m}{\omega!} |\omega|^j \varepsilon \omega \leq e^{-w} \sum_{\omega=0}^{\infty} \frac{m}{\omega!} |\omega|^j \varepsilon < \varepsilon;$$

thus $e^{-w}W_\omega(\omega) \Rightarrow e^{-w}$ and consequently $e^{-w}W_\omega(\omega) \rightarrow e^{-w}$.

2. Let $\varphi = 2$. Writing

$$W_\omega(\omega) = \sum_{\omega=0}^{\infty} \frac{1}{\omega!} (-1)^\omega,$$

we obtain $e^{-w}W_\omega(\omega) \Rightarrow e^{-2w}$.

For $0 < \omega = \operatorname{Re}(\omega)$ we have

$$e^{-w} = e^{-w}W_\omega(\omega) + (-1)^\omega e^{-w} e^{2k} \frac{1}{(k+1)!},$$

where $0 \leq \theta = \theta(k, \omega) \leq 1$; hence we obtain

$$|e^{-w} - e^{-w}W_\omega(\omega)| = e^{-w} e^{2w} e^{-w} \leq \frac{1}{k!} e^{-w} e^w \leq \frac{1}{k!} e^{-2w};$$

According to Stirling's formula $k! = k^k e^{-k/2} \sqrt{2\pi} \omega$, $a_k \rightarrow \sqrt{2\pi}$, we get

$$\frac{k^k e^{-k}}{k!} = \frac{1}{a_k \sqrt{2\pi}} \rightarrow 0.$$

Thus we have proved that $e^{-w}W_\omega(\omega) \Rightarrow e^{-2w}$ and consequently $e^{-w}W_\omega(\omega) \rightarrow e^{-2w}$.

3. Now we prove that every function $e^{-w}W_\omega(\omega)$, where $W_\omega(\omega)$ is a polynomial, can be approximated by functions of the form $e^{-w}W_\omega(\omega)$, where $W_\omega(\omega)$ is a polynomial.

It suffices to prove this for $W_\omega(\omega) = w^m$, $m = 0, 1, \ldots$. We have

$$e^{-2w}w^m = e^{-w}w^m e^{-w} = e^{-w}w^m \sum_{f=0}^{m} \frac{1}{f!} \left(-\frac{w}{2}\right)^f.$$

Writing

$$W_\omega(\omega) = w^m \sum_{f=0}^{\infty} \frac{1}{f!} \left(-\frac{w}{2}\right)^f,$$

we obtain $e^{-w}W_\omega(\omega) \Rightarrow e^{-2w}w^m$. For $0 < \omega = \operatorname{Re}(\omega) < +\infty$ we have

$$e^{-2w}w^m = e^{-w}W_\omega(\omega) + e^{-w} \frac{1}{k!} \left(-\frac{1}{2}\right)^k e^{w}w^k,$$

where $0 \leq \theta \leq 1$.

Hence we get

$$|e^{-2w}w^m - e^{-w}W_\omega(\omega)| \leq \frac{1}{2^k k!} e^{-w} e^{-2w} \frac{k^k e^{-k}}{k!} \leq \frac{1}{2^k k!} \left(k + m\right) \frac{k^k e^{-k}}{k!} \leq \frac{1}{2^k k!} \left(k + m\right) \frac{k^k e^{-k}}{k!} \leq \frac{1}{2^k k!} \left(k + m\right) \frac{k^k e^{-k}}{k!} \leq \frac{1}{2^k k!} \left(k + m\right) \frac{k^k e^{-k}}{k!} \leq a_{k+1} \rightarrow 0 \text{ if } k \rightarrow +\infty.$$

Thus we obtain $e^{-w}W_\omega(\omega) \Rightarrow e^{-2w}w^m$ and consequently $e^{-w}W_\omega(\omega) \rightarrow e^{-2w}$.

4. Now we show that every function $e^{-w}W_\omega(\omega)$ can be approximated by functions of the form $e^{-w}W_\omega(\omega)$, where $W_\omega(\omega)$ and $W(\omega)$ denote polynomials.

Indeed, in virtue of 2. there exists a sequence of polynomials $W_\omega(\omega)$ such that $e^{-w}W_\omega(\omega) \rightarrow e^{-w}W_\omega(\omega)$.

After substituting $w/2$ for $w$ we establish $e^{-2w}W_\omega(\omega/2) \rightarrow e^{-w}W_\omega(\omega)$, and multiplying by $e^{-w}W_\omega(\omega)$ we get

$$e^{-2w}W_\omega(\omega)W_\omega(\omega/2) \rightarrow e^{-w}W_\omega(\omega).$$

5. In virtue of 3. and 4. we establish that every function $e^{-w}W_\omega(\omega)$ can be approximated by functions of the form $e^{-w}W_\omega(\omega)$, where $W_\omega(\omega)$, $W(\omega)$ are polynomials.

6. Now the proof of Lemma 5 can be completed by the method of mathematical induction. Let $i < q \ll i+1$, where $i = 0, 1, \ldots$. According
to 1° the assertion of the Lemma is true for \( i = 0 \). Suppose that the assertion of the lemma is true for \( i \) and let \( i+1 < q \leq i+2 \). In virtue of the inductive hypothesis and \( i < q-1 \leq i+1 \) we conclude that there exists a sequence of polynomials \( P_k(w) \) such that \( e^{-w} P_k(w) \to e^{-w} \). After multiplication by \( e^{-w} \) we obtain \( e^{-w} P_k(w) \to e^{-w} \). In virtue of 5° there exists a sequence of polynomials \( W_k(w) \) such that \( e^{-w} W_k(w) \to e^{-w} \), q.e.d.

Elementary Lemmas 2 and 5 enable us to prove the following theorem of Ryll-Nardzewski [see [9]]:

**Theorem 4.** The Borel power method is perfect.

Proof. Let \( p(w) = e^w \). According to Lemma 2 we have to prove that the set of functions of the form \( W(w) + te^w \) is dense in the space \( H(p, E_{+\infty}) \), where \( W(w) \) is a polynomial.

Let \( f \in H(p, E_{+\infty}) \). Hence the function \( e^{-w} f(w) \) is an element of the space \( H_0 \). According to Lemma 5 there exists a sequence of polynomials \( W_k(w) \) and a sequence of numbers \( \alpha_k \) such that the sequence of functions \( e^{-w} W_k(w) + \alpha_k \) tends to \( e^{-w} f(w) \) uniformly on every compact subset of the open complex plane and uniformly in the interval \( 0 < w = \Re(w) < +\infty \). Hence \( W_k(w) + \alpha_k p(w) \) tends to \( f(w) \) uniformly on every compact subset of the open complex plane and \( W_k(w) + \alpha_k p(w) \) tends to \( f(w) \) uniformly in the interval \( 0 < w = \Re(w) < +\infty \), but this means that the sequence of functions \( W_k(w) + \alpha_k p(w) \) tends to \( f(w) \) in the sense of convergence in the space \( H(p, E_{+\infty}) \), q.e.d.

**Definition 10.** Let \( \chi(u) \) be a positive continuous function in the interval \( 0 < u < r \) (e.g. we may take \( \chi(u) = p(w) \) where \( p \not\in \mathcal{F} \), if hypothesis \( (\beta) \) is fulfilled). We shall say that \( \chi(u) \) has the \((M)\)-property if for every number \( 0 < q < r \) and for every function \( \alpha(u) \) continuous on the right such that \( \chi(u) < +\infty \), the condition

\[
\sup_{0 < u < r} \chi(u) < +\infty
\]

implies \( \alpha(u) = \alpha \) for \( 0 < u < r \).

According to Mikusiński's theorem of bounded moments the function \( \chi(u) = 1 \) has the \((M)\)-property if \( r < +\infty \) (see [7] and [8]).

**Lemma 6.** Let \( p \in \mathcal{F} \), \( D \in \mathcal{V'} \), and let condition \( (\beta) \) be fulfilled. If \( D \) is a simply connected domain and the function \( p(u) \) has the \((M)\)-property, then the method \( M(p, D) \) is perfect.

**Proof.** Let \( \varepsilon = (1, 1, 1, \ldots) \), \( \varepsilon_m = (0, \ldots, 0, 1, 0, \ldots) \) for \( m = 0, 1, \ldots \) and \( \varepsilon_m = 0 \) for \( m = 0, 1, \ldots \). To prove this lemma we show that \( \phi(x) = 0 \) for every \( x \in M^*(p, D) \). In virtue of formula (12), § 1, we have

\[
\phi(x) = \Phi(m(p, D; x, \cdot)) \int_0^r \frac{m(p, D; x, u)}{p(u)} \, du + a M(p, D; x).
\]

Putting \( x = \varepsilon_m \) we get

\[
(*) \int_0^r \frac{u_m}{p(u)} \, du = -\Phi(f_m), \quad \text{where} \quad f_m(w) = w_m, \quad m = 0, 1, \ldots
\]

Since \( \Phi \) is a linear functional over \( H(D) \), there exist a constant \( C \) and a compact subset \( Z \) of \( D \) such that

\[
|\Phi(f)| \leq C \sup \{ |f(w)| \} \quad \text{for} \quad f \in H(D).
\]

Writing

\[
e = \sup |w|
\]

we obtain

\[
|\Phi(f_m)| \leq C \phi^m, \quad m = 0, 1, \ldots
\]

Thus in virtue of \((*)\) we get

\[
\int_0^r \frac{u_m}{p(u)} \, du = O(\phi^m).
\]

According to the hypothesis of the Lemma, \( \alpha(u) = \alpha \) for \( \alpha < u < r \). The functional \( \phi(x) \) may now be written in the form

\[
\phi(x) = \Phi(m(p, D; x, \cdot)) + a M(p, D; x),
\]

where

\[
\Phi(f) = \Phi(f) + \int_0^r \frac{f(u)}{p(u)} \, du \quad \text{for} \quad f \in H(D).
\]

Equality \((*)\) implies \( \Phi(f_m) = 0 \) for \( m = 0, 1, \ldots \) and consequently \( \Phi(W) = 0 \) for every polynomial \( W \).

According to the theorem of Runge we get \( \phi(f) = 0 \) for every function \( f \in H(D) \) since \( \Phi \) is a linear functional over \( H(D) \). Thus we have,

\[
\phi(x) = a M(p, D; x) \quad \text{for} \quad x \in M^*(p, D).
\]

Putting \( x = \varepsilon \) we finally get \( a = 0 \), q.e.d.
Lemma 7. Let \( \chi(u) \) be a positive continuous function on the interval \( 0 \leq u < r \) such that
\[
\sup_{0 \leq u < r} \frac{u^n}{\chi(u)} < +\infty
\]
for \( n = 0, 1, \ldots \), where \( 0 < r < +\infty \). Then \( \chi(u) \) has the (M)-property.

Taking \( \chi(u) = 1 \) we get Mikusiński's theorem of bounded moments (see [7] and [8]). The above lemma may be deduced from the theorem of Mikusiński but we shall give a new proof which is based on the following well-known approximation theorem of Mergellian (see [6]):

Theorem of Mergellian. Let \( g(w) \) be a function defined on a compact subset \( F \) of the open complex plane \( \mathbb{C} \) and suppose that the following conditions are satisfied:
1. The function \( g(w) \) is continuous on \( F \).
2. The function \( g(w) \) is holomorphic in \( \text{Int}(F) \).
3. The set \( \mathbb{C} \setminus F \) is connected.

Then for every \( \varepsilon > 0 \) there exists a polynomial \( W(w) \) such that
\[
|g(w) - W(w)| < \varepsilon
\]
for every \( w \in F \).

Proof of Lemma 7. Suppose that
\[
\int_{\varepsilon}^{r} \frac{u^n}{\chi(u)} \, du = O(\varepsilon^n),
\]
where \( 0 < \varepsilon < r \) and the function \( \omega(u) \) is such as in definition 10.

Hence
\[
\int_{\varepsilon}^{r} \frac{u^n}{\chi(u)} \, du = O(\varepsilon^n).
\]

Thus we have
\[
\int_{\varepsilon}^{r} \frac{u^n}{\chi(u)} \, du = a_n \varepsilon^n,
\]
where \( |a_n| \leq C \) for \( n = 0, 1, \ldots \). Hence for an arbitrary polynomial
\[
W(u) = \sum_{n=0}^{\infty} a_n u^n
\]
we get
\[
\int_{\varepsilon}^{r} \frac{W(u)}{\chi(u)} \, du = \sum_{n=0}^{\infty} a_n \varepsilon^n.
\]

Let us fix the numbers \( \varepsilon_1, \varepsilon_2 \) so that \( \varepsilon < \varepsilon_1 < \varepsilon_2 < r \). Applying the inequalities of Cauchy
\[
|a_n| \leq \frac{1}{\varepsilon^{n+1}} \sup_{|w| < \varepsilon_1} |W(w)|
\]
we get
\[
\int_{\varepsilon}^{r} \frac{W(u)}{\chi(u)} \, du = \sum_{n=0}^{\infty} |a_n| \varepsilon^n \leq C \sum_{n=0}^{\infty} \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^n \sup_{|w| < \varepsilon_1} |W(w)|
\]
\[
\leq C \frac{\varepsilon_2}{\varepsilon_1 - \varepsilon} \sum_{n=0}^{\infty} (\varepsilon_2/\varepsilon_1)^n \sup_{|w| < \varepsilon_1} |W(w)|.
\]

Thus we have
\[
(*) \int_{\varepsilon}^{r} \frac{W(u)}{\chi(u)} \, du = C \frac{\varepsilon_2}{\varepsilon_1 - \varepsilon} \sum_{n=0}^{\infty} (\varepsilon_2/\varepsilon_1)^n \sup_{|w| < \varepsilon_1} |W(w)|.
\]

Now let \( f(u) \) be an arbitrary continuous function in the interval \( \varepsilon_1 \leq u < r \) such that \( f(\varepsilon_1) = 0 = f(u) \) for \( \varepsilon_1 \leq u < \varepsilon_2 \). Let us consider a function \( g(u) \) of the complex variable defined as follows:
\[
g(u) = \begin{cases} 
0 & \text{for } |u| \leq \varepsilon_1, \\
\frac{x(u) f(u)}{\chi(u)} & \text{for } \varepsilon_1 \leq u < r.
\end{cases}
\]

Applying the theorem of Mergellian to the function \( g(u) \) and the set \( F = \{ w : |w| \leq \varepsilon_1 \} \cup \{ w : \varepsilon_1 \leq w < r \} \), we can choose a sequence of the polynomials \( W_k(u) \) tending to \( g(u) \) uniformly on \( F \). It is easy to see that \( W_k(u)/\chi(u) \) tends to \( f(u) \) uniformly on the interval \( \varepsilon_1 \leq u < r \) since the function \( x(u)/\chi(u) \) is bounded for \( \varepsilon_1 \leq u < r \). In virtue of (*) we have
\[
\int_{\varepsilon}^{r} \frac{W_k(u)}{\chi(u)} \, du = C \frac{\varepsilon_2}{\varepsilon_1 - \varepsilon} \sum_{n=0}^{\infty} (\varepsilon_2/\varepsilon_1)^n \sup_{|w| < \varepsilon_1} |W_k(w)|.
\]

In the limit for \( k \to +\infty \) we finally obtain
\[
\int_{\varepsilon_1}^{r} f(u) \, du = 0.
\]

Since \( f(u) \) is an arbitrary function, we deduce that \( \omega(u) = \text{const} \) for \( \varepsilon_1 \leq u \leq \varepsilon_2 \) and since \( \varepsilon_1, \varepsilon_2 \) are arbitrary numbers we deduce that \( \omega(u) = \text{const} \) for \( \varepsilon \leq u \leq r \), q.e.d.

According to Lemmas 5, 7 and remark 3 we have the following theorem (see [11], Th. XI):

Theorem 5. Let \( p \in \mathcal{F}_r, D \in \mathcal{F}_r \) and let hypothesis (5) be satisfied for \( r < +\infty \). Then the method \( M(p, D) \) is perfect if and only if \( D \) is a simply connected domain.
LEMMA 8. Let \( f(u) \) be a continuous function for \( u \geq R > 0 \) such that
\[
f(R) = 0 = \lim_{u \to \infty} f(u).
\]

Then for every \( \varepsilon > 0 \) there exists a function \( h(w) \) of the form
\[
h(w) = \sum_{j=1}^{m} a_j e^{-\alpha_j w}
\]
where \( a_j > 0 \) such that
\[
|h(w) - f(u)| < \varepsilon \quad \text{for} \quad u \geq R,
\]
\[
|h(w)| < \varepsilon \quad \text{for} \quad |\text{Re}(w)| \leq R, \quad |\text{Im}(w)| \leq R.
\]

Proof. We apply the same method as in the proof of Lemma 4. Let the symbols \( K, a, \Omega \) have the same meaning as in the proof of Lemma 4. Let us consider a function \( g(\xi) \) defined as follows:
\[
g(\xi) = \begin{cases} 0 & \text{for} \quad \xi \in \Omega, \\ f\left(1 - \frac{1}{\alpha} \log \xi\right) & \text{for} \quad 0 < \xi = \text{Re}(\xi) \leq e^{-\alpha R}. \end{cases}
\]

Applying the approximation theorem of Mergelian to the function \( g(\xi) \) and to the set \( P = \Omega \cup \{\xi : 0 < \xi = \text{Re}(\xi) \leq e^{-\alpha R}\} \), we can choose a polynomial
\[
W(\xi) = \sum_{j=1}^{m} a_j \xi^j
\]
such that \( |W(\xi) - g(\xi)| < \varepsilon \) for \( \xi \in \mathbb{F} \). After the substitution \( \xi = e^{-w} \) we get the assertion of Lemma 8, q.e.d.

LEMMA 9. Let \( p \in \mathbb{F}_r \) and hypothesis \( (B) \) be satisfied for \( r = +\infty \), and let the power method \( \mathcal{M}(p) \) be perfect. If \( f \in \mathcal{H}_1 \) and
\[
\lim_{u \to \infty} f(u) = 0,
\]
then for every \( \varepsilon > 0 \) and for every compact subset \( Z \) of the open complex plane there exists such a polynomial \( W(w) \) that
\[
\left| \frac{W(w) - p(w)f(w)}{p(w)} \right| < \varepsilon \quad \text{for} \quad w \in \mathbb{C},
\]
\[
\frac{W(w)}{p(w)} - f(w) < \varepsilon \quad \text{for} \quad w = \text{Re}(w) \geq 0.
\]

Proof. Let
\[
\lambda = \sup_{w \in \mathbb{C}} |p(w)|.
\]

According to Lemma 2, since the function \( p(w)f(w) \) belongs to the space \( H(p, K_{+\infty}) \), there exist a polynomial \( W(w) \) and a number \( t \) such that
\[
(*) \quad \left| \frac{W(w) - t}\frac{W(w)}{p(w)} - p(w)f(w) \right| < \frac{\varepsilon}{1 + \lambda} \quad \text{for} \quad w \in Z,
\]
\[
(**) \quad \left| \frac{W(w)}{p(w)} - t\frac{W(w)}{p(w)} \right| < \frac{\varepsilon}{2(1 + \lambda)} \quad \text{for} \quad w = \text{Re}(w) \geq 0.
\]

Taking the limit in (**) if \( w = \text{Re}(w) \to +\infty \), we obtain \( |t| < \varepsilon/(2(1 + \lambda)) \). Hence we get for \( w \in Z \)
\[
\left| \frac{W(w) - p(w)f(w)}{p(w)} \right| < \left| \frac{W(w) + t}\frac{W(w)}{p(w)} - p(w)f(w) \right| + |t||p(w)|
\]
\[
< \frac{\varepsilon}{1 + \lambda} + \frac{\varepsilon}{2(1 + \lambda)} < \frac{\varepsilon}{1 + \lambda} (1 + \lambda) = \varepsilon.
\]

For \( w = \text{Re}(w) \geq 0 \) we get
\[
\left| \frac{W(w)}{p(w)} - f(w) \right| < \left| \frac{W(w)}{p(w)} + t\frac{W(w)}{p(w)} \right| + |t| < \frac{\varepsilon}{1 + \lambda} + \frac{\varepsilon}{2(1 + \lambda)} < \varepsilon,
\]
q.e.d.

LEMMA 10. Let \( p \in \mathbb{F}_r \), and hypothesis \( (B) \) be satisfied for \( r = +\infty \); let \( \chi(u) \) be a positive continuous function for \( 0 \leq u < +\infty \) such that
\[
\sup_{u \in \chi(u)} u^n < +\infty \quad \text{for} \quad n = 0, 1, \ldots
\]

If the power method \( M(p) \) is perfect and the inequality \( \chi(u) \geq p(u) \) is satisfied for \( u \geq 0 \), then the function \( \chi(u) \) has the \( \mathcal{M} \)-property.

Proof. Suppose that
\[
\int_{\chi(u)}^{+\infty} \frac{u^n}{\chi(u)} dw(u) = O(e^\varphi), \quad 0 < \varphi < +\infty,
\]
and fix the numbers \( \varepsilon_1, \varepsilon_2 \) so that \( \varphi < \varepsilon_1 < \varepsilon_2 < +\infty \). Repeating partially the proof of Lemma 7 we get
\[
(*) \quad \int_{\chi(u)}^{+\infty} \frac{W(u)}{\chi(u)} dw(u) \leq C \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2} \sup_{w \in \mathcal{D}_1} |W(w)|
\]
for an arbitrary polynomial \( W(u) \).

Now let \( f(u) \) be an arbitrary continuous function in the interval \( \varepsilon_1 \leq u < +\infty \) such that \( f(\varepsilon_1) = 0 = f(u) \) for \( \varepsilon_2 \leq u < +\infty \). Let
\[
g(u) = \frac{f(u)\chi(u)}{p(u)} \quad \text{for} \quad \varepsilon_1 \leq u < +\infty.
\]
Let $\varepsilon > 0$. According to Lemma 8 there exists a function

$$h(u) = \sum_{j=1}^{n} a_j e^{-\nu_j u}$$

such that

$$|h(u) - g(u)| < \varepsilon \quad \text{for} \quad u \geq \varepsilon_1,$$

$$|h(u)| < \varepsilon \quad \text{for} \quad |u| \leq \varepsilon_1.$$  

According to Lemma 9 there exists a polynomial $W(u)$ such that

$$\frac{|W(u)|}{p(u)} = |h(u)| < \varepsilon \quad \text{for} \quad u \geq \varepsilon_1,$$

$$|W(u)| < |W(u) - p(u)h(u)| + |p(u)h(u)| < \varepsilon + \varepsilon_1.$$  

Writing

$$l = \sup_{|w| < \varepsilon_1} |p(w)|,$$

we get for $|w| \leq \varepsilon_1$:

$$|W(w)| \leq |W(u) - p(u)h(u)| + |p(u)h(u)| < \varepsilon + \varepsilon_1.$$  

For $u \geq \varepsilon_1$ we get

$$\frac{W(u)}{p(u)} - f(u) = \frac{p(u)}{p(u)} \left( \frac{W(u)}{p(u)} - g(u) \right),$$

whence

$$\frac{W(u)}{p(u)} - f(u) \leq \frac{W(u)}{p(u)} - g(u) \leq \frac{W(u)}{p(u)} - h(u) + |h(u) - g(u)| < 2\varepsilon.$$  

Thus we have proved that for every $\varepsilon > 0$ there exists a polynomial $W(u)$ such that

$$|W(u)| < (1 + \lambda)\varepsilon \quad \text{for} \quad |w| \leq \varepsilon_1,$$

$$\frac{W(u)}{p(u)} - f(u) < 2\varepsilon \quad \text{for} \quad u \geq \varepsilon_1.$$  

Taking $\varepsilon = 1/k$ ($k = 1, 2, \ldots$) we get a sequence of polynomials $W_k(u)$ tending to zero uniformly in the circle $|w| \leq \varepsilon_1$ such that $W_k(u)/p(u)$ tends to $f(u)$ uniformly on the interval $\varepsilon_1 \leq u < +\infty$. In virtue of ($\ast$) we have

$$\int_{\varepsilon_1}^{+\infty} W_k(u) d\omega(u) \leq \frac{1}{\varepsilon_1 - \varepsilon_0} \sup_{|w| < \varepsilon_1} |W_k(w)|.$$  

Taking the limit as $k \to +\infty$, we get

$$\int_{\varepsilon_1}^{+\infty} f(u) d\omega(u) = 0.$$  

Hence we finally deduce that $m(u) = \text{const}$ for $u \geq \varepsilon_0$, q.e.d.

As a consequence of Lemmas 6 and 10 and Theorem 4 we get the following

**Theorem 6.** Let $p, q \in \mathcal{F}_r, D \in \mathcal{N}_r$ and let the functions $p, q$ fulfill condition (3) for $r = +\infty$. Then

(a) The power method $M(p)$ is perfect if and only if the function $p(u)$ has the $(M)$-property.

(b) If the power method $M(p)$ is perfect, $D$ is a simply connected domain and the inequality $q(u) \geq p(u)$ is satisfied for $u \geq 0$, then the method $M(q, D)$ is perfect; in particular, if $q(u) \geq e^u$ for $u \geq 0$, the method $M(q, D)$ is perfect.

Note. Let us denote by $\hat{M}^*(p) (p \in \mathcal{F}_r; r \leq +\infty)$ the set of all real sequences $x \in \mathcal{M}^*(p)$ and put $\hat{M}(p; x) = M(p; x)$ for $x \in \mathcal{M}^*(p)$. Thus $\hat{M}(p)$ is a “real” power method. It is easy to see that the “real” power method $\hat{M}(p)$ is perfect if and only if the method $M(p)$ is perfect.

§ 4. Methods “extracted” from methods $M(p, D)$

**Definition 11.** Let $D, \mathcal{N}_r, p \in \mathcal{F}_r$, and let $\delta = (u_k)$ be a sequence of real numbers such that $u_k \in D, \lim u_k = r$.

By $M(p, D, \delta)$ we denote a method defined as follows. The field $\mathcal{M}^*(p, D, \delta)$ of the method $M(p, D, \delta)$ is the set of all complex-valued sequences $x = (u_k)$ such that conditions (a), (b) from Definition 6, § 1, are satisfied and

(c') there exists a finite limit

$$\lim_{k} \frac{m(p, D; x; u_k)}{p(u_k)}.$$  

The generalized limit $M(p, D, u; x)$ of the sequence $x \in \mathcal{M}^*(p, D, \delta)$ (corresponding to the method $M(p, D, \delta)$) is defined as the limit

$$M(p, D, u; x) = \lim_{k} \frac{m(p, D; x; u_k)}{p(u_k)}.$$  

**Remark 4.** The methods $M(p, D, \delta), M(p, D)$ are consistent and $\mathcal{M}^*(p, D, \delta) \subset \mathcal{M}^*(p, D, \delta)$.

In the space $\mathcal{M}^*(p, D, \delta)$ we now introduce the family of pseudonorms

$$(p_{x}) \left\| x \right\|_x = \sup_{u \in D} \left| \frac{m(p, D; x; u)}{p(u)} \right|;$$

$$(p_{\infty}) \left\| x \right\|_\infty = \sup_{u \in D} \left| m(p, D; x; u) \right|, Z - \text{compact subsets of } D.$$
It is easy to verify that under the hypothesis \( p^{(n)}(0) \neq 0 \) \((n = 0, 1, \ldots)\) the space \( M(p, D, \delta) \) is a \((E_l)\)-space under the family of pseudonorms \((p_n)\). Considering the mapping \( U \) of the space \( M^*(p, D, \delta) \) into the product \( H(D) \times \mathbb{C} \):

\[
U(x) = \left( m(p, D; x, \cdot), \left( \frac{m(p, D; x, u_k)}{p(u_k)} \right) \right) \in H(D) \times \mathbb{C},
\]

we deduce that the space \( M^*(p, D, \delta) \) is separable and the general form of linear functionals over \( M^*(p, D, \delta) \) is given by the formula

\[
\varphi(x) = \Phi(m(p, D; x, \cdot)) + \sum_{k=0}^{\infty} a_k \frac{m(p, D; x, u_k)}{p(u_k)} + aM(p, D, \delta; x),
\]

where

\[
\sum_{k=0}^{\infty} |a_k| < +\infty.
\]

\( \varphi \) is a linear functional over \( H(D) \).

Thus we have the following

**Lemma 11.** Let \( D \in \mathcal{C}^r \), \( p \in \mathcal{P}_r \), \( p^{(n)}(0) \neq 0 \) \((n = 0, 1, \ldots)\) and let \( \delta = (u_k) \) be a sequence such as in Definition 11. Then \( M(p, D, \delta) \) is a separable \((E_l)\)-space under the totality of pseudonorms \((p_n)\); the general form of linear functionals over \( M^*(p, D, \delta) \) is given by formula \((1)\).

**Theorem 7.** Let the hypotheses of Lemma 11 be satisfied, where the function \( p \) fulfills conditions \((3)\), \((4)\). If the power method \( M(p) \) is perfect and \( D \) is a simply connected domain, then the method \( M(p, D, \delta) \) is also perfect (see [11], Th. XXII).

**Proof.** To prove the above theorem it suffices to consider the case of the increasing sequence \( \delta = (u_k) \).

Let \( \varepsilon = (1, 1, 1, \ldots), \varepsilon_m = (0, 0, 0, 1, 0, \ldots) \) \(m = 0, 1, \ldots\), and \( \varphi(x) = \varphi(\varepsilon_m) = 0 \) for \( m = 0, 1, \ldots \). We have to show that \( \varphi(x) = 0. \) Putting \( x = \varepsilon_m \) in formula \((1)\), we have

\[
(*) \quad \sum_{k=0}^{\infty} a_k \frac{u_k^m}{p(u_k)} = -\Phi(f_{\varepsilon_m}) \quad \text{where} \quad f_m(x) = u^m; \quad m = 0, 1, \ldots
\]

As in the proof of Lemma 6 we deduce that there exist constants \( C, \alpha < \beta < r \) such that

\[
|\Phi(f_{\varepsilon_m})| \leq C \beta^m \quad \text{for} \quad m = 0, 1, \ldots
\]

In virtue of \((*)\) we get

\[
(\star) \quad \sum_{k=0}^{\infty} a_k \frac{u_k^m}{p(u_k)} = O(\beta^m).
\]

Let \( \omega(u) \) be a function defined as follows:

\[
\omega(u) = \begin{cases} 0 & \text{for } 0 \leq u < u_k, \\ a_0 + a_1 + \ldots + a_k & \text{for } u_k \leq u < u_{k+1}; \quad k = 0, 1, \ldots
\end{cases}
\]

Thus \( \omega(u) \) is continuous on the right and such that

\[
\omega(u_k) = \omega(u_{k-1}) = 0,
\]

\[
\sum_{k=0}^{\infty} |a_k| < +\infty.
\]

We have

\[
\sum_{k=0}^{\infty} a_k \frac{u_k^m}{p(u_k)} = \int_{x}^{u} \frac{u^n}{p(u)} \, da(u).
\]

In virtue of \((**)\) we obtain

\[
\int_{x}^{u} \frac{u^n}{p(u)} \, da(u) = O(\beta^m).
\]

According to Theorem 6 \(\alpha\) or Lemma 7 we establish that \( \omega(u) = \text{const} \) for \( 0 \leq u < r \) and consequently \( a_k = 0 \) for \( k > \delta_0 \). Thus \( \varphi(x) \) may be written in the form

\[
\varphi(x) = \Phi(m(p, D; x, \cdot)) + \sum_{k=0}^{\delta_0} a_k \frac{m(p, D; x, u_k)}{p(u_k)} + aM(p, D, \delta; x),
\]

\[
\varphi(x) = \Psi[m(p, D; x, \cdot)] + aM(p, D, \delta; x),
\]

where

\[
\Psi(f) = \Phi(f) + \sum_{k=0}^{\delta_0} a_k \frac{f(u_k)}{p(u_k)} \quad \text{for} \quad f \in H(D).
\]

In virtue of \((**)\) we have \( \Psi(f_m) = 0 \) for \( m = 0, 1, \ldots \) and \( \Psi(W) = 0 \) for every polynomial \( W \). According to theorem of Runge we get \( \Psi(f) = 0 \) for every \( f \in H(D) \), since \( \Psi \) is a linear functional on \( H(D) \). Thus we have

\[
\varphi(x) = aM(p, D, \delta; x).
\]

Putting \( x = e \) we get \( a = 0 \), \( q, \varepsilon, \varepsilon_0, \ldots\).

\section{Relations between the generalized power methods \( M(p, D) \) and the Taupolni methods}

**Theorem 8.** Let \( D \in \mathcal{C}^r \), \( p \in \mathcal{P}_r \), and let \( \delta = (u_k) \) be a sequence such as in Definition 11. There exists a sequence of row-finite matrix methods
\[ C_0, C_1, C_2, \ldots \text{ such that} \]
\[ M(p, D; \xi; x) = \lim_{k} C_k(x) \]
for every \( x \in M^*(p, D; \xi) \).

Proof. Let \( D_0 \) be such a domain that \( D_{k+1} < D_k \), which can be conformally mapped onto the circle \( \{ z : |z| < 1 \} \). The existence of such a domain \( D_0 \) is the consequence of Riemann’s theorem. Let \( w = h(z) \) be a conformal mapping of the circle \( \{ z : |z| < 1 \} \) onto the domain \( D_k \) such that \( h(0) = 0 \). It can be shown by induction that for \( t = 0, 1, \ldots \) there exists such a system of functions \( g_{\alpha}(z), g_{\beta}(z), \ldots, g_{\gamma}(z) \) holomorphic for \( |z| < 1 \) that the formula
\[ \frac{d^t}{dz^t}f(h(z)) = \sum_{\alpha=0}^{t} \left( \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} g_{\beta}(z) f^{(\alpha)}(h(z)) \right) \]
is satisfied for \( |z| < 1 \) and for every function \( f(\omega) \) holomorphic in \( D_k \).

Writing \( a_{\alpha} = g_{\alpha}(0) \) we obtain
\[ f(h(z)) = \sum_{\alpha=0}^{\infty} a_{\alpha} \frac{1}{\alpha !} z^\alpha \]
for \( |z| < 1 \).

Let \( a_{\alpha} = h^{-1}(u_\alpha) \) for \( k = 0, 1, \ldots \) and let us put \( f(\omega) = m(p, D; \omega, \xi) \).

We have \( f^{(\alpha)}(0) = \alpha! p_{\alpha} \omega \alpha \) and consequently
\[ M(p, D, \xi; x) = \lim_{k} \frac{m(p, D; \xi; x)}{p_{\alpha}(0)} = \lim_{k} \left( \frac{f(h(z))}{p(z)} \right) = \lim_{k} \left( \frac{\frac{d^\alpha}{dz^\alpha}f(h(z))}{p(z)} \right) \]
for \( |z| < 1 \).

Let \( C_k = h^{-1}(u_k) \) for \( k = 0, 1, \ldots \) and let us put \( f(\omega) = m(p, D; \omega, \xi) \).

We have \( f^{(\alpha)}(0) = \alpha! p_{\alpha} \omega \alpha \) and consequently
\[ M(p, D; \xi; x) = \lim_{k} \left( \frac{\frac{d^\alpha}{dz^\alpha}f(h(z))}{p(z)} \right) \]
is convergent for \( m = 0, 1, \ldots \) Consequently there exist indices \( n_m \) where
\[ n_m \geq \alpha \text{ such that} \]
\[ \frac{|c_{m,n}|}{p_{\alpha}(u_k \omega \alpha)} \leq 1 \quad \text{for} \quad n > n_m; \quad m = 0, 1, \ldots \]
Let us consider the row-finite matrix method \( C \) where
\[ C(x) = \lim_{n} \sum_{m=0}^{\infty} c_{m,n} \omega_m \omega \]
for \( n = 0, 1, \ldots, m = 0, 1, \ldots \) and \( k = 0, 1, \ldots \), we finally obtain
\[ M(p, D; \xi; x) = \lim_{k} \left( \lim_{m=0}^{\infty} \sum_{n=0}^{m} c_{m,n} \omega_m \omega \right) = \lim_{k} C_k(x) \]
where \( C_k = (c_{m,n}) \), q.e.d.
and consequently $p_n|x|/|θ| ≤ λ$ for $n = 0, 1, \ldots$. We have

$$C_m(x) = C_m(x) - \sum_{n=0}^{∞} c_{m,n} t_n,$$

whence

$$|C_m(x) - C_m(x)| ≤ \sum_{n=0}^{∞} |c_{m,n} t_n| = \frac{1}{m} \sum_{n=0}^{∞} \frac{|θ/n|}{η} \to 0$$

if $m \to +∞$. This shows that $x ∈ C^∗$ and $C(x) = O(x)$, q.e.d.

§ 6. The $U(p)$ methods

Remark 5. Let $p ∈ S_r$. If $D_1, D_2 ∈ M_r$, then the methods $M(p, D_1)$, $M(p, D_2)$ are consistent.

Proof. Let $x ∈ M^*(p, D_1) ∩ M^*(p, D_2)$. We have $m(p, D_1; x, w) = m(p, D_2; x, w)$ for $0 ≤ w = Re(w) < r$, since $m(p, D_1; x, w)$ and $m(p, D_2; x, w)$ are analytic functions of the real variable on the interval $0 < w = Re(w) < r$ and the above identity is satisfied in a neighbourhood of $0$, q.e.d.

Remark 6. If $D_1, D_2 ∈ M_r$, then there exists a $D_3 ∈ M_r$ such that $D_3 ≤ D_1 ∩ D_2$, and consequently $M^*(p, D_3) ∪ M^*(p, D_3) ≤ M^*(p, D_3)$ where $p ∈ S_r$.

Proof. For every $w ∈ (0, r)$ there exists an open circle $S_w = \{w: |w − u| < θ_w\} ⊆ D_1 ∩ D_2$. It suffices to take $D_3 = \bigcup_{θ_w < r} S_w$, q.e.d.

Owing to Remarks 5 and 6 we can define the following method of limitation $U(p)$:

Definition 13. Let $p ∈ S_r$. The field of the method $U(p)$ is defined as the set

$$U^*(p) = \bigcup_{D_κ ∈ M_r} M^*(p, D),$$

The generalised limit $U(p; x)$ of the sequence $x ∈ U^*(p)$ (corresponding to the method $U(p)$) is defined by the formula $U(p; x) = M(p, D_κ; x)$, where $x ∈ M^*(p, D_κ; D_κ ∈ M_r).$

Theorem 11. Let $p ∈ S_r$ and conditions (β), § 3, be fulfilled. The method $U(p)$ is a permanent method of limitation in the sense of definition 3, but $U(p)$ is not an $(F)$-method (definition 5).

Proof. Suppose that $U^*(p)$ is an $(F)$-space under a norm $||.||$.

Let us consider the sets $A_k = U^*(p)$, where

$$A_k = \{x = (t_n; x) ∈ U^*(p); |t_n| ≤ \frac{k}{p_n} \text{ for } n = 1, 2, \ldots; \ k = 1, 2, \ldots\}.$$

The sets $A_k$ are closed, since $p_n(x) = t_n$ are continuous functionals on $U^*(p)$.

We show that

$$U^*(p) = \bigcup_{k=1}^{∞} A_k.$$

Indeed, let $x ∈ U^*(p)$, whence $x ∈ M^*(p, D)$ where $D ∈ M_r$. Thus the power series

$$\sum_{n=0}^{∞} p_n x_n w^n$$

has a positive radius of convergence:

$$\lim_{n \to +∞} \sqrt[n]{p_n} |x| < +∞.$$

Hence there exists a $k$ such that $\sqrt[n]{p_n} |x| ≤ k$ for $n = 1, 2, \ldots$ and consequently $x ∈ A_k$.

According to a theorem of Baire there exist a natural number $k$ and a ball $K = \{x: |x − x_0| < θ\} ⊆ A_k; x_0 = (|θ|)$. Putting $y = (s_n)$, where $s_n = (-k−1)^n/p_n$, we have

$$\sum_{n=0}^{∞} p_n s_n w^n = \frac{1}{1 + (k+1)w}$$

and consequently $y ∈ M^*(p, D)$, where

$$D = \{|w| < r; \ w = -\frac{1}{k+1}\} ∈ M_r;$$

hence $y ∈ U^*(p)$. For sufficiently great integer $m$ we have

$$\frac{1}{m} y + s_n x ≤ K$$

and consequently

$$\frac{1}{m} |s_n| ≤ \frac{k}{p_n} \text{ for } n = 1, 2, \ldots$$

Hence

$$\frac{1}{m} |s_n| ≤ \frac{k}{p_n} + |θ|^n ≤ 2 \frac{k}{p_n}; \ \frac{1}{m} \leq \frac{k}{k+1}.$$

Taking the limit for $n \to +∞$, we obtain a contradiction, q.e.d.
Remark 7. Let \( p \) be such a function as in Theorem 11. A sequence \( \delta_n \) \((\delta_n > 0)\) is a rate of growth of the method \( U(p) \) if and only if
\[
\lim_{n \to \infty} \frac{\delta_n}{P_n} = 0.
\]

Proof. Let
\[
\lim_{n \to \infty} \frac{\delta_n}{P_n} = 0
\]
and let \( \sigma = (\delta_n) \ast U^*(p) \). We have
\[
\lim_{n \to \infty} P_n \sigma_n^\alpha = 0
\]
for certain \( \alpha > 0 \), since the power series
\[
\sum_{n=1}^{\infty} p_n \sigma_n^\alpha n^\alpha
\]
has a positive radius of convergence. For \( n \geq n_0 \), we have \( \delta_n / P_n \leq c \); hence \( \delta_n / P_n \leq c \) \( \Rightarrow \) \( \sigma_n^\alpha \to 0 \) if \( n \to +\infty \).

Let \( \delta_n \) be a rate of growth of the method \( U(p) \) and let \( e > 0 \). It is easy to see that
\[
\frac{1}{P_n(-e)^n} U^*(p)
\]
and consequently the sequence \( \{\delta_n / P_n(-e)^n\} \) is bounded: \( \delta_n / P_n e^\alpha \leq C \). Hence
\[
\frac{n}{\delta_n} \leq C \frac{1}{\alpha} \frac{1}{\sqrt[C]{n}}, \quad \lim_{n \to \infty} \frac{n}{\delta_n} \leq e
\]
but this means that
\[
\lim_{n \to \infty} \frac{\delta_n}{P_n} = 0,
\]
q.e.d.

Corollary. If the hypothesis of Remark 7 is satisfied, then the method \( U(p) \) has no strict rate of growth.

References


Exposé par la Rédaction le 5. 7. 1955