THEOREM. *For a Banach space E the following statements are equivalent*:

(i) *E is reflexive.*

(ii) *E has property $\mathscr{S}$.*

(iii) *E has property $w\mathscr{S}$.*

In the arguments of [2] the essential positivity of T plays a fundamental role. Our proof is different from that of [2], being based on a profound result of A. Pełczyński ([3], theorem 2) concerning basic sequences.

Proof of the theorem. For (i) $\Rightarrow$ (ii), see [2]. (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Assume that E has property $w\mathscr{S}$ and let $\{x_n\}$ be an arbitrary basic sequence (i. e. a basis of a closed linear subspace) in E. Then the closed linear subspace $E_1 = [x_n]$ of E has property $w\mathscr{S}$ (by the theorem of S. Mazur [1], according to which the $\sigma(E, E^*)$-limit of any $\sigma(E, E^*)$-convergent sequence in E_1 belongs to E_1). Hence, by [2], theorem 3, the basis $\{x_n\}$ of E_1 must be boundedly complete (1). Thus every basic sequence in E is boundedly complete, whence, by [3], theorem 2, E is reflexive, which completes the proof.

(1) I. e. for every sequence of scalars $\{a_n\}$ such that $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < \infty$, the series $\sum_{i=1}^{\infty} a_i x_i$ converges.

References

[1] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), p. 70-84.

[2] T. Nishiura and D. Waterman, *Reflexivity and summability*, ibidem 23 (1963), p. 53-57.

[3] A. Pełczyński, *A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces"*, ibidem 21 (1962), p. 371-374.

INSTITUTE OF MATHEMATICS, RUMANIAN ACADEMY OF SCIENCES

Reçu par la Rédaction le 14. 12. 1964

STUDIA MATHEMATICA, T. XXVI. (1965)

A remark on the preceding paper of I. Singer

(From a letter to R. Sikorski)

by

A. PEŁCZYŃSKI (Warszawa)

The results of Nishiura and Waterman [2], and Singer [4] suggest the following

THEOREM. *Let W be a weakly closed bounded subset of a Banach space E. Then the following conditions are equivalent*:

(o) *W is weakly compact*;

(oo) *for every sequence (z_n) of elements of W there is a matrix $(c_{m,n})$ such that*

1) $c_{m,n} \geqslant 0$ *and* $c_{m,n} = 0$ *for* $n > n(m)$ $(n, m = 1, 2, \ldots)$,

2) $\sum_{n=1}^{n(m)} c_{m,n} = 1$ $(m = 1, 2, \ldots)$,

3) *the sequence* $\left(\sum_{n=1}^{n(m)} c_{m,n} z_n\right)$ *is convergent*;

(ooo) *for every sequence (z_n) of elements of W there is a regular matrix $(c_{m,n})$ such that the sequence $\left(\sum_{n=1}^{\infty} c_{m,n} z_n\right)$ is weakly convergent to an element of E.*

Proof. (o) $\rightarrow$ (oo). Let (z_n) be an arbitrary sequence in W. According to the Eberlein-Šmulian theorem ([1], p. 48) the sequence (z_n) contains a subsequence (z_{n_k}) which is weakly convergent to an element z of W. Then a theorem of Mazur ([1], p. 40) implies the existence of finite averages

$$w_m = \sum_{k=1}^{k(m)} t_{m,k} z_{n_k}$$

such that $\|z - w_m\| < m^{-1}$ $(m = 1, 2, \ldots)$. Let us set $c_{m,n} = t_{m,k}$ for $n = n_k$ $(k = 1, 2, \ldots, k(m);\ m = 1, 2, \ldots)$ and $c_{m,n} = 0$ in the other case. Then the matrix $(c_{m,n})$ has the desired properties 1)-3).

(oo) $\rightarrow$ (ooo). This implication is trivial.

non (o) $\rightarrow$ non (ooo). It follows from [3] that non (o) implies the existence of a basic sequence (z_n) of elements of W and a linear functional $z^* \epsilon E^*$ such that $\liminf_n z^* z_n > 0$. Since the sequence (z_n) is bounded ((z_n) being replaced, if necessary, by suitable subsequence), one can assume that there exists a limit $\lim_n z^* z_n > 0$. Let us suppose that for this sequence (z_n) there is a regular matrix $(c_{m,n})$ such that the sequence $\left(\sum_{n=1}^{\infty} c_{m,n} z_n\right)$ weakly converges to an element z in E. Let (z_n^*) denote the sequence of linear functionals in E^* biorthogonal to (z_n). Then (by the regularity of $(c_{m,n})$) we have,

$$z_p^* z = \lim_m z_p^* \left(\sum_{n=1}^{\infty} c_{m,n} z_n\right) = \lim_m \sum_{n=1}^{\infty} c_{m,n} z_p^* z_n = \lim_m c_{m,p} = 0 \quad (p = 1, 2, \ldots).$$

Thus

$$z = \sum_{n=1}^{\infty} z_n^* z \cdot z_n = 0$$

(because z belongs to the closed linear subspace spanned by the basic sequence (z_n)). Therefore

$$0 = z^* z = \lim_m z^* \left(\sum_{n=1}^{\infty} c_{m,n} z_n\right) = \lim_m \sum_{n=1}^{\infty} c_{m,n} z^* z_n.$$

But this leads to a contradiction, because the regularity of $(c_{m,n})$ implies

$$0 < \lim_n z^* z_n = \lim_m \sum_{n=1}^{\infty} c_{m,n} z^* z_n.$$

References

[1] M. M. Day, *Normed linear spaces*, New York 1962.

[2] T. Nishiura and D. Waterman, *Reflexivity dnd summability*, Studia Math. 23 (1963); p. 53-57.

[3] A. Pełczyński, *A proof of Eberlein-Šmulian theorem by an application of basic sequences*, Bull. Acad. Polon. Sci., Série math., astr. et phys., 12 (1964), p. 543-548.

[4] J. Singer, A remark on reflexivity and summability, Studia Math. 26 (1965), p. 113-114.

Reçu par la Rédaction le 4. 1. 1965