A remark on reflexivity and summability

by

I. SINGER (Bucharest)

Let us recall that a summability method \( T \) is a real matrix \( (c_{mn}) \),
\( m = 1, 2, \ldots, n = 1, 2, \ldots \) The \( T \)-means of a sequence \( \{a_n\} \) in a Banach
space \( E \) are
\[
T_n = \sum_{m=1}^{\infty} c_{mn} a_n.
\]

\( T \) is said to be regular if \( a_n \) real, \( a_n \to x \) (finite), implies that \( T_n \) exists
and \( T_n \to x \). According to the Toeplitz-Silverman theorem, \( T \) is regular
if and only if
1) \( \sum_{n=1}^{\infty} |c_{mn}| < M \) for all \( m \),
2) \( c_{mn} \to 0 \) as \( m \to \infty \), for all \( n \), and
3) \( \sum_{n=1}^{\infty} c_{mn} \to 1 \) as \( m \to \infty \).

A regular method \( T \) is said to be essentially positive [2], if
4) \( \sum_{n=1}^{\infty} |c_{mn}| \to 1 \) as \( m \to \infty \).

A Banach space \( E \) is said to have property \( \mathcal{S} \) (w\( \mathcal{S} \)) [2] if for every
bounded sequence in \( E \) there exists a regular method \( T \) and a subsequence
whose \( T \)-means converge strongly (weakly); or, equivalently [2], if for
every bounded sequence \( \{a_n\} \) in \( E \) there exists a regular method \( T \) such
that the \( T \)-means of \( \{a_n\} \) converges strongly (weakly).

Recently, T. Nishihara and D. Waterman have proved ([2], theorem 2) that for a Banach space \( E \) the following statements are equivalent:
(i) \( E \) is reflexive.
(ii) \( E \) has property \( \mathcal{S} \) with essentially positive \( T \).
(iii) \( E \) has property w\( \mathcal{S} \) with essentially positive \( T \).

The purpose of the present Note is to show that in this result the
essential positivity of \( T \) can be omitted, i.e. that we have the following
THEOREM. For a Banach space $E$ the following statements are equivalent:

(i) $E$ is reflexive;
(ii) $E$ has property $\mathcal{S}$;
(iii) $E$ has property $w\mathcal{S}$.

In the arguments of [2] the essential positivity of $T$ plays a fundamental role. Our proof is different from that of [2], being based on a profound result of A. Pelczyński ([3], theorem 2) concerning basic sequences.

Proof of the theorem. For (i) $\Rightarrow$ (ii), see [2]. (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Assume that $E$ has property $w\mathcal{S}$ and let $(x_n)$ be an arbitrary basic sequence (i.e., a basis of a closed linear subspace) in $E$. Then the closed linear subspace $E_1 = [x_n]$ of $E$ has property $w\mathcal{S}$ (by the theorem of S. Mazur [1]), according to which the $\sigma(E, E')$-limit of any $\sigma(E, E')$-convergent sequence in $E_1$ belongs to $E_1$. Hence, by [3], theorem 3, the basis $(x_n)$ of $E_1$ must be boundedly complete (1). Thus every basic sequence in $E$ is boundedly complete, whence, by [3], theorem 2, $E$ is reflexive, which completes the proof.

(1) I.e., for every sequence of scalars $(a_n)$ such that $\sup_n \left| \sum_{k=1}^{n} a_k x_k \right| < \infty$, the series $\sum_{k=1}^{\infty} a_k x_k$ converges.

References


Institute of Mathematics, Romanian Academy of Sciences

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A remark on the preceding paper of I. Singer

(From a letter to R. Sikorski)

by

A. PELCZYŃSKI (Warszawa)

The results of Nishura and Waterman [2], and Singer [4] suggest the following

**Theorem.** Let $W$ be a weakly closed bounded subset of a Banach space $E$. Then the following conditions are equivalent:

(o) $W$ is weakly compact;

(oo) for every sequence $(w_n)$ of elements of $W$ there is a matrix $(c_{m,n})$ such that

1) $c_{m,0} = 0$ and $c_{m,n} = 0$ for $n > n(m)$ ($n, m = 1, 2, \ldots$),

2) $\sum_{k=1}^{n} c_{m,n} = 1$ ($m = 1, 2, \ldots$),

3) the sequence $\left( \sum_{k=1}^{n} c_{m,n} w_k \right)$ is convergent;

(oo0) for every sequence $(w_n)$ of elements of $W$ there is a regular matrix $(c_{m,n})$ such that the sequence $\left( \sum_{k=1}^{n} c_{m,n} w_k \right)$ is weakly convergent to an element of $E$.

Proof. (o) $\Rightarrow$ (oo). Let $(w_n)$ be an arbitrary sequence in $W$. According to the Eberlein-Šmulian theorem ([1], p. 48) the sequence $(w_n)$ contains a subsequence $(w_{n_k})$ which is weakly convergent to an element $x$ of $W$. Then a theorem of Mazur ([3], p. 48) implies the existence of finite averages

$$w_m = \sum_{k=1}^{m} t_{m,k} w_{n_k}$$

such that $||w_m|| < m^{-1}$ ($m = 1, 2, \ldots$). Let us set $c_{m,n} = t_{m,k}$ for $n = n_k$ ($k = 1, 2, \ldots, k(m); m = 1, 2, \ldots$) and $c_{m,n} = 0$ in the other case. Then the matrix $(c_{m,n})$ has the desired properties 1)-3).

(oo) $\Rightarrow$ (oo0). This implication is trivial.

non (o) $\leftrightarrow$ non (oo). It follows from [3] that non (o) implies the existence of a basic sequence $(x_n)$ of elements of $W$ and a linear functio-