A note on parabolic fractional and singular integrals

by

C. SADOSEK (Buenos Aires)

Introduction. A. P. Calderón and A. Zygmund have considered in [3] singular integral operators given by convolution with kernels that are homogeneous of degree $-n$ and have mean value zero on the $n$-dimensional unit sphere, and they proved that, under suitable conditions, these are continuous operators from $L^p$ to $L^q$, $1 < p < \infty$, where the norms are taken with respect to the Lebesgue measure $dz$. Assuming this result and the boundedness of the kernel on the unit sphere, Stein [10] completed it by proving that the operators are continuous from $L^p$ to $L^q$ with weighted measures $|z|^n dz$ (This was known for $n = 1$, see [1] and [3]).

The purpose of this paper is to obtain an analogue of this last theorem for parabolic singular transforms that we introduce in §3. (See theorem 3 (1) below). We use a different method of proof. For this we define in §2 a “parabolic” fractional integral operator and prove some of its properties of continuity, which may be of independent interest.

1. Preliminaries. We begin with some notation and basic definitions.

In the following we shall denote by $(x, t) = (x_1, \ldots, x_n, t), (y, s) = (y_1, \ldots, y_n, s)$ points in the Euclidean $(n+1)$-dimensional half space $E_{n+1} = \mathbb{R}_+ \times (0, \infty)$. $\mathbb{R}^n$ will be understood as $L^p(E_{n+1})$, the class of complex valued measurable functions $f(x, t)$ defined on $E_{n+1}$ such that $\|f\|_p = \left(\int \int f(x, t)^p dz dt\right)^{1/p}$ is finite. An integral without specification of the domain of integration will be understood as being taken over the entire $E_{n+1}$. $C$ with various subscripts will stand for a constant, not necessarily the same at each occurrence, depending only on the variables displayed.

Dependence on the dimension, though, will not be indicated.

We introduce the metric $[x, t] = (|x|^m + t^m)^{1/m}$, where $m$ is a fixed positive integer.

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(1) This result was announced in Notices, Amer. Math. Soc. 13, no. 1, 65T-45.
A linear operator $T$, defined on functions in $L^p$, is said to be of type $(p, q, 1 \leq p, q \leq \infty)$ if there exists a constant $C_{pq}$ such that $\|Tf\|_q \leq C_{pq}\|f\|_p$ for all $f$.

For an $(n+1)$-tuple $P = (p_1, p_2, \ldots, p_{n+1})$, a mixed or vectorial norm $P$ is defined as

$$\|f\|_P = \left( \int \left( \int f(x_1, x_2, \ldots, t^{p_{n+1}} d\nu_1, d\nu_2, \ldots, d\nu_{n+1}) \right)^{p_{n+1}} dt \right)^{1/p_{n+1}}$$

[see [3]]. We say that an operator $T$ is of vectorial type $(P, Q)$ for $P = (p_1, p_2, \ldots, p_{n+1}), Q = (q_1, q_2, \ldots, q_{n+1})$, $1 \leq p_i, q_i \leq \infty$, $i = 1, 2, \ldots, n+1$, when $\|Tf\|_Q \leq C_{pq}\|f\|_P$, independently of $f$.

The distribution function of $f$ is defined by $D(f); \sigma = D_{\sigma}(f)$ is the (Lebesgue) measure $\{(x, t); |f(x, t)| > \sigma\}$. $D_{\sigma}(f)$ is non-increasing and continuous from the right. The non-increasing rearrangement of $f$ onto $(0, \infty)$ is then defined by $f^*(\tau) = \inf \{\sigma > 0; D_{\sigma}(f) < \tau\}$, $\tau > 0$. $f^*(\tau)$ is also continuous from the right and has the same distribution function as $f$.

A sublinear operator $T$, defined on functions in $L^p$, is said to be of weak type $(p, q, 1 \leq p < \infty, 1 \leq q \leq \infty)$ if there exists a constant $C_{pq}$ such that $D_{\sigma}(f) \leq C_{pq}\|f\|_p$ holds independently of $f$.

As

$$\sup_{\tau > 0} \sigma(D_{\sigma}(f)) = \sup_{\tau > 0} \tau^{1/p} f^*(\tau),$$

the weak $(p, q)$ condition of an operator $T$ can be stated as

$$\sup_{\tau > 0} \tau^{1/p} (Tf)^*(\tau) = \sup_{\tau > 0} \tau^{1/p} f^*(\tau),$$

in the following paragraph we shall use a property of $f^*$ that we are going to prove next.

Lemma 1. $(fg^*)(\tau_1 + \tau_2) \leq f^*(\tau_1)g^*(\tau_2)$.

Proof. By definition, $f^*(\tau_1) = \inf\{\sigma; D_{\sigma}(f) \leq \tau_1\}$ and $(fg^*)(\tau_1 + \tau_2) = \inf\{\sigma; D_{\sigma}(g) \leq \tau_1 + \tau_2\}$. As the distribution functions are non-increasing, if $\sigma \geq f^*(\tau_1)$, then $D_{\sigma}(f) \leq \tau_1$, and if $\beta \geq g^*(\tau_2)$, then $D_{\beta}(g) \leq \tau_2$. But $(\|f\|_p > \sigma) \leq (f > \sigma) + (|f| > \beta)$. So $D_{\sigma}(f) \leq D_{\sigma}(f) + D_{\beta}(g)$.

Then, it is $D_{\sigma}(f) \leq \tau_1 + \tau_2$ for all $\alpha, \beta$ such that $\alpha \geq f^*(\tau_1)$ and $\beta \geq g^*(\tau_2)$, and this implies $(fg^*)(\tau_1 + \tau_2) \leq \sigma\delta$ for all such $\alpha, \beta$. So $(fg^*)(\tau_1 + \tau_2) \leq f^*(\tau_1)g^*(\tau_2)$.

2. Parabolic fractional integrals. In one dimension we know that the fractional integral operator that we denote by $H_{\nu_1}; \theta < \gamma < 1$,

$$H_{\nu_1,f}(x) = \int_{-\infty}^{+\infty} f(y) |x - y|^{\gamma - 1} dy,$$

is of type $(p, \gamma)$, where $1/p - 1/\gamma = \gamma < 1/p < 1$ (see, for instance, [12], vol. II, p. 142) and of weak type $(1, 1/(1-\gamma))$ ([11], Th. 6, p. 242).

Let us now consider the one-dimensional operator

$$(T_{\nu_1,f})(x) = |x|^{-\gamma} (H_{\nu_1,f})(x) = |x|^{-\gamma} \int_{-\infty}^{+\infty} f(y)|x|^{-\gamma} cot \gamma dy.$$

Lemma 2. $T_{\nu_1,\gamma}; 0 < \gamma < 1$, is an operator of type $(p, p)$ for $1 < p < \infty$ and of type $(1, 1)$.

Proof. It is sufficient to show that $T_{\nu_1,\gamma}$ is of weak type $(p, p)$ for $1 < p < \infty$ and then apply Marcinkiewicz theorem [11].

Using the fact proved in Lemma 1 with $\tau_1 = \tau_2 = \tau/2$ we have

$$(T_{\nu_1,\gamma}f)^*(\tau) \leq (\tau/2)^{-\gamma} (H_{\nu_1,\gamma}f)^*(\tau/2)$$

and

$$\sup_{\tau > 0} \tau^{1/p} (T_{\nu_1,\gamma}f)^*(\tau) \leq 2^{1/p} \sup_{\tau > 0} \tau^{1/p} (H_{\nu_1,\gamma}f)^*(\tau/2).$$

Taking into account that $1/p - \gamma = 1/r$ and that $H_{\nu_1,\gamma}$ is of weak type $(p, r)$ for $1 < p < 1/r$, inequality (1) becomes

$$\sup_{\tau > 0} \tau^{1/p} (T_{\nu_1,\gamma}f)^*(\tau) \leq C_{\nu_1}\|f\|_p,$$

and the conclusion of the theorem follows.

In the $(n+1)$-dimensional case, we define, for $0 < \gamma < n$-1, the following “parabolic” fractional integral operators:

$$(H_{\nu_1,f}(x, t) = \int f(y, x)|x - y, t - a|^{\gamma - n + m} dy ds,$$

$$(T_{\nu_1,f}(x, t) = (x, t)|^{-\gamma} (H_{\nu_1,f}(x, t)) = (x, t)|^{-\gamma} \int f(y, x)|x - y, t - a|^{\gamma - n + m} dy ds.$$
But under the conditions of the hypothesis the conclusion follows.

(b) As before, we have

\[
[\gamma, \gamma'] \geq C_{\infty} \|\sqrt{\gamma} + \gamma'\|^{\gamma - \gamma'}
\]

so that

\[
(T_t f)(\gamma, t) \leq C_{\infty} \|\sqrt{\gamma} + \gamma'\|^{\gamma - \gamma'} H(t, \gamma, \gamma')\]

for \(f \geq 0\).

Combining what we know about the types of these one-dimensional operators, we infer that \(T_t\) is of type \((p, p)\) for \(1 < p < (n+1)/\gamma\).

The properties of type \((p, p)\) of the parabolic fractional integrals given in Theorem 1 are all that will be used in \(\S 3\).

However, it should be noticed that the following proposition, that is the natural analogue of Sobolev's lemma, holds:

**Theorem 2.** (a) The operator \(H_t\) is of type \((p, p)\) where \(1/p = 1/q + (n+1)/\gamma\).

(b) The operator \(H_t\) is of weak type \((1, (n+m)/(n+m-\gamma))\).

**Proof.** To prove (b) it is sufficient to consider \(f \in L^p\) such that \(\|f\|_p = 1\).

We now write \([\gamma, t] < \varepsilon\) and zero elsewhere and \(G(x, t) = \varepsilon < \varepsilon\). Then \(G \in L^p\) and \(G \in L^q\) for \(1 < q < \varepsilon\). We choose \(\varepsilon\) such that \(\|G\|_q = n-\gamma\).

By the definition of \(G\) its \(L^q\)-norm is equal to a constant times \(n-\gamma\).

Replacing in (2) and (3) we obtain

\[
D(H_t f; a) \leq D(G * f; a/2)
\]

Also it is

\[
D(G * f; a/2) \leq (2/a) \|G\|_q \leq (2/a) \|G\|_q
\]

By the definition of \(G\) its \(L^q\)-norm is equal to a constant times \(n-\gamma\).

Replacing in (2) and (3) we obtain

\[
D(H_t f; a) \leq C_{\infty} \|\sqrt{\gamma} + \gamma'\|^{\gamma - \gamma'}
\]

and this inequality is the weak type \((1, (n+m)/(n+m-\gamma))\) condition of the operator \(H_t\).

It may be proved in a similar way that \(H_t\) is of weak type \((p, p)\) for \(1 < p < (n+m)/\gamma\).

Then (a) follows as an application of Marcinkiewicz theorem.
and the last two integrals are easily seen to be equal, taking into account the “homogeneity” property (4) of \( k(x, t) \).

So, it is natural to impose conditions on the kernel to be satisfied on the set \( \{(x, t) : [x, t] = 1\} \).

Remark. The hypotheses of [7] are not the more general under which the theorem of type \((p, p)\), \(1 < p < \infty\), of the parabolic singular integrals holds (see [4]). Nevertheless we shall not specify less restrictive hypothesis here as in our main theorem we already assume the type \((p, p)\) of the singular integrals.

**Theorem 3.** If

\[
(Kf)(x, t) = \lim_{s \to 0} \int \int k(x-y, t-s)f(y, s)dyds
\]

is such that \( \|Kf\|_p \leq C_p \|f\|_p \), \(1 < p < \infty\), and \( \|k(x', t')\| \leq B \) whenever \([x', t'] = 1\), then \( K \) is of weighted type \((p, p)\), i.e.,

\[
\|Kf\|_p \leq C_{p,p} \|f\|_p
\]

for \(- (n+1)/p < \beta < (n+1)/p'\), where \( C_{p,p} \) is a constant depending only on \( p, p', \beta, m \) and \( n \). If \(- (n+1) < \beta \leq 0\), \( K \) is of weighted weak type \((1, 1)\).

**Proof.** By the first hypothesis we infer that

\[
\int \int |Kf(y, s)|^p dyds \leq C_{p,p} \int \int |f(x, t)|^p dydt.
\]

Then, it will be enough to prove that

\[
\int \int |Kf(y, s)|^p dyds \leq C_{p,p} \int \int |f(x, t)|^p dydt.
\]

is similarly bounded. Being \( x' = x/[x, t] \) and \( t' = t/[x, t] \), we consider the difference

\[
|Kf(y, s)[y, s] - [x, t]Kf(x, t)| = \int \int |(x-y, t-s)|^{-\alpha-n} k((x-y)', (t-s))f(y, s)[y, s]dyds - \\
\int \int |(x, t)|^{-\alpha-n} k((x-y)', (t-s))f(y, s)dyds
\]

\[
= \int k((x-y)', (t-s))f(y, s)[y, s]dyds - \int k((x-y)', (t-s))f(y, s)[y, s]dyds
\]

\[
\leq \int Bf(y, s)[y, s]dyds \leq C_{p,p} \int \int |f(x, t)|^p dydt.
\]

So, the theorem will be proved if we show that, under the stated conditions, the operator

\[
UF(x, t) = \int F(y, s)[y, s]dyds
\]

is of type \((p, p)\) and of weak type \((1, 1)\).

**Lemma 3.** Let

\[
K(x, t; y, s) = |1 - ((x, t)/[y, s])^p|[x-y, t-s]^{-\alpha-n}
\]

be the kernel of the operator \( U \), given by

\[
UF(x, t) = \int K(x, t; y, s)F(y, s)dyds.
\]

Then \( U \) is of type \((p, p)\) for \( 1 < p < \infty \) and \(- (n+1)/p < \beta < (n+1)/p'\) and of weak type \((1, 1)\) for \(- (n+1) < \beta < 0\).

**Proof.** We shall show that the operator \( U \) is dominated by a sum of operators \( T_w \)'s, that is, that its kernel \( K \) is dominated by sums of the kernels of these operators.

**First Case.** Let \(- \beta = \gamma > 0\). Then

\[
K(x, t; y, s) = \frac{[x, t]^{-\gamma}}{[x, t][y, s]^{-\gamma}}[x-y, t-s]^{-\alpha-n}.
\]

(I) Let \([y, s] < 2[x-y, t-s]\).

Let \((x, t) = (x-y, t-s)\) or \((x, t) = (x+y, s)\). Then

\[
[y + z, z + w] = [y, s] + z > 2(y, s) + z, \quad \text{and} \quad \frac{[y + z, z + w]}{[y, s]} \geq \frac{2(z-1)}{2z}.
\]

Then \([y, s] < 2[z-1][y, s] + 2[z-1][y, s]^{-1}\).

(II) Let \([y, s] > 2[x-y, t-s]\).

It is true that \([y, s] > 2[x, t]-2[y, s] \text{ and that } [y, s] > 2[y, s] - 2[x, t]\).

(III) Let \([y, s] > 2[x, t] > [y, s]\) or \(1 < (x, t)/[y, s] < 1\).

By the mean-value theorem we have

\[
[x, t]^{-\gamma} - [y, s]^{-\gamma} = \frac{[x, t]-[y, s]}{(z, z)},
\]

where \((z, z)\) is intermediate between \((x, t)\) and \((y, s)\). From inequality (6) it follows that

\[
|[x, t]^{-\gamma} - [y, s]^{-\gamma}| \leq C_{\gamma} [x-y, t-s]^{-\gamma}.
\]
which implies that
\[ K(x, t; y, s) \leq C \frac{[x, t]^{-\gamma}[x-y, t-s]}{[x, t][x-y, t-s]^{\gamma+n}} = C \frac{[x, t]^{-\gamma}[x-y, t-s]^{-(n+\gamma-\gamma)}}{[x, t][x-y, t-s]^{\gamma+n}}. \]

In this case \( UF \) is dominated by \( T_{\mathcal{F}} \). This will be of type \((p, p)\) if \( \gamma > 1 \) because this implies \( 1/(n+1) < \gamma/(n+1) < 1/p \). If \( \gamma < 1 \) we bound \( UF \) by \( T_{\mathcal{F}} \):
\[ [x-y, t-s] \leq [x, t] + 2[x, t] \]
\[ K(x, t; y, s) \leq C \frac{[x, t]^{-\gamma}[x-y, t-s]}{[x, t][x-y, t-s]^{\gamma+n}} \]
\[ \leq C \frac{[x, t]^{-\gamma}[x-y, t-s]^{-(n+\gamma-\gamma)}}{[x, t][x-y, t-s]^{\gamma+n}} \leq C \frac{[x, t]^{-\gamma}[x-y, t-s]^{-(n+\gamma-\gamma)}}{[x, t][x-y, t-s]^{\gamma+n}}. \]

In this case \( UF \) is dominated by \( T_{\mathcal{F}} \). So, \( UF \) is dominated by \( T_{\mathcal{F}} \) or by \( T_{\mathcal{F}}^{**} \).

Second case. Let now \( \beta > 0 \). We have
\[ \| UF \|_{p, \mathcal{F}} = \sup_{[0, 1]} \int UF(x, t)g(x, t)\,dxdt \]
\[ = \sup_{y, s} \int \left[ 1 - \left( \frac{[x, t]}{[y, s]} \right)^{\gamma+n} \right]^{-(n+\gamma-\gamma)} f(y, s)\,dyds \int \frac{g(x, t)\,dxdt}{[x-y, t-s]^{\gamma+n}} \]
\[ = \sup_{y, s} \frac{f(y, s)}{[x-y, t-s]^{\gamma+n}} \int \frac{g(x, t)\,dxdt}{[x-y, t-s]^{\gamma+n}}. \]

The last bracket is in the conditions of the preceding case with
\[ -(n+1)/p' < \beta < 0, \]
which is the hypothesis. So the conclusion follows.

References