

Germs and their Operational Calculus

by

J. MIKUSIŃSKI (Warszawa)

1. Definition and general properties of germs. We say that two functions f and g , defined in an interval $0 \leq t < T$, are *equivalent*, and we write $f \sim g$, if there is an interval $0 \leq t < t_0$ ($t_0 \leq T$) in which $f(t) = g(t)$ holds. The set $x = \{f\}$ of all functions k which are equivalent to a given function f will be called the *germ* of f . If we know the germ of a function, we know more than its value at $t = 0$, but less than its behaviour in any right neighbourhood of this point. Evidently, $f \sim g$ implies $\{f\} = \{g\}$, and conversely.

For real valued f and g , we write $\{f\} \leq \{g\}$, if there is an interval $0 \leq t < t_0$ in which $f(t) \leq g(t)$ holds. This definition of inequality of germs is consistent, i.e. does not depend on the choice of representants of germs. By a *representant* of a germ $\{f\}$ we understand the function f or any equivalent function. Evidently, the relation of inequality between germs has the following properties:

1° $x \leq x$.

2° If $x \leq y$ and $y \leq z$, then $x \leq z$.

3° If $x \leq y$ and $y \leq x$, then $x = y$.

Let \mathcal{G} denote the set of all real germs, i.e. of the germs of real-valued functions. Then, for any pair x, y from \mathcal{G} , there is in \mathcal{G} a third germ z such that $x \leq z$ and $y \leq z$. In other words, \mathcal{G} is a partially ordered set.

We define the arithmetic operations on germs as follows:

$$(1) \quad \{f\} + \{g\} = \{f + g\};$$

$$(2) \quad \{f\}\{g\} = \{fg\}.$$

It is immediate that these definitions are consistent and that \mathcal{G} is a commutative ring. If we identify numbers γ with the germs $\{\gamma\}$ (of constant functions whose values are γ), then \mathcal{G} becomes a commutative algebra and can be considered as an extension of the algebra of real numbers.

Note that there are positive germs which are less than any positive number, e.g. the germ of the linear function $f(t) = t$. Such germs can

be called *positive infinitesimals*. Similarly we can introduce *negative infinitesimals*. More generally, the germ of any function f such that

$$\lim_{t \rightarrow 0^+} f(t) = 0$$

(except for the trivial case $f(t) \equiv 0$) can be considered as an infinitesimal; it is greater than any negative number and less than any positive number.

Note finally that the ring \mathcal{G} has divisors of zero and therefore can not be completed to a field.

2. Convolution ring of germs. A germ is said to be *continuous* if it has, among its representants, a continuous function. The set of real continuous germs is a subring of \mathcal{G} , when adopting definitions (1) and (2). This subring has also divisors of zero.

However, we can also define the product, alternatively, as a convolution rather than by (2). Then we obtain a ring without divisors of zero. By the convolution of germs we understand

$$(3) \quad \{f\} * \{g\} = \{f * g\},$$

where

$$(4) \quad (f * g)(t) = \int_0^t f(t-\tau)g(\tau) d\tau.$$

It is easy to see that this definition is consistent and that, under addition (1) and multiplication (3), the set of continuous germs is a ring. The fact that this ring has no divisors of zero follows easily from Titchmarsh's theorem on convolution.

In the preceding definitions of this section, it is irrelevant whether the germs are real or complex. By a *complex germ* $\{f\}$ we understand the set of all functions k which are equivalent to a given complex valued function f . Let \mathcal{C} denote the ring of complex continuous germs with convolution as multiplication. Through the whole remaining part of this paper the product xy of germs x, y will be understood in the sense of convolution. The meaning of powers x^n ($x \in \mathcal{C}$, n positive integer) is obvious.

We shall say that a power series of $x \in \mathcal{C}$

$$(5) \quad a_1x + a_2x^2 + \dots \quad (a_n \text{ complex numbers})$$

converges to an element $u \in \mathcal{C}$, if for $x = \{f\}$ the series

$$(6) \quad a_1f + a_2f^2 + \dots$$

(where f^n denotes the n -th convolution power of the function f) converges, in an interval $0 \leq t < t_0$, uniformly to a function r such that $u = \{r\}$.

We shall show that this definition of convergence is consistent, i.e. that the limit u , if exists, does not depend on the choice of the representant f of the germ x . In fact, let $x = \{f\} = \{g\}$. Then, by definition, we have $f \sim g$, i.e. there exists an interval $0 \leq t < t_1$ in which $f(t) = g(t)$ holds. If (6) converges to r uniformly in $0 \leq t < t_0$, then it does so in the interval $0 \leq t < t_2$ with $t_2 = \min(t_0, t_1)$. Subsequently, the series $a_1g + a_2g^2 + \dots$ converges also to r uniformly in $0 \leq t < t_2$. Thus we obtain the same limit $\{r\}$, no matter what is the representant of x .

Note that the definition of convergence of a power series, given above, is different from the usual way. Traditionally, one defines first a convergence of sequences and then the convergence of series as sequences of partial sums. Such a traditional way fails in our case. In fact, it is natural to require, for purposes of Analysis, that the concept of convergence of sequences should satisfy the following conditions:

- (i) If $x_n = x$, then $x_n \rightarrow x$.
- (ii) If $\lambda_n \rightarrow 0$ (λ_n numbers), then $\lambda_n x \rightarrow 0$.
- (iii) If $|x_n| \leq y_n$ and $y_n \rightarrow 0$, then $x_n \rightarrow 0$.

It is clear in the above relation, how a product λx_n (λ number, $x \in \mathcal{C}$) and the modulus $|x|$ ($x \in \mathcal{C}$) should be understood. Namely,

$$(7) \quad \lambda \{f\} = \{\lambda f\},$$

$$(8) \quad |\{f\}| = \{|f\}|.$$

It is also easily seen that definitions (7) and (8) are consistent and that

$$\lambda(\kappa x) = (\lambda\kappa)x, \quad \lambda(x+y) = \lambda x + \lambda y, \quad (\lambda + \kappa)x = \lambda x + \kappa x, \\ |x+y| \leq |x| + |y|,$$

where λ, κ are numbers and $x, y \in \mathcal{C}$.

Now, let $x_n = f_n$, where

$$f_n(t) = \begin{cases} t & \text{for } 0 \leq t < \frac{1}{n}, \\ \frac{1}{n} & \text{for } \frac{1}{n} \leq t < \infty, \end{cases}$$

and let $x = \{f\}$, where $f(t) = 1$ for $0 \leq t < \infty$. Then (7) and (ii) imply that

$$y_n = \frac{1}{n} x \rightarrow 0$$

and, moreover, we have $|x_n| = x_n \leq y_n$. Thus, in view of (iii) we would have $x_n \rightarrow 0$. But, evidently, $x_n = x_1 \neq 0$ for every $n = 1, 2, \dots$, thus (i) would imply $x_n \rightarrow x_1 \neq 0$ and the limit of x_n would not be unique.

Thus we can introduce in \mathcal{G} no general convergence of sequences such that the limits of convergent sequences are unique and that properties (i), (ii) and (iii) are preserved. In spite of this fact it is sensible to define a convergence of power series, as has been done above.

Power series of germs have ordinary properties:

If

$$(9) \quad \alpha_1 x + \alpha_2 x^2 + \dots = u,$$

then, for every number λ ,

$$(10) \quad \lambda \alpha_1 x + \lambda \alpha_2 x^2 + \dots = \lambda u.$$

If (9) holds and, moreover,

$$\beta_1 x + \beta_2 x^2 + \dots = v,$$

$$(11) \quad (\alpha_1 + \beta_1)x + (\alpha_2 + \beta_2)x^2 + \dots = u + v,$$

and

$$(12) \quad \gamma_2 x^2 + \gamma_3 x^3 + \dots = uv$$

with $\gamma_n = \alpha_1 \beta_{n-1} + \dots + \alpha_{n-1} \beta_1$.

The proofs of (10) and (11) are immediate. In order to prove (12), we write $x = \{f\}$ and assume that (6) converges to r uniformly in $0 \leq t < t_0$ and

$$(13) \quad \beta_1 f + \beta_2 f^2 + \dots$$

converges to s uniformly in $0 \leq t < t_1$. Then both series (6) and (13) converge uniformly in $0 \leq t < t_2$ with $t_2 = \min(t_0, t_1)$, to r and s respectively. Then series (12) converges to rs uniformly in $0 \leq t < t_2$. Consequently (12) holds.

3. Distribution germs and operator germs. The germs considered so far will be called, in the sequel, *function germs*, in order to distinguish them from other kinds of germs we are going to introduce now.

Replacing in the preceding definition of germs the word *function* by *distribution* we obtain entities which may be called *distribution germs*. We consider first distributions defined in an interval $0 \leq t < T$ and we say that two such distributions f and g are *equivalent*, if $f = g$ holds in some interval $0 \leq t < t_0$. The set of all distributions which are equivalent to a given distribution is a *distribution germ*. In this definition, there is a small inexactitude, because distributions are usually defined in open sets, and in our case we need distributions which are defined in an interval which is half closed. This objection is not very serious and can be removed by a proper definition of distributions in such an interval. Instead of doing that we shall introduce a more general concept of germs, namely *operator germs*.

We shall first recall the definition of an operator in an interval $0 \leq t < T$. We consider the set C_T of complex-valued continuous functions in $0 \leq t < T$. The set C_T is a ring under ordinary addition and convolution (4) as multiplication. We extend C_T to a quotient ring M_T and call elements of M_T *operators* in $0 \leq t < T$. In particular, M_∞ is a field (see [3]) and, for finite T , M_T is a ring having divisors of zero (see [4]).

Two operators f and g will be said *equivalent*, if $f = g$ holds in some interval $0 \leq t < t_0$ ($t_0 \leq T$). The meaning of the last expression needs an explanation. The operators f and g can be represented in the form

$$(14) \quad f = \frac{p_1}{q_1} \quad \text{and} \quad g = \frac{p_2}{q_2},$$

where p_1, q_1, p_2 and q_2 belong to C_T and, besides, q_1, q_2 are functions which do not vanish identically in any interval $0 \leq t < t_1$. We say that

$$(15) \quad f = g \quad \text{in} \quad 0 \leq t < t_0,$$

if

$$(16) \quad p_1 q_2 = p_2 q_1 \quad \text{in} \quad 0 \leq t < t_0,$$

where the products are understood as convolutions. The meaning of the last equality is fully determined, for both sides of this equality are continuous functions. It remains to show that the definition of (15) does not depend on the choice of representation (14). Let us take another representation

$$(17) \quad f = \frac{p_3}{q_3} \quad \text{and} \quad g = \frac{p_4}{q_4}.$$

Then we have to show that (16) implies

$$(18) \quad p_3 q_4 = p_4 q_3 \quad \text{in} \quad 0 \leq t < t_0.$$

In fact, (14) and (17) imply

$$(19) \quad p_1 q_3 = p_3 q_1 \quad \text{and} \quad p_2 q_4 = p_4 q_2 \quad \text{in} \quad 0 \leq t < T.$$

Multiplying the first of equalities (19) by q_2 we obtain, in view of (16),

$$p_2 q_1 q_3 = p_3 q_1 q_2 \quad \text{in} \quad 0 \leq t < t_0.$$

Hence, by Titchmarsh's theorem on convolution,

$$p_2 q_3 = p_3 q_2 \quad \text{in} \quad 0 \leq t < t_0.$$

Multiplying the last equality by q_4 , we obtain, in view of the second of equalities (19),

$$p_4 q_2 q_3 = p_3 q_2 q_4 \quad \text{in} \quad 0 \leq t < t_0,$$

which implies (18), by Titchmarsh's theorem.

Thus the definition of equivalency of operators f and g is consistent. The set $x = \{f\}$ of all operators k which are equivalent to a given operator f will be called the *germ of the operator* f . The concept of an operator germ is more general than that of a function germ, since every continuous function is a particular operator. Operators of the form

$$(20) \quad f = \frac{p}{q}, \quad \text{where} \quad q(t) = t^n,$$

are distributions of finite order. The operator germs which have (20) as their representant can be called *distribution germs*. Distribution germs are then a particular case of operator germs. If for instance $p(t) = q(t) = t$, then $\{f\}$ is the germ of the Dirac delta distribution.

Since integrable functions are distributions, they are also operators. Thus germs of integrable functions are another particular case of operator germs.

The set M_0 of all operator germs is a linear space.

4. An alternative definition of operator germs. Since the ring \mathcal{C} has no divisors of zero, we can extend it to a quotient field \mathcal{F} . In other terms, elements of \mathcal{F} are quotients of germs x/y with $y \neq 0$. The field \mathcal{F} contains \mathcal{C} as its subring. Also, it contains isomorphically the field of complex numbers; we shall simply say that complex numbers are elements of \mathcal{F} . (They are not elements of \mathcal{C} , i.e. the field of complex numbers cannot be imbedded isomorphically in the ring \mathcal{C} ; this follows from the definition of product, which is convolution in \mathcal{C} .)

To every pair of functions p, q from C_T such that q does not vanish identically in the neighbourhood of 0 there corresponds uniquely an

element of \mathcal{F} of the form $\frac{\{p\}}{\{q\}}$. Similarly, there corresponds uniquely

an element of M_0 of the form $\left\{ \frac{p}{q} \right\}$. Thus the pair p, q determines a cor-

respondence between elements of \mathcal{F} and M_0 . We shall show that this correspondence is one-to-one. To this end we have to show that

$$(21) \quad \frac{\{p_1\}}{\{q_1\}} = \frac{\{p_2\}}{\{q_2\}}$$

implies

$$(22) \quad \left\{ \frac{p_1}{q_1} \right\} = \left\{ \frac{p_2}{q_2} \right\},$$

and conversely. Suppose first that (21) holds. Then $\{p_1\}\{q_2\} = \{p_2\}\{q_1\}$, i.e. $\{p_1q_2\} = \{p_2q_1\}$. This means that $p_1q_2 = p_2q_1$ holds in some interval $0 \leq t < t_0$. Thus we have $p_1/q_1 = p_2/q_2$, provided the fractions are con-

sidered as operators in $0 \leq t < t_0$. This implies (22). Suppose now, conversely, that (22) holds. Then the operators p_1/q_1 and p_2/q_2 are equal in some interval $0 \leq t < t_0$. This means that $p_1q_2 = p_2q_1$ holds in $0 \leq t < t_0$. Hence we obtain successively $\{p_1q_2\} = \{p_2q_1\}$, $\{p_1\}\{q_2\} = \{p_2\}\{q_1\}$, and finally (21). Thus the correspondence between \mathcal{F} and M_0 is one-to-one. It is not difficult to verify that this correspondence is an isomorphism. Thus we can identify the elements of \mathcal{F} with the elements of M_0 and call them equally operator germs.

The definition given in this section has the advantage that we see at once that the set of operator germs is a field. On the other hand, the definition in section 3 is adapted better to introducing the concept of convergence of power series.

5. Power series of operator germs. As in section 2, we can consider power series (5) where x is, actually, an operator germ. Imitating the previously given definition of convergence, we shall say now that series (5), where x is an operator germ, converges to an operator germ u , if for $x = \{f\}$ (where f is an operator) the corresponding power series (6) of operators converges, in some interval $0 \leq t < t_0$, to an operator r such that $u = \{r\}$.

In order to make this definition sensible, we have to say what a convergence of a series of operators in an interval means, when that interval is less than the interval in which the operators are defined. Let u_1, u_2, \dots be a sequence of operators defined in an interval $0 \leq t < T$. We say that such a sequence converges in $I_0: 0 \leq t < t_0$ ($0 < t_0 \leq T$) to an operator u defined in I_0 , if there is a function q , continuous in I_0 , such that all the products qu and qu_n are continuous functions in I_0 and the sequence qu_n converges to qu almost uniformly in I_0 . This definition is consistent, i.e. the limit, if exists, is determined uniquely. Now, we say that a series of operators $v_1 + v_2 + \dots$, defined in $0 \leq t < T$, converges to v in a subinterval $0 \leq t < t_0$, if the sequence of partial sums $u_n = v_1 + \dots + v_n$ converges to v in that subinterval. Thus the definition of convergence in $0 \leq t < t_0$ of a series of operators is traditional in this sense that we first define a convergence of a sequence and then the convergence of a series. On the other hand, the definition of convergence of a power series of operator germs is based directly on the convergence of series of operators, but it can be extended neither to sequences nor to arbitrary series which are not power series. Power series of operators have the same properties as expressed by formulae (10), (11) and (12).

From properties of power series of operators it is also easy to deduce other properties of power series of operator germs. For instance:

(I) *If a series $a_0 + a_1\lambda + a_2\lambda^2 + \dots$ of the complex variable λ has a positive radius of convergence, then the series $a_0 + a_1x + a_2x^2 + \dots$ converges for every germ x of an integrable function.*

E.g., the series $1+x+x^2+\dots$ converges for every x which is a germ of an integrable function. As particularly interesting, we can consider the series

$$(23) \quad 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots$$

This series also converges for every x which is the germ of an integrable function. However, if x is an arbitrary operator germ, then series (23) can be either convergent or not convergent. An operator germ x for which (23) is convergent can be called, as in the Operational Calculus, a *logarithm*. It is natural to denote the sum of (23) by e^x . In other words, series (23) can be considered as a definition of the exponential e^x . The exponential e^x exists for operator germs which are logarithms, and does not exist for other operator germs.

Note that in Operational Calculus the exponential is determined more generally by means of the differential equation $y'(\lambda) = xy(\lambda)$. Such a definition cannot be used in the Calculus of Germs, for there is there no general notion of limit, and therefore no general notion of derivative.

(II) If x is a logarithm, then λx is a logarithm, no matter what is the complex number λ .

In other words, if e^x exists, then $e^{\lambda x}$ exists for every complex number λ .

In fact, by multiplication of power series we find that if e^x and e^y exist for some operator germs x and y , then also e^{x+y} exists and we have

$$(24) \quad e^{x+y} = e^x \cdot e^y.$$

In particular, if e^x exists, then $e^{2x} = e^x \cdot e^x$. By induction we find that then also e^{nx} exists for every natural number n . Thus (II) follows immediately from

(III) If the series

$$(25) \quad a_0 + a_1 \lambda x + a_2 \lambda^2 x^2 + \dots,$$

where a_n are fixed complex numbers and x is a fixed operator germ, converges for some complex number $\lambda = \lambda_0$, then that series converges also for every complex number λ such that $|\lambda| < |\lambda_0|$.

In order to prove (III), let us represent x in the form $x = \{f\}$, where f is an operator. There is an interval $0 \leq t < t_0$ in which the series $a_0 + a_1 \lambda_0 f + a_2 \lambda_0^2 f^2 + \dots$ converges operationally. By a known theorem ([3], p. 166), the series $a_0 + a_1 \lambda f + a_2 \lambda^2 f^2 + \dots$ converges operationally in $0 \leq t < t_0$ for $|\lambda| < |\lambda_0|$. This implies that (25) converges as a power series of operator germs for every $|\lambda| < |\lambda_0|$.

From (24) and (III) it follows that the set of all logarithms is a linear subspace of M_0 .

6. Remarks on operators. From recent results of P. Antosik and of Boehme, we can easily obtain the following theorems (IV) and (V):

(IV) If $0 < T_1 < T_2 < \dots$ and $T_n \rightarrow T$ (where T may be finite or infinite) and there is a sequence of operators x_1, x_2, \dots such that x_n is defined in $0 \leq t < T_n$ and that $x_n = x_{n+1}$ holds in $0 \leq t < T_n$, then there is an operator x defined in $0 \leq t < T$ such that $x_n = x$ holds in $0 \leq t < T_n$ for every index n .

In fact, after Antosik [1], operators in open intervals $0 \leq t < T_n$ can be identified with operators in the corresponding closed intervals $0 \leq t \leq T_n$. I.e., each of them can be represented as a convolution quotient $x_n = p_n/q_n$, where p_n and q_n are continuous functions in $0 \leq t \leq T_n$. These functions can be extended continuously to the infinite interval $0 \leq t < \infty$. In this way also x_n may be considered as operators in $0 \leq t < \infty$ with the property that $x_n = x_{n+1}$ holds in $0 \leq t \leq T_n$. After Boehme [2] there is a continuous function q in $0 \leq t < \infty$ such that the convolution quotients $g_n = q/q_n$ are continuous functions. Evidently, we have $x_n = p_n g_n / q$ in $0 \leq t < \infty$, which implies that $p_n g_n = p_{n+1} g_{n+1}$ in $0 \leq t \leq T_n$. Thus $p_n g_n$ is a sequence of continuous functions which converge almost uniformly in $0 \leq t < T$. Its limit p is a continuous function in $0 \leq t < T$. The operator $x = p/q$ has the required properties.

(V) Let the meaning of T_m and T be the same as in (IV). If a sequence of operators x_n defined in $0 \leq t < T$ has the property that, when restricted to any of subintervals $0 \leq t < T_m$, converges in that subinterval, then it converges in the whole interval $0 \leq t < T$.

In fact, without loss of generality we can replace open intervals $0 \leq t < T_m$ by closed intervals $0 \leq t \leq T_m$. There are functions q_m , continuous in $0 \leq t < T$, such that the products $q_m x_n$ are continuous functions in $0 \leq t \leq T_m$ and that, for any fixed m , the sequence $q_m x_n$ converges uniformly in $0 \leq t \leq T_m$, as $n \rightarrow \infty$. In view of [2], there is a continuous function q in $0 \leq t < T$ such that the convolution fractions

$\frac{q}{q_n}$ are continuous functions in $0 \leq t < T$. Consequently, the sequence

$q x_n = \frac{q}{q_n} \cdot q_m x_n$ consists of continuous functions and converges uniformly

in each of intervals $0 \leq t \leq T_m$, i.e. almost uniformly in $0 \leq t < T$. Its limit p is a continuous function in $0 \leq t < T$. This means that x_n

converges operationally to $\frac{p}{q}$ in $0 \leq t < T$.

We shall also need the following theorem:

(VI) Let $x_1+x_2+\dots$ be a convergent series of operators in $0 \leq t < T$ and β_1, β_2, \dots an absolutely convergent sequence of complex numbers, i.e., such that $|\beta_2-\beta_1|+|\beta_3-\beta_2|+\dots < \infty$. Then the operator series $\beta_1x_1+\beta_2x_2+\dots$ is also convergent in $0 \leq t < T$.

In fact, there is a function g , continuous in $0 \leq t < T$, such that $qx_1+qx_2+\dots$ is a sequence of continuous functions which converges almost uniformly in $0 \leq t < T$. We have

$$\beta_{m+1}qx_{m+1}+\dots+\beta_nqx_n = \beta_{m+1}(qx_{m+1}+\dots+qx_n) + (\beta_{m+2}-\beta_{m+1})(qx_{m+2}+\dots+qx_n) + \dots + (\beta_n-\beta_{n-1})qx_n.$$

Given any subinterval $0 \leq t < T_0$ and a positive number ε , there exists an index n_0 such that $|qx_{k+1}+\dots+qx_n| < \varepsilon$ holds in that interval for $n > k > n_0$. Moreover, since β_n converges absolutely, there is a number M such that $|\beta_2-\beta_1|+|\beta_3-\beta_2|+\dots < M$ and $|\beta_n| < M$ for every n . Thus we have

$$|\beta_{m+1}qx_{m+1}+\dots+\beta_nqx_n| \leq (|\beta_{m+1}|+|\beta_{m+2}-\beta_{m+1}|+\dots+|\beta_n-\beta_{n-1}|)\varepsilon < 2M\varepsilon$$

for $n > k > n_0$ and $0 \leq t < T_0$. Since the subinterval $0 \leq t < T_0$ is arbitrary, this proves that the series $\beta_1qx_1+\beta_2qx_2+\dots$ converges almost uniformly in $0 \leq t < T$. Hence the series of operators $\beta_1x_1+\beta_2x_2+\dots$ is convergent.

(VII) Let us consider a power series

$$(26) \quad \alpha_0 + \alpha_1 f + \alpha_2 f^2 + \dots$$

where f is an operator in $0 \leq t < T$ and α_n are complex numbers for which the sequence $\beta_n = n\alpha_n/\alpha_{n-1}$ is absolutely convergent. If series (26) is convergent (operationally) in $0 \leq t < T$, then it remains so, if we replace f by any operator g in $0 \leq t < T$ which has the same germ as f .

In fact, let T_0 be any number with $0 < T_0 < T$ and let m be a positive integer such that $f = g$ holds in the interval $0 \leq t < t_0 = T_0/m$. Furthermore, let $g = f+k$ in $0 \leq t < T$. Then $k = 0$ in $0 \leq t \leq t_0$. This implies that $k^n = 0$ in $0 \leq t \leq T_0$ for $n \geq m$. Consequently

$$g^n = f^n + \binom{n}{1} f^{n-1} k + \dots + \binom{n}{m-1} f^{n-m+1} k^{m-1} \quad \text{for } 0 \leq t \leq T_0.$$

Hence

$$\begin{aligned} \sum_0^\infty a_n g^n &= \sum_0^\infty a_n f^n + k \sum_1^\infty \binom{n}{1} a_n f^{n-1} + \dots + k^{m-1} \sum_{m-1}^\infty \binom{n}{m-1} a_n f^{n-m+1} \\ &= \sum_0^\infty a_n f^n + \frac{k}{1!} \sum_1^\infty \beta_n a_{n-1} f^{n-1} + \dots + \frac{k^{m-1}}{(m-1)!} \sum_{m-1}^\infty (\beta_n \dots \beta_{n-m+1}) a_{n-m+1} f^{n-m+1} \end{aligned}$$

and after a proper change of indices

$$\sum_0^\infty a_n g^n = \sum_0^\infty a_n f^n + \frac{k}{1!} \sum_0^\infty \beta_{n+1} a_n f^n + \dots + \frac{k^{m-1}}{(m-1)!} \sum_0^\infty (\beta_{n+1} \dots \beta_{n+m-1}) a_n f^n$$

in $0 \leq t \leq T_0$. Now, it is easy to show that, for every fixed p , the sequence $\beta_{n+1} \dots \beta_{n+p}$ is absolutely convergent. Hence by (VI) all the series on the right-hand side of the last equality are convergent. This implies that the series on the left-hand side converges in $0 \leq t \leq T_0$. Since T_0 is arbitrary, the series converges in the whole interval $0 \leq t < T$, by (V).

Theorem (VII) applies in particular to the series

$$(27) \quad 1 + \frac{1}{1!} f + \frac{1}{2!} f^2 + \dots$$

In this case, the preceding argument can be made more heuristic, when we remark that the existence of the exponentials e^f and e^k implies the existence of the exponential $e^g = e^{f+k}$.

In connection with (VII) some conjectures arise. (1) Does theorem (VII) remain true if the assumption on the coefficients α_n is released? (2) Can we assert that the convergence of (27) in a subinterval $0 \leq t < t_0$ implies the convergence of (27) in the whole interval $0 \leq t < T$? Or, in other words, is the convergence of (27) a consequence of the convergence of the corresponding sequence of germes?

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