ainsi $h(z) = g(z)+if(z)$ est analytique dans le disque $|z| < 1$, soit

$$h(z) = \log a + \sum_{1}^{\infty} h_n z^n.$$

Pour $t \in F$, $g(re^{it})$ tend uniformément, quand $r \to 1$, vers une limite $\geqslant 0$. Pour N assez grand, on a donc

$$(2) \qquad \mathscr{R}\left(\sum_{1}^{N} h_n \left(1-\frac{n}{N}\right) e^{int}\right) \geqslant \frac{1}{2}\log\frac{1}{a} \qquad (t \in E).$$

Posons enfin

$$q(t) = \mathscr{I}\left(\sum_{1}^{N} h_n \left(1-\frac{n}{N}\right) e^{int}\right) = \sigma_N\big(f(e^{it}), t\big),$$

$$p(t) = \frac{2}{\pi} e^{-iNt} q(t).$$

On a bien, d'après (1)

$$\|p\|_{C(T)} < 1,$$

et

$$S_N(p, t) = \frac{2}{\pi} e^{-iNt} \frac{1}{2i} \sum_{1}^{N} h_n \left(1-\frac{n}{N}\right) e^{int},$$

donc d'après (2)

$$|S_N(p, t)| \geqslant \frac{1}{\pi}\log\frac{1}{a} \qquad \text{pour} \qquad t \in F,$$

et le lemme est démontré.

Reçu par la Rédaction le 1. 7. 1965

STUDIA MATHEMATICA, T. XXVI. (1966)

Weak*-sequential closure and the characteristic of subspaces of conjugate Banach spaces*

by

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1. The notion of *characteristic* of a subspace of a conjugate Banach space is due to Dixmier [2] and has been studied recently by Petunin [7], [8]. If X is a Banach space and M is a subspace of X^*, and $U_r = \{f \in X^*: \|f\| \leqslant r\}$, then the characteristic $\nu(M)$ of M is the least upper bound of all real numbers r such that $M \cap U_1$ is w*-dense in U_r. In [3] there was defined a number $\mu(M)$ (denoted there by C_M) depending on the w*-sequential closure $K_X(M)$ of M. The purpose of the present paper is to establish certain relationships between $\mu(M)$ and $\nu(M)$. In particular, it is shown that if $K_X(M) = X^*$ and X is separable, then $\nu(M) = 1/\mu(M)$. Also there is given a strengthening of a result of Banach concerning iterated w*-sequential closure.

The results in this paper are contained in the author's dissertation written at Florida State University under the direction of Professor R. D. McWilliams, but the proofs have been somewhat simplified.

2. If X is a Banach space and M is a subspace of X^*, let $K_X(M)$ be the set of all functionals in X^* which are weak*-limits of sequences in M. If $f \in K_X(M)$, let

$$\varphi_M(f) = \inf\{\sup_n \|f_n\| : \{f_n\} \subset M,\ \text{w*-lim} f_n = f\}.$$

In [4], McWilliams showed that φ_M is a norm on $K_X(M)$ and that $K_X(M)$ is closed in the norm-topology of X^* if and only if there exists a real number C such that $\varphi_M(f) \leqslant C\|f\|$ for all $f \in K_X(M)$. Now let

$$\mu(M) = \inf\{C: \varphi_M(f) \leqslant C\|f\| \text{ for all } f \in K_X(M)\}.$$

Thus $1 \leqslant \mu(M) \leqslant \infty$.

* Supported in part by National Science Foundation Grant GP-2179.

THEOREM 1. *If X is a separable Banach space and M is a subspace of X^*, then*

$$(2.1)\qquad \begin{aligned} \nu(M) &\leqslant \frac{1}{\mu(M)} \quad \text{if} \quad \mu(M) < \infty. \\ \nu(M) &= 0 \quad \text{if} \quad \mu(M) = \infty. \end{aligned}$$

Proof. Suppose $\mu(M) < \infty$, and suppose r is a real number such that $M \cap U_1$ is w*-dense in U_r. Let $\varepsilon > 0$ be given. There exists a non-zero $f \in X^*$ such that

$$\varphi_M(f) \geqslant (\mu(M) - \varepsilon)\|f\|.$$

If $g = rf/\|f\|$, then $g \in U_r$ and since $M \cap U_1$ is w*-dense in U_r and X is separable, it follows that $K_X(M \cap U_1)$ contains U_r. Hence

$$1 \geqslant \varphi_M(g) = \frac{r}{\|f\|}\varphi_M(f) \geqslant r(\mu(M) - \varepsilon).$$

Since ε was arbitrary, it follows that $1/\mu(M) \geqslant r$. Therefore, since $\nu(M)$ is the least upper bound of all such numbers r, we must have $\nu(M) \leqslant 1/\mu(M)$.

Now if $\mu(M) = \infty$ and r is a positive real number, there exists a non-zero $f \in K_X(M)$ such that $\varphi_M(f) > \|f\|/r$. Then if $g = rf/\|f\|$, it follows that $g \in U_r$ but g is not in the w*-closure of $M \cap U_1$. Since r was arbitrary, it follows that $\nu(M) = 0$.

Without the assumption of separability, (2.1) need not be satisfied. McWilliams [5] has exhibited a Banach space X (with non-separable conjugate X^*) such that $K_{X^*}(K_{X^*}(J_X X))$ (where J_X denotes the canonical mapping of X into X^{**}) is not norm-closed. Thus $\mu(K_{X^*}(J_X X)) = \infty$ although $\nu(K_{X^*}(J_X X)) = 1$.

The next theorem deals with an inequality in the opposite direction from that in Theorem 1. Following Banach ([1], p. 213), if M is a subspace of X^*, let $K_0(M) = M$, and if β is any ordinal, let

$$K_\beta(M) = \begin{cases} K_X(K_{\beta-1}(M)) & \text{if } \beta \text{ is non-limiting}, \\ \bigcup_{\alpha<\beta} K_\alpha(M) & \text{if } \beta \text{ is limiting}. \end{cases}$$

To simplify notation, let $\mu(K_\beta(M)) = \mu(\beta)$. It is not necessary to consider any ordinal beyond the first uncountable ordinal Ω since $K_\Omega(M)$ is weak*-sequentially closed.

Now for each ordinal $\beta \leqslant \Omega$, let a non-negative real number D_β be defined as follows. Let $D_1 = 1$ and if D_α has been defined for each ordinal $\alpha < \beta$, let

$$D_\beta = \begin{cases} D_{\beta-1}\cdot\dfrac{1}{\mu(\beta-1)} & \text{if } \beta \text{ is non limiting and } \mu(\beta-1) \neq \infty, \\ 0 \text{ if } \beta \text{ is non-limiting and } \mu(\beta-1) - \infty, \\ \inf_{\alpha<\beta} D_\alpha \text{ if } \beta \text{ is limiting.} \end{cases}$$

Note that $D_\alpha \geqslant D_\beta$ if $\alpha \leqslant \beta$.

THEOREM 2. *If X is any Banach space and M is a subspace of X^* such that $K_\beta(M) = X^*$ for some ordinal β, then*

(a) $\nu(M) \geqslant D_\beta\cdot\dfrac{1}{\mu(M)}$ *if* $\mu(M) < \infty$,

and

(b) $\mu(M) = \infty$ *if* $\nu(M) = 0$ *and* $D_\beta \neq 0$.

Proof. We first prove by transfinite induction that for each ordinal β,

(2.2) if $D_\beta > 0$, $\mu(M) < \infty$, and ε is a positive real number such that $0 < \varepsilon < D_\beta$, then $M \cap U_1$ is weak*-dense in

$$U_{\lambda(\beta,\varepsilon)} \cap K_\beta(M), \quad \text{where} \quad \lambda(\beta,\varepsilon) = (D_\beta - \varepsilon)\cdot\frac{1}{\mu(M)+\varepsilon}.$$

If $f \in K_X(M) \cap U_{\lambda(1,\varepsilon)}$, there exists a sequence $\{g_n\} \subset M$ such that w*-$\lim g_n = f$ and

$$\sup_n \|g_n\| \leqslant (\mu(M)+\varepsilon)\|f\| \leqslant 1-\varepsilon.$$

Hence $f \in \text{w}^*(M \cap U_1)$, and (2.2) holds for $\beta = 1$.

Now suppose (2.2) holds for all $\alpha < \beta$.

Case 1. Suppose β is non-limiting. If $f \in K_\beta(M) \cap U_{\lambda(\beta,\varepsilon)}$ there exists $\{g_n\} \subset K_{\beta-1}(M)$ such that w*-$\lim g_n = f$ and $\sup_n \|g_n\| \leqslant (\mu(\beta-1)+\delta)\|f\|$, where δ is a positive real number with the property that $(\mu(\beta-1)+\delta)(D_\beta - \varepsilon) = D_{\beta-1} - \varepsilon/2$. If V is an arbitrary neighborhood of f in the w*-topology, there exists a positive integer m such that $g_m \in V$. Since $g_m \in K_{\beta-1}(M) \cap U_{\lambda(\beta-1,\varepsilon/2)}$, there exists, by the induction hypothesis, a $g \in M \cap U_1$ such that $g \in V$. Hence $f \in \text{w}^*(M \cap U_1)$.

Case 2. Suppose β is limiting. If $f \in K_\beta(M) \cap U_{\lambda(\beta,\varepsilon)}$, then $f \in \bigcup_{\alpha<\beta} K_\alpha(M)$ and hence there exists $\alpha < \beta$ such that $f \in K_\alpha(M) \cap U_{\lambda(\beta,\varepsilon)}$. Since $D_\beta \leqslant D_\alpha$, it follows that $f \in K_\alpha(M) \cap U_{\lambda(\alpha,\varepsilon)}$. Now if V is an arbitrary neighborhood of f in the w*-topology, it follows from the induction hypothesis that there exists $g \in M \cap U_1$ such that $g \in V$. Thus $f \in \text{w}^*(M \cap U_1)$.

Thus if (2.2) holds for all $\alpha < \beta$, then (2.2) holds for β as well. We now have that (2.2) holds for all ordinals β by the principle of transfinite induction.

Now if $D_\beta \neq 0$, $\mu(M) < \infty$, and $K_\beta(M) = X^*$, it follows from (2.2) and the definition of $\nu(M)$ that

$$\nu(M) \geqslant (D_\beta - \varepsilon)\frac{1}{\mu(M)+\varepsilon}$$

for each $\varepsilon > 0$. Hence

$$\nu(M) \geqslant D_\beta \frac{1}{\mu(M)}.$$

It is clear that if $\nu(M) = 0$ and $D_\beta \neq 0$, then $\mu(M)$ is necessarily infinite.

COROLLARY 1. *If $K_X(M) = X^*$, then $\nu(M) \geqslant 1/\mu(M)$.*

COROLLARY 2. *If $K_X(M) = X^*$ and X is separable, then $\nu(M) = 1/\mu(M)$. If X is separable and $K_X(M) \subset K_2(M) = X^*$, then $\mu(M) = \infty$.*

Proof. The first statement follows from Theorem 1 and Corollary 1. The second statement follows from the fact that $\nu(M) = 0$ if $K_X(M) \neq X^*$ ([1], p. 213) and Theorem 2.

If M is a subspace of X^* for which $K_\beta(M) = X^*$ for some ordinal β, then M is necessarily w*-dense in X^*. We now prove a partial converse to this statement.

THEOREM 3. *If X is separable and M is a subspace of X^*, then M is w*-dense in X^* only if $K_\beta(M) = X^*$ for some ordinal β.*

Proof. Suppose M is a w*-dense subspace of X^*. Then $K_\beta(M)$ is w*-sequentially closed for some ordinal $\beta \leqslant \Omega$. (Actually we may choose $\beta < \Omega$; [1], p. 213). Since X is separable and $K_\beta(M)$ is w*-sequentially closed, we must have $w^*\big(K_\beta(M)\big) = K_\beta(M)$ ([2], Thm. 3, p. 1060). Hence, since M is w*-dense, we have $X^* = w^*(M) \subset w^*\big(K_\beta(M)\big) = K_\beta(M) \subset X^*$, and the proof is complete.

Theorem 3 need not hold if X is not separable. For example, if we regard l^1 as a subspace of $(m)^* = (l^1)^{**}$, then $K_{(m)}(l^1) = l^1$ but $w^*(l^1) = (m)^*$.

If X is separable and $\mu(M) = 1$ for every subspace M of X^*, then it follows from Theorem 3 and Theorem 2 that $\nu(M) = 1$ for every w*-dense subspace M of X^* and hence that X is reflexive [7]. This is related to a question asked in [3], and is a special case of a theorem proved recently in [6] by a different approach.

3. We now make observations about a theorem of Banach concerning iterated w*-sequential closure. Banach's Theorem ([1], p. 209) is as follows:

If $X = (c_0)$, then for every positive integer n there is a subspace M_n of X^* such that $K_n(M_n)$ is not w-sequentially closed.*

Remark 1. *In the statement of Banach's Theorem, each M_n may be taken to be w*-dense in X^*.*

Proof. Since $X = (c_0)$, X^* is isometrically isomorphic to l^1. For each pair r, s of positive integers, let $p_{rs} = (c_i)$ where

$$c_i = \begin{cases} 1 & \text{if} \quad i = 2^r(2s-1), \\ 0 & \text{otherwise}. \end{cases}$$

If $f = (b_i) \in l^1$, let $Tf = (d_i)$ be the element of l^1 defined by

$$d_i = \begin{cases} 0 & \text{if} \quad i = 2j \text{ for some } j, \\ b_j & \text{if} \quad i = 2j-1. \end{cases}$$

Then T is an isometric isomorphism of l^1 onto a proper subspace of l^1. If G is a subspace of l^1, let $\{h_r\}$ be a sequence in the unit ball H_1 of $T(G)$ such that $\{h_r\}$ is norm-dense in H_1. For each pair r, s of positive integers let $f_{rs} = h_r + rp_{rs}$ and let S_G be the subspace spanned by $\{f_{rs}: r, s$ positive integers$\}$. Banach proved that if $K_{n-1}(G)$ is not w*-sequentially closed, then $K_n(S_G)$ is not w*-sequentially closed. There exists a subspace G of l^1 such that $K_1(G) \neq G$ and G is w*-dense (for example, the subspace of l^1 spanned by the elements $(1, 1, 0, 0, \ldots)$, $(1, 0, 1, 0, \ldots)$, etc.); thus let $M_0 = G$ and for each positive integer n let $M_n = S_{M_{n-1}}$. The conclusion of Banach's Theorem now follows by induction.

Since a subspace M of X^* is w*-dense if and only if it is total (that is, if $x \in X$ with $x \neq 0$, there exists $f \in M$ such that $f(x) \neq 0$), to establish the remark it suffices to show that if G is total, then S_G is total. Suppose $x = (a_i)$ is a non-zero element of (c_0). Either $a_{2k} = 0$ for all positive integers k, or there exists an even positive integer i such that $a_i \neq 0$. In each case it will be shown that if G is total, then there exists $f \in S_G$ such that $f(x) \neq 0$.

Case 1. Suppose there exists an even positive integer i such that $a_i \neq 0$. Since i is even, there exist positive integers r and s such that $i = 2^r(2s-1)$. Since $\lim_j a_j = 0$, there exists a positive integer n such that if $j > n$, then $|a_j| < |a_i|/2$. Choose m so that $2^r(2m-1) > n$ and let $j = 2^r(2m-1)$. If $f = f_{rs} - f_{rm}$, then $f \in S_G$ and

$$|f(x)| = |ra_i - ra_j| > r|a_i| - \frac{r|a_i|}{2} > 0.$$

Case 2. Suppose that for each k, $a_{2k} = 0$. Let $x_0 = (b_i)$ where $b_i = a_{2i-1}$ for each i. Then $x_0 \in (c_0)$ and $x_0 \neq 0$ since $x \neq 0$. If G is total, there exists $g \in G$ with $\|g\| \leqslant 1$ such that $g(x) \neq 0$. Then $Tg \in H_1$ and $Tg(x) = g(x_0) \neq 0$. Since $\{h_r\}$ is dense in H_1, there exists a positive integer r such that $\|h_r - Tg\| < |Tg(x)|/2\,\|x\|$, and hence

$$0 < |Tg(x)| - \frac{|Tg(x)|}{2} \leqslant |h_r(x)|.$$

If $f = h_r + rp_{r1}$, then $f \in S_G$ and $|f(x)| = |h_r(x) + rp_{r1}(x)| = |h_r(x)| > 0$.

THEOREM 4. *There exists a separable Banach space X whose conjugate X^* contains a subspace M such that M is w^*-dense in X^* and such that for each positive integer n, $K_n(M) \neq K_{n+1}(M)$.*

Proof. Let $\{S_k\}_{k=1}^{\infty}$ be the collection of disjoint sets of positive integers for which $S_k = \{n : n = 2^{k-1}(2j-1)\}$. For each positive integer n, let T_n be the transformation on l^1 defined so that if $f = (b_i) \in l^1$, then $T_n f = (c_i)$ where

$$c_i = \begin{cases} 0 & \text{if} \quad i \notin S_n, \\ b_j & \text{if} \quad i = 2^{n-1}(2j-1). \end{cases}$$

Then T_n is, for each n, an isometric isomorphism of l^1 onto a proper subset of itself.

Let $X = (c_0)$. By Banach's Theorem and Remark 1, there exists, for each n, a w*-dense subspace M_n of $X^* = l^1$ for which $K_{n+1}(M_n) \neq K_n(M_n)$. Now let M be the subspace of X^* spanned by $\bigcup_{n=1}^{\infty} T_n(M_n)$.

If $x = (a_i)$ is a non-zero element of (c_0), then there exists a pair k, h of positive integers such that $a_p \neq 0$ where $p = 2^{k-1}(2h-1)$. Let $x_k = (b_{kj})_{j=1}^{\infty}$ where $b_{kj} = a_t$, $t = 2^{k-1}(2j-1)$. Then $x_k \neq 0$ and since M_k is w*-dense, there exists $f_k \in M_k$ such that $f_k(x_k) \neq 0$. Since $T_k f_k \in M$ and $T_k f_k(x) = f_k(x_k) \neq 0$, it follows that M is total and hence w*-dense in X^*. It is now a routine matter to show that $K_{n+1}(M) \neq K_n(M)$ for each positive integer n.

Banach remarked ([1], p. 213) that for each ordinal $\alpha < \Omega$, there exists a subspace M_α of l^1 such that $K_\alpha(M_\alpha)$ is not w*-sequentially closed, but that if X is any separable space and M a subspace of X^*, then $K_\alpha(M)$ is w*-sequentially closed for some $\alpha < \Omega$. It follows from this that the words "for each positive integer n" cannot be replaced by the words "for each ordinal $\alpha < \Omega$" in the statement of Theorem 4.

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Reçu par la Rédaction le 15. 7. 1965