ainsi $h(x) = g(x) + if(x)$ est analytique dans le disque $|x| < 1$, soit

$$h(x) = \log a + \sum_{k=1}^{\infty} h_k x^k.$$ 

Pour $t \in E$, $g(\rho e^{it})$ tend uniformément, quand $r \to 1$, vers une limite $\geq 0$. Pour $N$ assez grand, on a donc

$$\mathcal{A} \left( \sum_{n=1}^{N} h_n \left( 1 - \frac{n}{N} \right) e^{i t} \right) \geq \frac{1}{2} \log \frac{1}{a} (t \in E).$$

Posons enfin

$$q(t) = \mathcal{A} \left( \sum_{n=1}^{N} h_n \left( 1 - \frac{n}{N} \right) e^{i t} \right) = \sigma f(e^{it}, t),$$

$$p(t) = \frac{2}{\pi} e^{-\pi t} (t).$$

On a bien, d'après (1)

$$\|p(t)\| < 1,$$

et

$$S_\mu(p, t) = \frac{2}{\pi} e^{-\pi t} \sum_{n=1}^{N} h_n \left( 1 - \frac{n}{N} \right) e^{i n t},$$

donc d'après (2)

$$|S_\mu(p, t)| \geq \frac{1}{\pi} \log \frac{1}{a} \text{ pour } t \in E,$$

et le lemme est démontré.

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Weak*-sequential closure and the characteristic of subspaces of conjugate Banach spaces

by

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1. The notion of characteristic of a subspace of a conjugate Banach space is due to Dixmier [2] and has been studied recently by Petunin [7], [8]. If $X$ is a Banach space and $M$ is a subspace of $X^*$, and $U_r = \{ f \in X^* : \| f \| < r \}$, then the characteristic $\nu(M)$ of $M$ is the least upper bound of all real numbers $r$ such that $M \cap U_r$ is $w^*$-dense in $U_r$. In [3] there was defined a number $\mu(M)$ (denoted there by $C_M$) depending on the $w^*$-sequential closure $K_X(M)$ of $M$. The purpose of the present paper is to establish certain relationships between $\mu(M)$ and $\nu(M)$. In particular, it is shown that if $K_X(M) = X^*$ and $X$ is separable, then $\nu(M) = 1/\mu(M)$. Also there is given a strengthening of a result of Banach concerning iterated $w^*$-sequential closure.

The results in this paper are contained in the author's dissertation written at Florida State University under the direction of Professor R. D. McWilliams, but the proofs have been somewhat simplified.

2. If $X$ is a Banach space and $M$ is a subspace of $X^*$, let $K_X(M)$ be the set of all functionals in $X^*$ which are weak*-limits of sequences in $M$. If $f \in K_X(M)$, let

$$\varphi_M(f) = \inf \left\{ \sup_{n} \| f_n \| : \{ f_n \} \subset M, w^*-\lim f_n = f \right\}.$$ 

In [4], McWilliams showed that $\varphi_M$ is a norm on $K_X(M)$ and that $K_X(M)$ is closed in the norm-topology of $X^*$ if and only if there exists a real number $C$ such that $\varphi_M(f) \leq C \| f \|$ for all $f \in K_X(M)$. Now let

$$\mu(M) = \inf \{ C : \varphi_M(f) \leq C \| f \| \text{ for all } f \in K_X(M) \}.$$ 

Thus $1 \leq \mu(M) \leq \infty$.

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Theorem 1. If $X$ is a separable Banach space and $M$ is a subspace of $X^*$, then

$$v(M) \leq \frac{1}{\mu(M)}$$

if $\mu(M) < \infty$.

(2.1)

$$v(M) = 0$$

if $\mu(M) = \infty$.

Proof. Suppose $\mu(M) < \infty$, and suppose $r$ is a real number such that $M \cap U_r$ is $w^*$-dense in $U_r$. Let $\varepsilon > 0$ be given. There exists a non-zero $f \in X^*$ such that

$$\varphi_M(f) \geq \frac{1}{\mu(M)} v(M) - \varepsilon.$$ 

If $g = rf/\|f\|$, then $g \in U_r$ and since $M \cap U_1$ is $w^*$-dense in $U_r$ and $X$ is separable, it follows that $K_X(M \cap U_1)$ contains $U_r$. Hence

$$1 \geq \varphi_M(g) = \frac{r}{\|f\|} \varphi_M(f) \geq r(\mu(M) - \varepsilon).$$

Since $\varepsilon$ was arbitrary, it follows that $1/\mu(M) \geq r$. Therefore, since $v(M)$ is the least upper bound of all such numbers $r$, we must have $v(M) \leq 1/\mu(M)$.

Now if $\mu(M) = \infty$ and $r$ is a positive real number, there exists a non-zero $f \in K_X(M)$ such that $\varphi_M(f) > r/\|f\|$. Then if $g = rf/\|f\|$, it follows that $g \in U_r$, but $g$ is not in the $w^*$-closure of $M \cap U_1$. Since $\varepsilon$ was arbitrary, it follows that $v(M) = 0$.

Without the assumption of separability, (2.1) need not be satisfied.

McWilliams [5] has exhibited a Banach space $X$ (with non-separable conjugate $X^*$) such that $K_X = K_{w^*}(X)$ (where $J_x$ denotes the canonical mapping of $X$ into $X^{**}$) is not norm-closed. Thus $\mu(J_x(J(X))) = \infty$ although $\mu(K_x(J_x(X))) = 1$.

The next theorem deals with an inequality in the opposite direction from that in Theorem 1. Following Bana (1), p. 213), if $M$ is a subspace of $X^*$, let $K_{\beta}(M) = M$, and if $\beta$ is any ordinal, let

$$K_{\beta}(M) = \begin{cases} 
K_X(M_{\beta-1}(M)) & \text{if } \beta \text{ is non-limiting}, \\
\bigcup_{\alpha < \beta} K_{\alpha}(M) & \text{if } \beta \text{ is limiting}.
\end{cases}$$

To simplify notation, let $\mu(K_{\beta}(M)) = \mu(\beta)$. It is not necessary to consider any ordinal beyond the first uncountable ordinal $\Omega$ since $K_{\Omega}(M)$ is weak$^*$-sequentially closed.

Now for each ordinal $\beta \leq \Omega$, let a non-negative real number $D_\beta$ be defined as follows. Let $D_1 = 1$ and if $D_\beta$ has been defined for each

ordinal $\alpha < \beta$, let

$$D_\beta = \begin{cases} 
D_{\beta-1}, & \text{if } \beta \text{ is non-limiting and } \mu(\beta-1) < \infty, \\
0, & \text{if } \beta \text{ is non-limiting and } \mu(\beta-1) = \infty,
\end{cases}$$

and

$$D_\beta = \mu(\beta-1) - D_{\beta-1}$$

$\mu(\beta-1)$. Note that $D_\beta \geq D_\beta$ if $\alpha < \beta$.

Theorem 2. If $X$ is any Banach space and $M$ is a subspace of $X^*$ such that $K_{\beta}(M) = X^*$ for some ordinal $\beta$, then

(a) $v(M) \geq D_\beta - \mu(M) < \infty$, and

(b) $\mu(M) = \infty$ if $v(M) = 0$ and $D_\beta < 0$.

Proof. We first prove by transfinite induction that for each ordinal $\beta$,

(2.2) if $D_\beta > 0$, $\mu(\beta) < \infty$, and $\varepsilon$ is a positive real number such that $0 < \varepsilon < D_\beta$, then $M \cap U_\varepsilon$ is $w^*$-dense in $U_{\mu(M)}$ and $\lambda(\beta, \varepsilon) = (D_\beta - \varepsilon) \frac{1}{\mu(M) + \varepsilon}$.

If $x \in K_X(M) \cap U_{\mu(M)}$, there exists a sequence $(g_n) \in M$ such that $w^*$-lim $g_n = f$ and

$$\sup \|g_n\| \leq (\mu(M) + \varepsilon) \|f\| \leq 1 - \varepsilon.$$ 

Hence $f \in w^*(M \cap U_\varepsilon)$, and (2.2) holds for $\varepsilon = 1$.

Now suppose (2.2) holds for all $\alpha < \beta$.

Case 1. Suppose $\beta$ is non-limiting. If $x \in K_X(M) \cap U_{\mu(M)}$ there exists $(g_n) \in M$ such that $w^*$-lim $g_n = f$ and

$$\sup \|g_n\| \leq (\mu(\beta-1) + \varepsilon) \|f\|,$$ 

where $\varepsilon$ is a positive real number with the property that $\mu(\beta-1) + \varepsilon$ is a positive real number such that $g_n \in V$. Since $g_n \in K_{\beta-1}(M) \cap U_{\mu(M) + \varepsilon}$, there exists, by the induction hypothesis, a $g \in M \cap U_1$ such that $g \in V$. Hence $f \in w^*(M \cap U_1)$.

Case 2. Suppose $\beta$ is limiting. If $x \in K_X(M) \cap U_{\mu(M)}$ then $f \in K_{\beta}(M)$ and hence there exists $\alpha < \beta$ such that $f \in K_{\alpha}(M) \cap U_{\mu(M)}$. Since $D_\alpha \leq D_\beta$, it follows that $f \in K_{\alpha}(M) \cap U_{\mu(M)}$. Now if $V$ is an arbitrary neighborhood of $f$ in the $w^*$-topology, there exists a positive integer $m$ such that $g_m \in V$. Since $g_m \in K_{\alpha-1}(M) \cap U_{\mu(M) + \varepsilon}$, there exists, by the induction hypothesis, a $g \in M \cap U_1$ such that $g \in V$. Hence $f \in w^*(M \cap U_1)$.

Thus if (2.2) holds for all $\alpha < \beta$, then (2.2) holds for $\beta$ as well. We now have that (2.2) holds for all ordinals $\beta$ by the principle of transfinite induction.
Now if $D_{\theta} \neq 0$, $\mu(M) < \infty$, and $K_{\beta}(M) = X^*$, it follows from (2.2) and the definition of $\nu(M)$ that

$$
\nu(M) \geq (D_{\theta} - \epsilon) \frac{1}{\mu(M)} + \epsilon
$$

for each $\epsilon > 0$. Hence

$$
\nu(M) \geq D_{\theta} \frac{1}{\mu(M)}.
$$

It is clear that if $\nu(M) = 0$ and $D_{\theta} \neq 0$, then $\mu(M)$ is necessarily infinite.

Corollary 1. If $K_X(M) = X^*$, then $\nu(M) \geq 1/\mu(M)$.

Corollary 2. If $K_X(M) = X^*$ and $X$ is separable, then $\nu(M) = 1/\mu(M)$, if $X$ is separable and $K_X(M) = K_{\beta}(M) = X^*$, then $\mu(M) = \infty$.

Proof. The first statement follows from Theorem 1 and Corollary 1. The second statement follows from the fact that $\nu(M) = 0$ if $K_X(M) \neq X^*$ (11), p. 213) and Theorem 2.

If $M$ is a subspace of $X^*$ for which $K_{\beta}(M) = X^*$ for some ordinal $\beta$, then $M$ is necessarily $\omega^*$-dense in $X^*$. We now prove a partial converse to this statement.

Theorem 3. If $X$ is separable and $M$ is a subspace of $X^*$, then $M$ is $\omega^*$-dense in $X^*$ only if $K_{\beta}(M)$ is a subspace of $X^*$ for some ordinal $\beta$.

Proof. Suppose $M$ is a $\omega^*$-dense subspace of $X^*$. Then $K_{\beta}(M)$ is $\omega^*$-sequentially closed for some ordinal $\beta$. (Actually we may choose $\beta = \alpha_1$ [1], p. 213). Since $X$ is separable and $K_{\beta}(M)$ is $\omega^*$-sequentially closed, we must have $\omega^*(K_{\beta}(M)) = K_{\beta}(M)$. Then $\omega^*(K_{\beta}(M)) = K_{\beta}(M)$, and the proof is complete.

Theorem 3 need not hold if $X$ is not separable. For example, if we regard $\ell_2$ as a subspace of $\ell_2$, then $\omega^*(\ell_2) = \ell_2$, but $\omega^*(\ell_2) = (\ell_2)^*$. Hence, we change $\omega^*(\ell_2)$ to be a subspace of $\ell_2$.

Then $\omega^*(\ell_2)$ is not $\omega^*$-sequentially closed for some ordinal $\beta$. (Actually we may choose $\beta = \alpha_1$ [1], p. 213). Since $X$ is separable and $K_{\beta}(M)$ is $\omega^*$-sequentially closed, we must have $\omega^*(K_{\beta}(M)) = K_{\beta}(M)$). Then $\omega^*(K_{\beta}(M)) = K_{\beta}(M)$, and the proof is complete.

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teger r such that \(|h_r - Tg| < |Tg(x)|/2\|w\|\), and hence

\[0 < |Tg(x)| - |Tg(x)|/2 \leq |h_r(x)|.\]

If \(f = h_r + r \eta_1\), then \(f \in S_2\) and \([f] = |h_r(x) + r \eta_1(x)| = |h_r(x)| > 0\).

**Theorem 4.** There exists a separable Banach space \(X\) whose conjugate \(X^*\) contains a subspace \(M\) such that \(M\) is \(w^*\)-dense in \(X^*\) and such that for each positive integer \(n\), \(K_n(M) \neq K_{n+1}(M)\).

**Proof.** Let \(S_2 = \{n : n = 2^{n-1}(2^j - 1)\}\). For each positive integer \(n\), let \(T_n\) be the transformation on \(P\) defined so that if \(f = (h_k) \in P\), then \(T_n f = (h_i)\) where

\[h_i = \begin{cases} 0 & \text{if } i \notin S_n, \\ h_i & \text{if } i = 2^n - 1(2^j - 1). \end{cases}\]

Then \(T_n\) is, for each \(n\), an isometric isomorphism of \(P\) onto a proper subset of itself.

Let \(X = (e_n)\). By Banach's Theorem and Remark 1, there exists, for each \(n\), a \(w^*\)-dense subspace \(M_n\) of \(X^* = P\) for which \(K_{n+1}(M_n) \neq K_n(M_n)\). Let \(M\) be the subspace of \(X^*\) spanned by \(\bigcup_{n=1}^\infty T_n(M_n)\).

If \(x = (e_n)\) is a non-zero element of \((e_n)\), then there exists a pair \(k, h\) of positive integers such that \(a_k \neq 0\) where \(p = 2^{h-1}(2^k - 1)\). Let \(x_k = (b_{kj})_{n_j}\) where \(b_{kj} = a_k, t = 2^{h-1}(2^k - 1)\). Then \(x_k \neq 0\) and since \(M_k\) is \(w^*\)-dense, there exists \(f_k \in M_k\) such that \(f_k(x_k) \neq 0\). Since \(T_n f_k \in M\) and \(T_n f_k(x) = f_k(x) \neq 0\), it follows that \(M\) is total and hence \(w^*\)-dense in \(X\). It is now a routine matter to show that \(K_{n+1}(M) \neq K_n(M)\) for each positive integer \(n\).

Banach remarked ([11], p. 213) that for each ordinal \(\alpha < \Omega\), there exists a subspace \(M_\alpha\) of \(P\) such that \(K_\alpha(M_\alpha)\) is not \(w^*\)-sequentially closed, but that if \(X\) is any separable space and \(M\) a subspace of \(X^*\), then \(K_\alpha(M)\) is \(w^*\)-sequentially closed for some \(\alpha < \Omega\). It follows from this that the words "for each positive integer \(n\)" cannot be replaced by the words "for each ordinal \(\alpha < \Omega\)" in the statement of Theorem 4.

**References**