есть r-мерной в O. Отсюда и из теоремы 3.2 следует, что

$$a_1(\chi)\chi_1(x),\ \psi_1(x)\varphi_1(x)\chi_1(x),\ a_2(x)\chi_2(x),\ \ldots$$

есть основной последовательностью в O и поэтому

$$a_n(x)\chi_n(x) \sim \psi_n(x)\varphi_n(x)\chi_n(x).$$

Подобным образом доказывается, что $\psi_n(x)\beta_n(x) \sim \psi_n(x)\varphi_n(x)\chi_n(x)$. На основании транзитивности $a_n(x)\chi_n(x) \sim \psi_n(x)\beta_n(x)$; это дает равенство (6.4), что и требовалось доказать.

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On semi-groups of contractions in Hilbert spaces

by

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Suppose we are given a complex Hilbert space H. Let $f, g, h, \ldots$ stand for vectors of H and $\alpha, \beta, \gamma, \ldots$ for complex scalars. (f, g) is the inner product of f and g, $|f|$ is the norm of f. By $|V|$ we understand the norm of the linear bounded operator in H. V^* stands for the adjoint of V and I for the identity operator in H. By $V|Z$ we mean the restriction of the operator V to the subset $Z \subset H$. A contraction is a linear bounded operator V in H such that $|V| \leqslant 1$.

Let G be an abelian group. The inner group operations in G are written additively. Suppose that the semi-group G_+ orders G, that is

(i) $$G_+ \cup (-G_+) = G,$$

(ii) $$G_+ \cap (-G_+) = \{0\}.$$

We write $\xi \leqslant \eta$ if $\eta - \xi \in G_+$ and $\xi < \eta$ if $\xi \leqslant \eta$ but $\xi \neq \eta$.

A contraction valued function $T(\xi)$ determined for $\xi \in G_+$ is called a *semi-group of contractions* (s.g.c. for brevity) if

(iii) $$T(0) = I, \quad T(\xi+\eta) = T(\xi)T(\eta) \quad \text{for} \quad \xi, \eta \in G_+.$$

Let $U_\xi : K \to K$ be a unitary representation of G into the Hilbert space K and assume that $H \subset K$. Write P for the orthogonal projection of K onto H. We say that U_ξ is a *unitary dilation* [7] of the s.g.c. $T(\xi)$ if

(iv) $$T(\xi)f = PU_\xi f \quad \text{for} \quad f \in H \text{ and } \xi \in G_+.$$

The minimality condition $K = \bigvee_{\xi \in G} U_\xi H$ (1) determines U_ξ and K uniquely up to a unitary isomorphism. U_ξ is called then the *minimal unitary dilation* of s.g.c. $T(\xi)$.

A few examples are now in order.

Example 1. Let T be a contraction and $G = N$ — the additive group of integers. Then $T(n) = T^n$ ($T^0 = I$ by convention) for $n \geqslant 0$ is an s.g.c. G_+ is the set of non-negative integers.

(1) $\bigvee S_a$ stands for the closed linear span of the union of S_a.

EXAMPLE 2. Let $G = R$ the additive group of reals and let G_+ stand for the set of all non-negative reals. Any one-parameter semi-group $T(\xi)$ of contractions such that $T(h) \underset{h\to 0+}{\dashrightarrow} I$ weakly, may be extended to a strongly continuous s.g.c. if we put $T(0) = I$.

EXAMPLE 3. Let $a_1, \ldots, a_n$ be rationally independent real numbers. Consider the additive subgroup G of reals of the form $\xi = \sum_{k|1}^{n} n_k a_k$, with integer valued factors n_k. We define

$$G_+ = \{\xi \mid \xi = \sum_{k|1}^{n} n_k a_k \geqslant 0\}.$$

Suppose τ^n is the n-dimensional torus and let μ be a finite, positive Borel measure on τ^n. Let $H^2(\mu)$ be the closed linear $L^2(\mu)$ span of monomials

$$e^{i\langle\xi,x\rangle} = \exp\Big(i\sum_{k|1}^{n} n_k a_k x_k\Big) \qquad (0 \leqslant x_k < 2\pi)$$

where $\xi \in G_+$. Then the formula $\big(T(\xi)f\big)(x) = e^{i\langle\xi,x\rangle}f(x)$ determines a semi-group of isometries in $H^2(\mu)$.

B. Sz.-Nagy proved in [7] that semi-groups of Example 1 and of Example 2 do have unitary dilations. These results were extended in [6] to s.g.c. as considered in the present paper, that is to contraction valued representations of the semi-group which orders the group in question.

Assume now that G is a locally compact group. If the s.g.c. is weakly continuous in the topology of G, then there is a unique regular operator positive measure F, defined for Borel subsets of the dual group $\hat{G}$ and such that

$$\big(T(\xi)f, g\big) = \int_{\hat{G}} \xi(\lambda)\, d\big(F(\lambda)f, g\big), \quad f, g \in H, \quad \xi \geqslant 0.$$

$\xi(\lambda)$ is the continuous character of $\hat{G}$, corresponding to $\xi \in G_+$. In fact, the measure F is of the form $F = PE|H$, where E is the spectral measure of the minimal unitary dilation of $T(\xi)$.

Since a non-discrete locally compact ordered group is a direct sum of the usual group of reals and an ordered discrete group (see [9], p. 196), at the first step of investigations we restrict ourselves in the study of weakly continuous s.g.c. to the continuous case (Example 2) and separately to the discrete one. Our contribute to the continuous case is section 2 of the present paper. We prove there a certain prediction theoretic property of the measure F. The rest of the paper deals with archimedean ordered G, not isomorphic to the group of integers. This excludes s.g.c. of Example 1. Since G is archimedean ordered, it is isomorphic to a dense additive subgroup of reals. The group G of Example 3 is of this type.

1. Let G be still an arbitrary abelian ordered group and let U_ξ be the minimal unitary dilation of s.g.c. $T(\xi)$. The space H may be decomposed uniquely in the form $H = H_1 \oplus H_2$ in such a way that H_1 and H_2 are reducing subspaces for $T(\xi)$ and the following holds true:

(*) The function T_ξ defined by

$$T_\xi = \begin{cases} T(\xi) & \text{if} \quad \xi \in G_+, \\ T^*(-\xi) & \text{if} \quad \xi \in (-G_+), \end{cases}$$

when restricted to H_1 is a unitary representation of G,

(**) For every $f \in H_2, f \neq 0$

$$\inf_\xi |T_\xi f| < |f| \text{ (}^2\text{)}.$$

The above decomposition of H and $T(\xi)$ respectively is called a *canonical* one. The s.g.c. is called *completely non-unitary* if $H_1 = \{0\}$. H_2 is called the *completely non-unitary part* of the canonical decomposition. The space H_1 is characterized by

$$H_1 = \bigcap_{\xi \in G} U_\xi H = \{f \mid |T_\xi f| = |f| \text{ for all } \xi \in G\}.$$

U_ξ stands as usually for the minimal unitary dilation of $T(\xi)$. We say that $T(\xi)$ is *non-unitary* if $H_2 \neq \{0\}$.

Let S be an arbitrary linear subset of H and write

$$M_\pm(S) = \bigvee_{\xi>0} U_{\pm\xi} S, \qquad R_\pm(S) = \bigcap_{\xi>0} U_{\pm\xi} M_\pm(S).$$

If S is a one-dimensional space spanned by $f \neq 0$, we put f in place of S in the above notation.

It was proved in [6] that

(1.0) $$R_+(H) \cap R_-(H) = \bigcap_{\xi \in G} U_\xi H.$$

Assume now additionally that G is locally compact and $T(\xi)$ is a weakly continuous semi-group of contractions. Let E stand for the spectral measure of the minimal unitary dilation U_ξ of $T(\xi)$. We will consider the space $M_+(f) \vee \{f\} \vee M_-(f) = M(f)$. The correspondence

$$\sum a_i U_{\xi_i} f \longleftrightarrow \sum a_i \xi_i(\lambda)$$

(²) For references, see [8], IV.

may be extended to the unitary equivalence between $M(f)$ and $L^2(\mu_f)$ where $\mu_f(\sigma) = (E(\sigma)f,f)$. The restriction $U_\xi|M(f)$ is equivalent to multiplication by $\xi(\lambda)$ in $L^2(\mu_f)$. The copies of reducing spaces $R_+(f)$, $R_-(f)$ in $L^2(\mu_f)$ consist of functions vanishing outside suitable measurable sets. It follows then easily from the definitions of $R_+(f)$ and $R_-(f)$ that this sets differ at most by a set of measure zero. Hence

$$R_+(f) = R_-(f). \tag{1.1}$$

It results now by (1.0) that if $f \in H_2$, then

$$R_+(f) = R_-(f) = \{0\}. \tag{1.2}$$

2. Assume now that $G = R$ — the additive group of reals with usual topology and usual order. The weakly continuous s.g.c. $T(\xi)$ is that of Example 2 and satisfies

$$T(h) \underset{h\to 0+}{\longrightarrow} I \text{ strongly}. \tag{2.0}$$

An s.g.c. which satisfies (2.0) is called *of class* (C_0). The spectral measure E of the minimal unitary dilation of s.g.c. of class (C_0) is determined on Borel subsets of R. U_ξ when restricted to $M(f)$ may be regarded as a mean continuous weakly stationary stochastic process with supporting measure $\mu_f(\sigma) = (E(\sigma)f,f)$. Suppose that $f \in H_2$. Then, by (1.2), the remote past of this process is trivial. In other words, the corresponding process is purely non-deterministic. It is a classical result of prediction theory (see [4]) that the supporting measure μ of purely non-deterministic process is absolutely continuous with respect to the linear Lebesgue measure on line and

$$\int_{-\infty}^{+\infty} \frac{\log \frac{d\mu}{d\lambda}}{1+\lambda^2}\, d\lambda$$

is finite. This, when applied to $\mu = \mu_f$ shows that the following theorem holds true:

THEOREM 1. *Let $f \neq 0$ belong to the completely non-unitary part of the canonical decomposition corresponding to the one-parameter s.g.c. $T(\xi)$ of class (C_0). Suppose E is the spectral measure of the minimal unitary dilation of $T(\xi)$. Then*

$$\int_{-\infty}^{+\infty} \frac{\log \frac{d(E(\lambda)f,f)}{d\lambda}}{1+\lambda^2}\, d\lambda$$

is finite.

Remark. The above theorem is a continuous version of a result of [5].

3. Since now we assume that G is an everywhere dense subgroup of reals, G_+ is identified with the set of all non-negative reals in G. We regard, G as a discrete topological group. We will study s.g.c. for such groups.

To begin with, we note that zero is an (in ordinary topology) accumulation point of G_+. Consider the subset H' of H determined by

$$H' = \{f \mid \lim_{\substack{\xi\to 0+\\ \xi\in G}} (T(\xi)f,f) = (f,f)\}.$$

It is a simple matter to check that $f \in H'$ iff $\lim_{h\to 0+} |T(h)f-f| = 0$. Consequently, H' is a closed linear subspace of H. Moreover, H' reduces all of $T(\xi)$. Since G_+ is dense in the positive halfline and $|T(\xi)| \leqslant 1$, $\lim_{\xi\to\xi_0} T(\xi)f$ exists for every $\xi_0 \geqslant 0$ and $f \in H'$. It follows that $T(\xi)|H'$ may be extended uniquely to one parameter s.g.c. of class (C_0). Note that H' is the largest subspace which does have this property. In order to have a "purely discrete" case we will assume in the sequel without stating it explicitely that for every $f \in H, f \neq 0$

$$\limsup_{h\to 0+} |T(h)f-f| > 0.$$

Let U_ξ stand for the minimal unitary dilation of $T(\xi)$ and let K be the corresponding dilation space. There is no difficulty to show that

$$K = \bigvee_{\xi>0} (U_\xi - T(\xi))H \oplus H \oplus \bigvee_{\xi>0} (U_{-\xi} - T^*(\xi))H$$

and

$$\bigvee_{\xi\geqslant 0} U_\xi H = H \oplus \bigvee_{\xi>0} (U_\xi - T(\xi))H, \quad \bigvee_{\xi\geqslant 0} U_{-\xi} H = H \oplus \bigvee_{\xi>0} (U_{-\xi} - T^*(\xi))H.$$

The other type decomposition formulae are that of Wold type [4]. We notice first that $R_\pm(H)$ reduces U_ξ. It results from (1.0) that if $T(\xi)$ is non-unitary, then at least one of the spaces $R_+(H), R_-(H)$ is not the whole K. Define $Q_0^\pm$ as the orthogonal projection onto $M_\pm(H)$ and consider the subspace $D_\pm = \overline{(I-Q_0^\pm)H}$. Since $M_+(H)$ $(M_-(H))$ is invariant with respect to $U_\xi(U_{-\xi})$ for $\xi \geqslant 0$, we have $U_\xi D_\pm \perp U_\eta D_\pm$ for $\xi \neq \eta$. We put $L_\pm(H) = \bigoplus_G U_\xi D_\pm$. Obviously $L_\pm(H) \perp R_\pm(H)$. Write now

$$S_\pm(H) = K \ominus (R_\pm(H) \oplus L_\pm(H)).$$

$S_\pm(H)$ need not reduce to a zero subspace, which is the case for classical Wold decomposition. This is the result of Helson and Lowdenslager, who constructed in [2], II, a suitable example. Finally we have

$$K = R_\pm(H) \oplus L_\pm(H) \oplus S_\pm(H). \tag{3.0}$$

Let Q_ξ be the projection onto $U_\xi M_+(H)$. It was proved in [6] that

(3.1) $$|Q_\xi f| \leqslant |T^*(\xi)f| \quad \text{for} \quad \xi > 0, f \in H.$$

Suppose now that $\xi < \eta$ $(\xi, \eta \in G)$ and put

$$\varrho(\xi, \eta) = |U_\xi T^*(\xi)f - U_\eta T^*(\eta)f|^2.$$

Then

$$\varrho(\xi, \eta) = |T^*(\xi)f|^2 + |T^*(\eta)f|^2 - 2\mathrm{Re}(U_{\eta-\xi}T^*(\xi)f, PT^*(\eta)f)$$
$$= |T^*(\xi)f|^2 - |T^*(\eta)f|^2.$$

Since $\lim_{\xi\to 0+} |T^*(\xi)f|^2$ exists, we conclude therefore that

$$\lim_{\xi\to 0+} U_\xi T^*(\xi)f \quad (\xi \in G_+)$$

exists for every $f \in H$. It follows $\lim_{\xi\to 0+} U_{-\xi}T(\xi)f$ exists for each $f \in H$.

Note now that $I - T(\xi)T^*(\xi) \leqslant I - T(\eta)T^*(\eta)$ for $\xi < \eta$ $(\xi, \eta \in G_+)$. Hence, the strong limit

$$\lim_{\xi\to 0+} \big(I - T(\xi)T^*(\xi)\big) = A_+$$

exists. A_+ is a positive operator and

(3.2) $$|f|^2 - \lim_{\xi\to 0+} |T^*(\xi)f|^2 = (A_+ f, f).$$

The operator A_- is defined by $\lim_{\xi\to 0+} \big(I - T^*(\xi)T(\xi)\big) = A_-$.

The innovation part $L_\pm(H)$ of (3.0) is characterized by the following lemma:

LEMMA 1. *For every* $f \in H$

$$Q_0^+ f = \lim_{\xi\to 0+} U_\xi T^*(\xi)f, \quad Q_0^- f = \lim_{\xi\to 0+} U_{-\xi}T(\xi)f \quad (\xi \in G).$$

Proof. It is clear that it suffices to consider merely the case of Q_0^+. Thus, let $f \in H$ and write

$$g = \lim_{\xi\to 0+} U_\xi T^*(\xi)f.$$

Since $U_\xi T^*(\xi)f \in M_+(H)$ for $\xi > 0$, we have $Q_0^+ g = g$. Hence

(3.3) $$|f-g|^2 = |(I - Q_0^+)f|^2 + |Q_0^+ f - g|^2.$$

On the other hand, by (3.1), $|Q_\xi f|^2 \leqslant |T^*(\xi)f|^2$ which together with formula

(3.4) $$|(I - U_\xi T^*(\xi)f|^2 = |f|^2 - |T^*(\xi)f|^2$$

implies

$$|f-g|^2 \leqslant |f|^2 - |Q_0^+ f|^2 = |(I - Q_0^+)f|^2.$$

It results now from (3.3) and (3.4) that $Q_0^+ f = g$ as was to be proved.

COROLLARY. *Note that*

(3.5) $$|f - Q_0^+ f|^2 = |A_+^{1/2} f|^2.$$

Consequently, the correspondence $f - Q_0^+ f \leftrightarrow A_+^{1/2} f$ *is isometric and may be extended uniquely to the unitary one between closures of subspaces* D_+ *and* $A_+^{1/2} H$ *respectively.*

Let $\mathscr{A}(G_+)$ be the algebra of functions, generated in the sup norm by polynomials $\sum_{i|1}^{n} a_i \xi_i(\lambda)$ $(n \geqslant 0)$ with $\xi_i \geqslant 0$. Suppose m is the normalized Haar measure of $\hat{G}$ and put

$$\mathscr{A}_m = \Big\{u \,|\, u \in \mathscr{A}(G_+),\ \int u\,dm = 0\Big\}.$$

Consider a non-negative regular measure determined on Borel sets of $\hat{G}$. A generalized version of Szegö theorem, proved in [2], I, is the following one:

(3.6) $$\inf_{u \in \mathscr{A}_m} \int |1-u|^2 d\mu = \inf_{u \in \mathscr{A}_m} \int |1-u|^2 w\,dm = \exp\Big(\int \log w\,dm\Big);$$

w stands here for the Radon-Nikodym derivative of μ with respect to m. If $\log w$ is not m summable over $\hat{G}$, then the right side of (3.6) should be interpreted as zero and the left sides are in fact zero. Take now $\mu_f(\sigma) = \big(E(\sigma)f, f\big)$ where E is the spectral measure of the minimal unitary dilation of $T(\xi)$. We write w_f for the Radon-Nikodym derivative of μ_f with respect to m.

THEOREM 2. *Suppose that* $f \in H$. *Then, if* $\max\big((A_+ f, f), (A_- f, f)\big)$ *is positive,*

$$\int \log w_f\,dm \geqslant \log \max\big((A_+ f, f), (A_- f, f)\big).$$

Proof. The unique continuous extension of the isomorphism

$$\sum a_i \xi_i(\lambda) \leftrightarrow \sum a_i U_{\xi_i} f$$

establishes the unitary equivalence between $M(f)$ and $L^2(\mu_f)$. Under this equivalence $1 \leftrightarrow f$ and $U_\xi | M(f)$ is interpreted as multiplication by $\xi(\lambda)$. It follows that

$$\varrho = \big(\mathrm{distance}(f, M_+(f))\big)^2 = \inf \int |1-u|^2 d\mu_f = \inf \int |1-\bar{u}|^2 d\mu$$
$$= \big(\mathrm{distance}(f, M_-(f))\big)^2,$$

where inf is taken over $u \in \mathscr{A}_m$. Since $M_\pm(f) \subset M_\pm(H)$, we have $|(I-Q_0^\pm)f|^2 \leqslant \varrho$ which together with (3.2), (3.4) and (3.6) proves the assertion of the theorem.

COROLLARY. *Using arguments similar to those used in the proof of theorem 2 one shows that if* $\log w_f$ *is not in* $L^1(m)$, *then* $A_+f = A_-f = 0$. *Hence, if* E *is singular with respect to* m, *then* $A_+ = A_- = 0$.

It is of some interest to point out some consequences of theorem 2. Suppose that

$$c_f^2|f|^2 = \min\big(\lim_{\xi\to 0+}|T(\xi)f|^2, \lim_{\xi\to 0+}|T_i^*(\xi)f|^2\big) < |f|^2.$$

Then $\max\big((A_+f,f),(A_-,f)\big) = (1-c_f^2)|f|^2$ and by the theorem $\log w_f$ is summable over $\hat{G}$ and

$$\int \log w_f\, dm \geqslant \log\big((1-c_f^2)|f|^2\big).$$

This gives us the estimation of the contraction coefficient c_f^2 with the aid of μ_f. Unless $T(\xi)$ is identically zero for $\xi > 0$, no uniform estimates of this type may be given. Indeed, if $|T(\xi)| \leqslant c < 1$ for some constant c for $\xi > 0$, then $|T(\xi_n)| = |T(\xi_n/n)^n| \leqslant c^n \to 0$ for a suitable sequence $\xi_n \to 0$ ($\xi_n \in G_+$). If zero is not in the spectrum of A_+, then $k(f,f) \leqslant (A_+f,f) \leqslant |f|^2 - |T^*(\xi)f|^2$ for $f \in H$ and $\xi > 0$ and some $k \in (0,1\rangle$. It follows that $T(\xi) = 0$ for $\xi > 0$. We infer therefore that zero is in the spectrum of A_+ iff it is in that of A_- and each of these inclusions holds if and only if $T(\xi) \neq 0$ for some $\xi > 0$.

4. Let Λ be the union of all open null sets of the spectral measure E of the minimal unitary dilation U_ξ of the s.g.c. $T(\xi)$. The set $\hat{G}-\Lambda$ is called the *closed support* of E and we write $s(E) = \hat{G}-\Lambda$. If $s(E)$ is not the whole $\hat{G}$, then both operators A_+, A_- reduce to zero operator. Consequently, the innovation parts $L_\pm(H)$ are trivial. We will prove more, namely that if $s(E) \neq \hat{G}$, then not only $L_\pm(H) = \{0\}$ but also $S_\pm(H) = \{0\}$. In other words, the spectral measure of the minimal unitary dilation of a non-unitary s.g.c. can not have any gaps. For s.g.c. of example 1 the corresponding property has been proved in [8], III. It results easily from theorem 2 of [5]. The case of s.g.c. of class (C_0) may be reduced to the case of a single contraction, as well we can use theorem 1 of the present paper. The proof used in [8] exploits a certain approximation theorem of Runge. We will apply here arguments of similar character.

To begin with we note that $\mathscr{A}(G_+)$ includes constants and separates the points of $\hat{G}$. Moreover, $\mathscr{A}(G_+)$ is antisymmetric and thereby essential. Since the order in G is archimedean, $\mathscr{A}(G_+)$ is maximal (see [3]). It results now from theorems of [1] that the following property holds true:

(P) *If* Δ *is a closed proper subset of* $\hat{G}$, *then to every complex valued function* v *on* Δ, *continuous on* Δ, *there is a sequence* $u_n \in \mathscr{A}(G_+)$ *such that* $u_n \to v$ *uniformly on* Δ.

LEMMA 2. *Suppose that the closed support* $s(\mu_f)$ *of the numerical measure* $\mu_f(\sigma) = \big(E(\sigma)f,f\big)$ $(f \in H)$ *is a proper subset of* $\hat{G}$. *Then* $U_{-\xi}f \in M_+(f)$ *for* $\xi \geqslant 0$.

Proof. Since $\log w_f$ is not in $L^1(m)$, we have $f \in M_+(f)$. Using the functional interpretation involved in the proof of theorem 2 we need only to show that for $\xi > 0$

$$\overline{\xi(\lambda)} \in M_+(\mu_f) = \text{the closed linear span of } \eta(\lambda), \eta > 0, \text{ in } L^2(\mu_f).$$

We know that $1 \in M_+(\mu_f)$. Since $\overline{\xi(\lambda)}$ is continuous and $s(\mu_f)$ is not the whole space $\hat{G}$, we infer by (P) that there is a sequence $u_n \in \mathscr{A}(G_+)$ converging uniformly on $s(\mu_f)$ to $\overline{\xi(\lambda)}$. Hence

$$\|\bar{\xi} - u_n\|^2_{L^2(\mu_f)} = \int_{s(\mu_f)} |\xi(\lambda) - u_n(\lambda)|^2 d\mu_f \to 0$$

which shows that $U_{-\xi}f \in M_+(f)$, q.e.d.

We are able now to prove the following theorem:

THEOREM 3. *The closed support of the spectral measure of the minimal unitary dilation of a non-unitary s.g.c. is the whole group* $\hat{G}$.

Proof. We shall use previous notation. Suppose $s(E)$ is not the whole $\hat{G}$. Let $f \in H$. Then $s(\mu_f) \neq \hat{G}$. Consequently, by the lemma, $f \in U_\xi M_+(f)$ for $\xi > 0$. Hence

$$f \in R_+(f). \tag{4.0}$$

But $R_+(f) = R_-(f) \subset R_+(H) \cap R_-(H) =$ the unitary part of the canonical decomposition. It follows that $f \in \bigcap_G U_\xi H$ for each $f \in H$ which is in contradiction with the assumption.

Remark. Note that if $f \in R_+(f)$ for every $f \in H$, then $K = R_\pm(H)$, which implies that $L_\pm(H) = S_\pm(H) = \{0\}$.

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Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*

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Introduction. The purpose of this paper is to obtain conditions for the validity of statements on interpolation between the L^1 and the L^∞ of a measure space, and for analogous statements under the hypothesis of the theorem of Marcinkiewicz. To describe our aim more precisely, let us discuss briefly some basic notions concerning interpolation of linear operations. Given a topological vector space V and two Banach spaces A_1 and A_2 which are contained and continuously embedded in V, we will call the pair (A_1, A_2) an *interpolation pair*. The space A_1+A_2 consisting of elements of V of the form $x+y$ with $x \in A_1$ and $y \in A_2$ with the norm $\|x+y\| = \inf(\|x\|_{A_1}+\|y\|_{A_2})$ is also a Banach space and its embedding in V is continuous. Given two interpolation pairs (A_1, A_2) and (B_1, B_2), a linear mapping $T: A_1+A_2 \to B_1+B_2$ will be called *admissible* if it maps A_j continuously into B_j, $j = 1, 2$. The largest of the corresponding norms will be called the *norm of the admissible mapping T*. The class of admissible mappings with this norm is a Banach space. Given two Banach spaces A and B contained and continuously embedded in A_1+A_2 and B_1+B_2 respectively, we will say that A and B are *associated* if every admissible mapping T maps A into B. It is a consequence of the closed graph theorem that T does so continuously. If $A_j = B_j$, $j = 1, 2$, and A is associated with itself, we will say that A is *intermediate* between A_1 and A_2. If in addition every admissible T of norm 1 maps A into A with norm less than or equal to 1, A will be said to be *strictly intermediate* between A_1 and A_2. Every intermediate space can be renormed so as to become strictly intermediate. A pair of associated spaces A and B will be called *optimal* if whenever A' and B' are associated and $A \subset A'$, $B \supset B'$ it follows that $A = A'$ and $B = B'$. In other words, if the pair of associated spaces A and B is optimal, the statement that every admissible T maps A into B cannot be strengthened by either enlarging A or making B smaller. According to a result of N. Aronszajn if the pair A, B is optimal, then A is intermediate between A_1 and A_2 and B is intermediate between B_1 and B_2. This result of N. Aronszajn says even more,

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