есть r-мерной в V. Отсюда и из теоремы 3.2 следует, что
$$a_1(x_1)(2), v_1(x_1)v_2(x_1)X_2(x_1), a_2(x_2)X_2(x_2), \ldots$$
есть основной последовательностью в V и поэтому
$$a_n(x_n)X_n(x_n) \sim v_n(x_n)v_n(x_n)X_n(x_n).$$

Подобным образом доказывается, что
$$v_n(x_n)v_n(x_n) \sim v_n(x_n)v_n(x_n)X_n(x_n).$$
На основании транзитивности
$$a_n(x_n)X_n(x_n) \sim v_n(x_n)v_n(x_n);$$
это дает равенство (6.4), что и требовалось доказать.

Литература


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On semi-groups of contractions in Hilbert spaces
by
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Suppose we are given a complex Hilbert space H. Let \( f, g, h, \ldots \)
stand for vectors of H and \( a, b, c, \ldots \) for complex scalars. \( f, g \) is the
inner product of \( f \) and \( g \); \( |f| \) is the norm of \( f \). By \( |V| \) we understand
the norm of the linear bounded operator in \( H \). \( V^* \) stands for the
adjoint of V and \( J \) for the identity operator in H. By \( V \) we mean the restric-
tion of the operator V to the subset \( Z \subset H \). A contraction is a linear
bounded operator V in H such that \( |V| \leq 1 \).

Let \( G \) be an abelian group. The inner group operations in \( G \) are written
additively. Suppose that the semi-group \( G_+ \) orders \( G \), that is

(i) \( G_+ \circ (-G_+) = G \),
(ii) \( G_+ \circ (-G_+) = \{0\} \).

We write \( \xi \leq \eta \) if \( \xi \in G_+ \) and \( \xi \leq \eta \) if \( \xi \leq \eta \) but \( \xi \neq \eta \).

A contraction valued function \( T(\xi) \) determined for \( \xi \in G_+ \) is called
a semi-group of contractions (s.g.c. for brevity) if

(iii) \( T(0) = I \), \( T(\xi + \eta) = T(\xi)T(\eta) \) for \( \xi, \eta \in G_+ \).

Let \( U_f : K \to K \) be a unitary representation of \( G \) into the Hilbert
space \( K \) and assume that \( H \subset K \). Write \( P \) for the orthogonal projection
of \( K \) onto H. We say that \( U_f \) is a unitary dilatation \( T(\xi) \) of \( g \) if

(iv) \( T(\xi)f = PU_fJf \) for \( f \in H \) and \( \xi \in G_+ \).

The minimality condition \( K = \bigvee_{h \in H} U_hJ \) determines \( U_f \) and \( K \)
uniquely up to a unitary isomorphism. \( U_f \) is called then the minimal
unitary dilatation of s.g.c. \( T(\xi) \).

A few examples are now in order.

**Example 1.** Let \( T \) be a contraction and \( G = \mathbb{Z} \) – the additive
group of integers. Then \( T(n) = T^n \) (\( T^n = I \) by convention) for \( n \geq 0 \)
is an s.g.c. \( G_n \) is the set of non-negative integers.\( ^{(1)} \)

\( \bigvee S_n \) stands for the closed linear span of the union of \( S_n \).
EXAMPLE 2. Let \( G = \mathbb{R} \) the additive group of reals and let \( G_+ \) stand for the set of all non-negative reals. Any one-parameter semi-group \( T(t) \) of contractions such that \( T(\lambda t) = T(\lambda)T(t) \) weakly, may be extended to a strongly continuous s.g.c. if we put \( T(0) = I \).

EXAMPLE 3. Let \( a_1, \ldots, a_n \) be rationally independent real numbers.

Consider the additive subgroup \( G \) of reals of the form \( \xi = \sum \frac{n_k a_k}{n_k} \), with integer valued factors \( n_k \). We define

\[
G_+ = \{ \xi | \xi = \sum \frac{n_k a_k}{n_k} \text{ and } n_k > 0 \}.
\]

Suppose \( \mathbb{R}^n \) is the n-dimensional torus and let \( \mu \) be a finite, positive Borel measure on \( \mathbb{R}^n \). Let \( H^2(\mu) \) be the closed linear \( L^2(\mu) \) span of monomials

\[
\phi(x) = \exp \left( \sum \frac{n_k a_k x_k}{n_k} \right) \quad (0 \leq n_k < 2\pi)
\]

where \( \xi \in G_+ \). Then the formula \( T(\xi)f(\cdot) = \phi^{\xi}(\cdot)f(\cdot) \) determines a semi-group of isometries in \( H^2(\mu) \).

B. Sz.-Nagy proved in [7] that semi-groups of Example 1 and of Example 2 do have unitary dilations. These results were extended in [6] to s.g.c. as considered in the present paper, that is to contraction valued representations of the semi-group which orders the group in question.

Assume now that \( G \) is a locally compact group. If the s.g.c. is weakly continuous in the topology of \( G_+ \), then there is a unique regular operator positive measure \( F \), defined for Borel subsets of the dual group \( \hat{G} \) and such that

\[
[T(\xi)f, g] = \int_{\hat{G}} \phi(\lambda) d[F(\lambda)f, g], \quad f, g \in \mathcal{D}, \xi \geq 0.
\]

\( \phi(\lambda) \) is the continuous character of \( \hat{G} \), corresponding to \( \xi \in G_+ \). In fact, the measure \( F \) is of the form \( F = PE_\lambda H \), where \( E \) is the spectral measure of the minimal unitary dilation of \( T(\xi) \).

Since a non-discrete locally compact ordered group is a direct sum of the usual group of reals and an ordered discrete group (see [9], p. 190), at the first step of investigations we restrict ourselves in the study of weakly continuous s.g.c. to the continuous case (Example 2) and separately to the discrete one. Our contribute to the continuous case is section 2 of the present paper. We prove there a certain prediction theoretic property of the measure \( F \). The rest of the paper deals with archimedean ordered \( \mathcal{G} \), not isomorphic to the group of integers. This excludes s.g.c. of Example 1. Since \( G \) is archimedean ordered, it is isomorphic to a dense additive subgroup of reals. The group \( G \) of Example 3 is of this type.

I. Let \( G \) be still an arbitrary abelian ordered group and let \( U_t \) be the minimal unitary dilation of s.g.c. \( T(\xi) \). The space \( H \) may be decomposed uniquely in the form \( H = H_1 \oplus H_2 \) in such a way that \( H_1 \) and \( H_2 \) are reducing subspaces for \( T(\xi) \) and the following holds true:

(*) The function \( T_t : \mathbb{R} \rightarrow \mathbb{R} \)

\[
T_t = \begin{cases} T(\xi) & \text{if } \xi \in G_+ \\ T'(\xi) & \text{if } \xi \in -G_+ \end{cases}
\]

when restricted to \( H_1 \), is a unitary representation of \( G \).

(**) For every \( f \in \mathcal{H}, f \neq 0 \)

\[
\inf \{ |T_t f| \} < |f| (\ast)
\]

The above decomposition of \( H \) and \( T(\xi) \) respectively is called a canonical one. The s.g.c. is called completely non-unitary if \( H_1 = \{ 0 \} \), \( H_2 \) is called the completely non-unitary part of the canonical decomposition. The space \( H_2 \) is characterized by

\[
H_2 = \bigcap_{t > 0} U_t H = \{ f | |T_t f| < |f| \text{ for all } \xi \in G \}.
\]

\( U_t \) stands as usually for the minimal unitary dilation of \( T(\xi) \). We say that \( T(\xi) \) is non-unitary if \( H_2 \neq \{ 0 \} \).

Let \( S \) be an arbitrary linear subset of \( H \) and write

\[
M_+ (S) = \bigvee_{t > 0} U_t S, \quad M_- (S) = \bigvee_{t > 0} U_t M_+ (S).
\]

If \( S \) is a one-dimensional space spanned by \( f \neq 0 \), we put \( f \) in place of \( S \) in the above notation.

It was proved in [6] that

\[
R_+(H) \cap R_-(H) = \bigcup_{t > 0} U_t H.
\]

Assume now additionally that \( G \) is locally compact and \( T(\xi) \) is a weakly continuous semi-group of contractions. Let \( E \) stand for the spectral measure of the minimal unitary dilation \( U_t \) of \( T(\xi) \). We will consider the space \( \mathcal{M}_+ (f) \cup \mathcal{M}_- (f) \cup \mathcal{M}_m (f) \). The correspondence

\[
\sum a_k U_{k} f = \sum a_k \xi_k (\lambda)
\]

(*) For references, see [8], IV.
may be extended to the unitary equivalence between $M(f)$ and $L^1(\mu)$ where $\mu(x) = |E(e)\xi, f|$. The restriction $U_1(M(f))$ is equivalent to multiplication by $\xi(\lambda)$ in $L^1(\mu)$. The copies of reducing spaces $R_+(f)$, $R_-(f)$ in $L^1(\mu)$ consist of functions vanishing outside suitable measurable sets. It follows then easily from the definitions of $R_+(f)$ and $R_-(f)$ that this set differs at most by a set of measure zero. Hence

\[ R_+(f) = R_-(f). \]

(1.1)

It results now by (1.0) that $f \in H_1$, then

\[ R_+(f) = R_-(f) = \{0\}. \]

(1.2)

2. Assume now that $G = R$ — the additive group of reals with usual topology and usual order. The weakly continuous s.g.c. $T(\xi)$ is that of Example 2 and satisfies

\[ T(h) \to T(h) \text{ strongly.} \]

(2.0)

An s.g.c. which satisfies (2.0) is called of class $\mathcal{C}_0$. The spectral measure $E$ of the minimal unitary dilation of s.g.c. of class $\mathcal{C}_0$ is determined on Borel subsets of $R$. $U_1$ when restricted to $M(f)$ may be regarded as a mean continuous weakly stationary stochastic process with supporting measure $\mu(x) = |E(\xi)\xi, f|$. Suppose that $f \in H_1$. Then, by (1.2), the remote past of this process is trivial. In other words, the corresponding process is purely non-deterministic. It is a classical result of prediction theory (see [4]) that the supporting measure $\mu$ of purely non-deterministic process is absolutely continuous with respect to the linear Lebesgue measure on line and

\[ \int_\infty^\infty \frac{d\mu}{1 + \lambda^2} d\lambda \]

is finite. This, when applied to $\mu = \mu_1$, shows that the following theorem holds true:

**Theorem 1.** Let $f \neq 0$ belong to the completely non-unitary part of the canonical decomposition corresponding to the one-parameter s.g.c. $T(\xi)$ of class $\mathcal{C}_0$. Suppose $H$ is the spectral measure of the minimal unitary dilation of $T(\xi)$. Then

\[ \int_\infty^\infty \frac{d(E(\xi)\xi, f)}{1 + \lambda^2} d\lambda \]

is finite.

**Remark.** The above theorem is a continuous version of a result of [5].

3. Since now we assume that $G$ is an everywhere dense subgroup of reals, $G_0$ is identified with the set of all non-negative reals in $G$. We regard $G$ as a discrete topological group. We will study s.g.c. for such groups.

To begin with, we note that zero is an (in ordinary topology) accumulation point of $G$. Consider the subset $H'$ of $H$ determined by

\[ H' = \{ f \in H : \lim_{t \to 0 \pm} |T(\xi)\xi, f| = (f, f) \}. \]

It is a simple matter to check that $f \in H'$ iff $\lim_{t \to 0 \pm} |T(\xi)\xi, f| = 0$.

Consequently, $H'$ is a closed linear subspace of $H$. Moreover, $H'$ reduces all of $T(\xi)$. Since $G_0$ is dense in the positive halfline and $|T(\xi)| \leq 1$, $\lim_{t \to 0 \pm} T(\xi)f$ exists for every $\xi \in H'$ and $f \in H'$. It follows that $T(\xi)'(H')$ may be extended uniquely to one parameter s.g.c. of class $\mathcal{C}_0$. Note that $H'$ is the largest subspace which does have this property. In order to have a "purely discrete" case we will assume in the sequel without stating it explicitly that for every $f \in H$, $f \not= 0$

\[ \lim_{\lambda \to +\infty} \sup_{t \in H'} |T(\xi)\xi, f| > 0. \]

Let $U_1$ stand for the minimal unitary dilation of $T(\xi)$ and let $K$ be the corresponding dilation space. There is no difficulty to show that

\[ K = \bigvee_{t \in H} \{ U_1 - T(\xi) \} H \oplus H \bigvee_{t \in H} \{ U_{-t} - T(-t) \} H \]

and

\[ \bigvee_{t \in H} U_{-t} = H \oplus \bigvee_{t \in H} \{ U_{-t} - T(-t) \} H, \quad \bigvee_{t \in H} \quad \bigvee_{t \in H} \quad \bigvee_{t \in H} \quad \bigvee_{t \in H} \quad \bigvee_{t \in H} \]

The other type decomposition formulae are that of Wolf type [4]. We notice first that $R_\infty(H)$ reduces $U_1$. It results from (1.0) that if $T(\xi)$ is non-unitary, then at least one of the spaces $R_+(H)$, $R_-(H)$ is not the whole $K$. Define $Q_H$ as the orthogonal projection onto $M_+(H)$ and consider the subspace $D_\infty = (I - Q_H)H$. Since $M_+(H)$ ($M_-(H)$) is invariant with respect to $U_{\xi}(U_{-\xi})$ for $\xi \not= 0$, we have $U_{\xi}D_\infty \subseteq U_{\xi}D_\infty$ for $\xi \not= 0$. We put $L_\infty(H) = U_{t}D_\infty$. Obviously $L_\infty(H) \perp R_\infty(H)$. Write now

\[ S_\infty(H) = K \ominus (R_\infty(H) \oplus L_\infty(H)). \]

$S_\infty(H)$ need not reduce to a zero subspace, which is the case for classical Wolf decomposition. This is the result of Helson and Lowdenslager, who constructed in [2, II], a suitable example. Finally we have

\[ K = R_\infty(H) \oplus L_\infty(H) \oplus S_\infty(H). \]

(3.0)
Let $Q_\xi$ be the projection onto $U_1 M_+(H)$. It was proved in [6] that

$$|Q_\xi f|^2 \leq |T_\xi^*(\xi)f|^2 \quad \text{for} \quad \xi > 0, f \in H.$$  

Suppose now that $\xi < \eta$ (the $\eta \in G_+$) and put

$$\phi(\xi, \eta) = |U_1 T_\xi^*(\xi)f - U_1 T_\eta^*(\eta)f|^2.$$  

Then

$$\phi(\xi, \eta) = |T_\xi^*(\xi)f|^2 + |T_\eta^*(\eta)f|^2 - 2 \text{Re}(U_{1 - \xi} T_\xi^*(\xi)f, P T_\eta^*(\eta)f)$$

$$= |T_\xi^*(\xi)f|^2 - |T_\eta^*(\eta)f|^2.$$  

Since $\lim_{\xi \to \eta} |T_\xi^*(\xi)f|^2$ exists, we conclude therefore that

$$\lim_{\xi \to \eta} U_1 T_\xi^*(\xi)f \quad (\xi \in G_+),$$  

exists for every $f \in H$. It follows $\lim_{\xi \to \eta} U_{1 - \xi} T_\xi^*(\xi)f$ exists for each $f \in H$.

Note now that

$$I - T_\xi^*(\xi)f = I - T_\eta^*(\eta)f \quad \text{for} \quad \xi < \eta \quad (\xi, \eta \in G_+).$$  

Hence, the strong limit

$$\lim_{\xi \to \eta} (I - T_\xi^*(\xi)f) = A_+$$  

exists. $A_+$ is a positive operator and

$$|f|^2 - \lim_{\xi \to \eta} |T_\xi^*(\xi)f|^2 = (A_+, f, f).$$  

The operator $A_-$ is defined by

$$|f|^2 - \lim_{\xi \to \eta} |T_\xi^*(\xi)f|^2 = (A_-, f, f).$$  

The innovation part $I_\xi (H)$ of (3.0) is characterized by the following lemmas:

**Lemma 1.** For every $f \in H$

$$Q_\xi f = \lim_{\xi \to \eta} U_1 T_\xi^*(\xi)f, \quad Q_\xi f = \lim_{\xi \to \eta} U_{1 - \xi} T_\xi^*(\xi)f \quad (\xi \in G_+).$$  

**Proof.** It is clear that it suffices to consider merely the case of $Q_+$. Thus, let $f \in H$ and write

$$g = \lim_{\xi \to \eta} U_1 T_\xi^*(\xi)f.$$  

Since $U_1 T_\xi^*(\xi)f \in M_+(H)$ for $\xi > 0$, we have $Q_+ g = g$. Hence

$$|f|^2 - \lim_{\xi \to \eta} |T_\xi^*(\xi)f|^2 = (I - Q_+ f, f) = (I - Q_\xi f, f).$$  

On the other hand, by (3.1),

$$|Q_\xi f|^2 \leq |T_\xi^*(\xi)f|^2$$  

which together with formula

$$|f - Q_\xi f|^2 = |(I - Q_\xi f)f|^2 + |Q_\xi f|^2,$$  

implies

$$|f - g|^2 \leq |f|^2 - |Q_\xi f|^2 = |(I - Q_\xi f)f|^2.$$  

It results now from (3.3) and (3.4) that $Q_\xi f = g$ as was to be proved.

**Corollary.** Note that

$$|f - Q_\xi f|^2 = |Q_\xi f|^2.$$  

Consequently, the correspondence $f - Q_\xi f \leftrightarrow A_\xi^m f$ is isometric and may be extended uniquely to the unitary one between closures of subspaces $D_\xi$ and $A_\xi^m H$ respectively.

Let $\mathcal{A}(G_+)$ be the algebra of functions, generated in the sup norm by polynomials $\sum_{n \geq 0} a_n \xi^\zeta (\lambda) (a_n \geq 0)$ with $\xi, \zeta > 0$. Suppose $m$ is the normalized Haar measure of $G$ and put

$$\mathcal{A}_m = \{ a \in \mathcal{A}(G_+), \int u dm = 0 \}.$$  

Consider a non-negative regular measure determined on Borel sets of $G$. A generalized version of Szegö theorem, proved in [2], is the following one:

$$\inf_{w \in \mathcal{A}_m} \int |1 - u|^2 d\mu = \inf_{w \in \mathcal{A}_m} \int |1 - u|^{2m} dm = \exp \{ \log w \};$$  

where $w$ stands here for the Radon-Nikodym derivative of $\mu$ with respect to $m$. If $\mu$ is not $m$ summable over $G$, then the right side of (3.6) should be interpreted as zero and the left sides are in fact zero. Take now $\mu(x) = \mu(x)f$ where $E$ is the spectral measure of the minimal unitary dilation of $T(f)$. We write $w_f$ for the Radon-Nikodym derivative of $\mu_f$ with respect to $m$.

**Theorem 2.** Suppose that $f \in H$. Then, if $\max(\{A_+, f, f\}, \{A_-, f, f\})$ is positive,

$$\int \log w_f dm > \log \max(\{A_+, f, f\}, \{A_-, f, f\}).$$

**Proof.** The unique continuous extension of the isomorphism

$$\sum a_n \xi^\zeta (\lambda) \leftrightarrow \sum a_n U_n \xi^\zeta$$

establishes the unitary equivalence between $M(f)$ and $D_\xi (\mu_f)$. Under this equivalence $1 \leftrightarrow f$ and $U_1 M(f)$ is interpreted as multiplication by $\xi (\lambda)$. It follows that

$$\phi = \{ \text{distance}(f, M_+(f)) \}^2 = \int |1 - u|^2 d\mu = \int |1 - u|^{2m} d\mu$$

$$= \{ \text{distance}(f, M_-(f)) \}^2.$$.  

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where inf is taken over \( u \in \mathcal{U}_m \). Since \( M_\Sigma(f) \subset M_\Sigma(H) \), we have \( |(I - \Omega f) f|^2 \leq \varepsilon \) which together with (3.2), (3.4) and (3.6) proves the assertion of the theorem.

**Corollary.** Using arguments similar to those used in the proof of theorem 2 one shows that if \( \log \varepsilon \) is not in \( L^1(m) \), then \( A_\Sigma f = A_\Sigma f = 0 \). Hence, if \( E \) is singular with respect to \( m \), then \( A_\Sigma = A_\Sigma = 0 \).

It is of some interest to point out some consequences of theorem 2. Suppose that
\[
\phi_0 f^2 = \min_{\varepsilon > 0} \lim_{\varepsilon \to 0} \left( \| T(\varepsilon) f \|_2, \lim_{\varepsilon \to 0} \| T(\varepsilon) f \|_2 < |f|^2. \right)
\]

Then \( \max \{ |A_\Sigma f, A_\Sigma f | = (1 - \phi_0) |f|^2 \} \) and by the theorem \( \log \varepsilon \) is summable over \( \tilde{G} \) and
\[
\int \log \varepsilon \, d\mu = \log \left( (1 - \phi_0) |f|^2 \right). \]

This gives us the estimation of the contraction coefficient \( \phi_0 \) with the aid of \( \mu \). Unless \( T(\varepsilon) \) is identically zero for \( \varepsilon > 0 \), no uniform estimates of this type may be given. Indeed, if \( |T(\varepsilon)f| < \varepsilon < 1 \) for some constant \( c \) and \( \varepsilon > 0 \), then
\[
|T(\varepsilon_\delta) - T(\varepsilon_\delta) n |_* \leq \varepsilon \delta \to 0 \text{ for a suitable sequence } \delta > 0. \]

If zero is not in the spectrum of \( A_\Sigma \), then
\[
|\kappa f, f | = |A_\Sigma f, f | < |f|^2 - \alpha T(\varepsilon f) f \text{ for } f \in H \text{ and } \delta > 0 \text{ and some } \kappa > 0. \]

It follows that \( T(\varepsilon f) = 0 \) for \( \varepsilon > 0 \). We infer therefore that zero is in the spectrum of \( A_\Sigma \) iff it is in that of \( A_\Sigma \), and each of these inclusions holds if and only if \( T(\varepsilon) \neq 0 \) for some \( \varepsilon > 0 \).

4. Let \( \Lambda \) be the union of all open null sets of the spectral measure \( E \) of the minimal unitary dilation \( U_\delta \) of the s.g.c. \( T(\varepsilon) \). The set \( \tilde{G} - \Lambda \) is called the closed support of \( E \) and we write \( s(E) = \tilde{G} - \Lambda \). If \( s(E) \) is not the whole \( \tilde{G} \), then both operators \( A_\Sigma, A_\Sigma \) reduce to zero operator. Consequently, the innovation parts \( L_\Sigma(H) \) are trivial. We will prove more, namely that if \( s(E) \neq \tilde{G} \), then not only \( L_\Sigma(H) = \{ 0 \} \) but also \( S_\Sigma(H) = \{ 0 \} \).

In other words, the spectral theorem of the minimal unitary dilation of a non-unitary s.g.c. can have no gaps. For s.g.c. of class \( (C_0) \) may be reduced to the case of a single contraction, as we can use theorem 1 of the present paper. The proof used in [8] exploits a certain approximation theorem of Runge. We will apply here arguments of similar character.

To begin with we note that \( \mathcal{M}(G_\varepsilon) \) includes constants and separates the points of \( \tilde{G} \). Moreover, \( \mathcal{M}(G_\varepsilon) \) is antisymmetric and thereby essential. Since the order in \( \tilde{G} \) is archimedean, \( \mathcal{M}(G_\varepsilon) \) is maximal. (see [3]). It results now from theorems of [1] that the following property holds true:

\[ (P) \text{ If } \Lambda \text{ is a closed proper subset of } \tilde{G}, \text{ then to every complex valued } \]
\[ \text{ function } \omega \text{ on } \Lambda, \text{ continuous on } \Lambda, \text{ there is a sequence } u_n \in \mathcal{M}(G_\varepsilon) \text{ such that } \]
\[ u_n \to \omega \text{ uniformly on } \Lambda. \]

**Lemma 2.** Suppose that the closed support \( s(\mu) \) of the numerical measure \( \mu(\cdot) = |E(\sigma)| f, f H \) is a proper subset of \( \tilde{G} \). Then \( U_\delta f \in M_\Sigma(f) \) for \( \delta > 0 \).

**Proof.** Since \( \log \varepsilon \) is not in \( L^1(m) \), we have \( f \in M_\Sigma(f) \). Using the functional interpretation involved in the proof of theorem 2 we need only to show that for \( \delta > 0 \)
\[ \xi(\delta) \in M_\Sigma(\mu) = \text{the closed linear span of } \eta(\delta), \eta > 0, \text{ in } D(\mu). \]

We know that \( 1 \in M_\Sigma(\mu) \). Since \( \xi(\delta) \) is continuous and \( s(\mu) \) is not the whole space \( \tilde{G} \), we infer by (P) that there is a sequence \( v_n \in \mathcal{M}(G_\varepsilon) \) converging uniformly on \( s(\mu) \) to \( \xi(\delta) \). Hence
\[ \int \xi(\delta) - u_n(\delta) |d\mu \to 0 \]
which shows that \( U_\delta f \in M_\Sigma(f) \), q.e.d.

We are able now to prove the following theorem:

**Theorem 4.** The closed support of the spectral measure of the minimal unitary dilation of a non-unitary s.g.c. is the whole group \( \tilde{G} \).

**Proof.** We shall use previous notation. Suppose \( s(E) \) is not the whole \( \tilde{G} \). Let \( f \in H \). Then \( s(\mu) \neq \tilde{G} \).

Consequently, by the lemma, \( f \in \mathcal{M}(G_\varepsilon) \) for \( \delta > 0 \). Hence
\[ f \in M_\Sigma(f). \]

But \( R_\varepsilon(f) = R_\varepsilon(f) \subset B_\varepsilon(H) \subset \mathcal{E}(H) \) is the unitary part of the canonical decomposition. It follows that \( f \in U_\delta H \) for each \( f \in H \) which is in contradiction with the assumption.

**Remark.** Note that if \( f \in \mathcal{M}(G_\varepsilon) \) for every \( f \in H \), then \( K = B_\varepsilon(H) \), which implies that \( L_\Sigma(H) = S_\Sigma(H) = \{ 0 \} \).

**References**

Introduction. The purpose of this paper is to obtain conditions for the validity of statements on interpolation between the $L^p$ and the $L^q$ of a measure space, and for analogous statements under the hypothesis of the theorem of Marcinkiewicz. To describe our aim more precisely, let us discuss briefly some basic notions concerning interpolation of linear operations. Given a topological vector space $V$ and two Banach spaces $A_1$ and $A_2$ which are contained and continuously embedded in $V$, we will call the pair $(A_1, A_2)$ an interpolation pair. The space $A_1 + A_2$, consisting of elements of $V$ of the form $x+y$ with $x \in A_1$ and $y \in A_2$ with the norm $\|x+y\| = \inf (\|x\|_1 + \|y\|_2)$ is also a Banach space and its embedding in $V$ is continuous. Given two interpolation pairs $(A_1, A_2)$ and $(B_1, B_2)$, a linear mapping $T: A_1 + A_2 \to B_1 + B_2$ will be called admissible if it maps $A_j$ continuously into $B_j$, $j = 1, 2$. The largest of the corresponding norms will be called the norm of the admissible mapping $T$. The class of admissible mappings with this norm is a Banach space. Given two Banach spaces $A$ and $B$ contained and continuously embedded in $A_1 + A_2$ and $B_1 + B_2$ respectively, we will say that $A$ and $B$ are associated if every admissible mapping $T$ maps $A$ into $B$. It is a consequence of the closed graph theorem that $T$ does so continuously. If $A_j = B_j$, $j = 1, 2$, and $A$ is associated with itself, we will say that $A$ is intermediate between $A_1$ and $A_2$. If in addition every admissible $T$ of norm 1 maps $A$ into $A$ with norm less than or equal to 1, $A$ will be said to be strictly intermediate between $A_1$ and $A_2$. Every intermediate space can be renormed so as to become strictly intermediate. A pair of associated spaces $A$ and $B$ will be called optimal if whenever $A'$ and $B'$ are associated and $A \subset A'$, $B \supset B'$ it follows that $A = A'$ and $B = B'$. In other words, if the pair of associated spaces $A$ and $B$ is optimal, the statement that every admissible $T$ maps $A$ into $B$ cannot be strengthened by either enlarging $A$ or making $B$ smaller. According to a result of N. Aronszajn if the pair $A$, $B$ is optimal, then $A$ is intermediate between $A_1$ and $A_2$ and $B$ is intermediate between $B_1$ and $B_2$. This result of N. Aronszajn says even more, *This research was partly supported by the NSF grant GP-3984.