dar. Weil die rechtsstehende Reihe absolutkonvergent ist, gibt es dann in E'_b eine beschränkte Teilmenge B mit

$$\sum_{n=1}^{\infty} p_B(a_n f'_n) < +\infty$$

(vgl. [4], S. 135, Cor. 4, oder [7], 1.5.8).

Nun bestimmen wir die rationalen Zahlen a_{mn} so, daß

$$p_B(a_n f'_n - a_{mn} f'_n) \leqslant m^{-2}$$

gilt. Da es zu jeder auf E'_b stetigen Halbnorm p eine positive Zahl ϱ mit

$$p(b) \leqslant \varrho p_B(b) \quad \text{für alle } b \in E'(B)$$

gibt, hat man

$$p(a - a_m) \leqslant \sum_{n=1}^{m} p(a_n f'_n - a_{mn} f'_n) + \sum_{n=m+1}^{\infty} p(a_n f'_n)$$

$$\leqslant \varrho \Big[m^{-1} + \sum_{n=m+1}^{\infty} p_B(a_n f'_n) \Big].$$

Deshalb gilt in E'_b, wie behauptet, die Beziehung

$$\lim a_m = a.$$

Da wir soeben gezeigt haben, daß E'_b folgenseparabel ist, muß der nach dem Darstellungssatz zu einem Folgenraum Λ isomorphe (F)-Raum E auf Grund von [6], S. 421, (4), reflexiv sein. Nun folgt aber aus Satz 1 und Satz 2, daß die Systeme $\mathfrak{V}_{\mu 1}(E)$ und $\mathfrak{V}_m(E)$ fundamental sind. Somit ist E nach Satz 3 nuklear.

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The extended spectrum of completely non-unitary contractions and the spectral mapping theorem

by

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Let H^∞ be the algebra of bounded functions, analytic in the open unit disk $\Delta = \{z: |z| < 1\}$. Suppose T is a completely non-unitary contraction [5] in the complex Hilbert space $\mathscr{H}$. There was developed in [6] the functional calculus for H^∞ and such T. More precisely, it was shown that there is a unique representation $u \to u(T)$ $(u \in H^\infty)$ of H^∞ into a certain operator algebra on $\mathscr{H}$, such that

(i) $u_0(T) = I$ for $u_0(z) = 1$, $u_1(T) = T$ for $u_1(z) = z$.

(ii) $|u(T)| \leqslant \sup\limits_{|z|<1} |u(z)|$ for $u \in H^\infty$ (1).

(iii) If $u_n(e^{i\lambda}) \to u(e^{i\lambda})$ boundedly, almost everywhere on $\langle 0, 2\pi)$, then $u_n(T) \to u(T)$ strongly.

The restriction of the mapping $u \to u(T)$ to $u \in A$, the algebra of functions analytic on Δ and continuous on $\overline{\Delta}$, coincides with the functional calculus of J. von Neumann for T and S-analytic functions with $S = \overline{\Delta}$ (for details see [1]). Let $\sigma(V)$ stand for the spectrum of the operator V. It is known (see [1]) that for von Neumann calculus the spectral mapping theorem holds true, i.e.

$$(*) \qquad \sigma[u(T)] = u[\sigma(T)] \quad \text{for} \quad u \in A.$$

It is then natural to ask, how the things are going on with $\sigma[u(T)]$ and $u[\sigma(T)]$ in case where $u \in H^\infty$.

The present paper attempts to give a certain solution of this problem. We always assume, if otherwise not stated explicitely, that T is completely non-unitary contraction.

1. We notice first that $u(e^{i\lambda})$ for $u \in H^\infty$ is defined only almost everywhere by formula

$$u(e^{i\lambda}) = \lim_{r \to 1-} u(re^{i\lambda}).$$

(1) $|V|$ stands for the norm of the linear bounded operator $|V|$ in $\mathscr{H}$.

Consequently, the symbol $u[\sigma(T)]$ makes no sense in general. The reason is that the boundary part of $\sigma(T)$, namely $\sigma(T)\cap C$ (C stands for the unit circle of the complex plane) may be non-empty and included in the set of those $z\in C$ for which $\lim\limits_{r\to 1-} u(rz)$ does not exist. It is not clear what we should mean by $u(z)$ in this case. One can hope that it is possible to prove a certain limit form of (*). This seems to be a rather delicate question, as shown by the following example.

Let $\mathscr{H} = L_2(0,1)$ and let T_t be a strongly continuous one parameter semi-group of contractions in $\mathscr{H}$ defined by

$$(T_t f)(x) = \begin{cases} 0 & \text{if} \quad x < t \text{ and } 0 \leqslant t < 1, \\ f(x-t) & \text{if} \quad x \geqslant t \text{ and } 0 \leqslant t < 1, \\ 0 & \text{if} \quad t \geqslant 1. \end{cases}$$

Let T be the cogenerator of T_t. Then $T_t = u_t(T)$ where

$$u_t(z) = \exp\left(t\,\frac{z+1}{z-1}\right) \in H^\infty.$$

Since $u_t(T) = 0$ for $t = 1$,

$$\sigma[u_t(T)] = \{0\} \quad \text{for} \quad t = 1. \tag{1.0}$$

The set L of limit values of $\exp\big((z+1)/(z-1)\big)$ at $z = 1$ equals to $\bar{\Delta}$. It was shown in [2] that $\sigma(T) = \{1\}$. It follows now that $u[\sigma(T)]$ may be not in general replaced in (*) by the set of limit values of u at $\sigma(T)$. However $0\in L$. In terms of maximal ideals of H^∞ this last property means that there is an element m in the space $\mathscr{M}(H^\infty)$ of maximal ideals of H^∞ such that $m(u_t) = 0$ for $t = 1$. Such m belongs to the fiber $\mathscr{F}_1$ corresponding to $z = 1$ and one can hope that a certain subset of $\mathscr{F}_1$ should imitate in some way $\sigma(T) = \{1\}$ in a reasonable generalization of (*). This point of view requires a certain modification of the idea of the spectrum of a contraction. What we have in mind may be better understood by considering the algebra A and the related image algebra arising from the von Neumann calculus.

Let T be now an arbitrary contraction and let $\mathscr{A}_T$ be the uniformly closed operator algebra generated by $(zI-T)^{-1}$ for $z\notin\sigma(T)$. Obviously $u(T)\in\mathscr{A}_T$ for $u\in A$ and the maximal ideals space of $\mathscr{A}_T$ may be identified with $\sigma(T)$ (see [1]). We have

$$\widehat{u(T)}(z) = u(z) \quad \text{for} \quad z\in\sigma(T),\ u\in A \ (^2) \tag{1.1}$$

(²) By $u\to\hat{u}$ we always mean the Gelfand representation of a considered commutative Banach algebra with unit.

under this identification. On the other hand, it was proved in [1] that the spectrum of $u(T)$ ($u\in A$), considered as an element of $\mathscr{A}_T$, equals exactly to $\sigma[u(T)]$. The Gelfand formula

$$\hat{V}\big(\sigma(T)\big) = \text{spectrum of } V \text{ as an element of } \mathscr{A}_T$$

together with (1.1) implies $\sigma[u(T)] = u[\sigma(T)]$. Two facts should be pointed out. The first is that the maximal ideals space of $\mathscr{A}_T$, namely $\sigma(T)$ is naturally embbeded in that of algebra A. The other is that $\mathscr{A}_T$ may be obtained as an algebra generated just by $(zI-T)^{-1}$ ($z\notin\sigma(T)$). To have the first property we essentially needed such rich algebra. $\mathscr{A}_T$ may not be replaced, for instance, by the uniformly closed algebra generated by polynomials in T. The reason is that if $\sigma(T)$ separates the complex plane, then the suitable homeomorphic image of maximal ideals space of polynomially generated algebra does not. The example below, which is of its own interest, shows that such situation may realy happen.

Example. It is sufficient to show that for some T and some $z\notin\sigma(T)$, $(zI-T)^{-1}$ is not a uniform limit of polynomials in T. Let T be a contraction of class C_0 and such that $\sigma(T)$ is the whole unit circle. It was proved in [7] that such T exists. We will show that T^{-1} is not a uniform limit of polynomials in T.

Suppose, for contrary, that there is a sequence of polynomials p_n such that $p_n(T)\to T^{-1}$ in the operator norm. Since $\sigma(T) = C$,

$$|q(T)| = \max_C |q(z)|$$

for $q\in A$ (see Prop. 1.2.1 of [1]). Consequently $p_n(z)\to f(z)\in A$, uniformly on $\bar{\Delta}$. Hence $f(T) = T^{-1}$. Put now $g(z) = f(z)z-1$. Then $g(T) = 0$. Since T belongs to C_0, the minimal function m_T of T is a divisor, in H^∞, of g. On the other hand, by theorem 7 of [7]

$$m_T(z) = k\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right), \qquad |k| = 1,$$

and the closed support of the singular measure μ is the whole unit circle. It follows then that for every $t\in\langle 0, 2\pi)$ there is a sequence $z_n\to e^{it}$ ($|z_n|<1$) such that $m_T(z_n)\to 0$. Consequently $g(z_n)\to 0 = g(e^{it})$ because g is continuous ($g\in A$). Hence $g(z)$ is identically zero which implies that $1/z = f(z)$ for z belonging to the unit circle. But this is in contradiction with the fact that f belongs to A.

Going back to our previous considerations, we observe that what we needed in fact for the proof of (*) in case of algebra A, were (1.1) and the Gelfand representation formula. Both this points admit a natural generalization in case of algebra H^∞, in place of A.

2. Let T be now a completely non-unitary contraction in the complex Hilbert space $\mathscr{H}$. We have for disposition operators $u(T)$ with $u \in H^\infty$. By analogy with $\mathscr{A}_T$ we define $\mathscr{H}_T^\infty$ as the uniformly closed algebra generated by $(zI - u(T))^{-1}$ with $u \in H^\infty$ and $z \notin \sigma[u(T)]$. Notice that

$$u(T) = \frac{1}{2\pi i} \int_{|z|=\varrho} z(zI - u(T))^{-1} dz$$

with $\varrho = |u(T)|+1$. This shows that $u(T) \in \mathscr{H}_T^\infty$ for $u \in H^\infty$. Let $\mathscr{M}(\mathscr{H}_T^\infty)$ and $\mathscr{M}(H^\infty)$ be the spaces of maximal ideals of the algebras $\mathscr{H}_T^\infty$ and H_T^∞ respectively.

We define a mapping $\varphi: \mathscr{M}(\mathscr{H}_T^\infty) \to \mathscr{M}(H^\infty)$ by formula

$$\widehat{u(T)}(m) = \hat{u}(\varphi(m)), \quad m \in \mathscr{M}(\mathscr{H}_T^\infty),\ u \in H^\infty, \tag{2.0}$$

$\widehat{u(T)}$ stands for the Gelfand representation of $u(T)$ considered as an element of $\mathscr{H}_T^\infty$. We will show that φ is one-to-one. Indeed, suppose $\varphi(m_1) = \varphi(m_2)$ $(m_1, m_2 \in \mathscr{M}(\mathscr{H}_T^\infty))$ and consider the operators $(zI - u(T))^{-1}$ and $(zI - u(T))$ $(z \notin \sigma[u(T)])$ which both are in $\mathscr{H}_T^\infty$. Then

$$\widehat{(zI-u(T))^{-1}}(m_i)\widehat{(zI-u(T))}(m_i) = 1, \quad i = 1, 2,$$

for $z \notin \sigma[u(T)]$. But $\widehat{u(T)}(m_1) = \hat{u}(\varphi(m_1)) = \hat{u}(\varphi(m_2)) = \widehat{u(T)}(m_2)$. Hence

$$\widehat{(zI-u(T))^{-1}}(m_1) = \widehat{(zI-u(T))^{-1}}(m_2) \quad (z \notin \sigma[u(T)]). \tag{2.1}$$

Since u is arbitrary and $(zI - u(T))^{-1}$ generate $\mathscr{H}_T^\infty$, (2.1) implies that $m_1 = m_2$, q.e.d.

It is easy to see that φ is continuous. Consequently, φ establishes a homeomorphic correspondence between $\mathscr{M}(\mathscr{H}_T^\infty)$ and a certain closed subset of $\mathscr{M}(H^\infty)$. We denote this set by σ_0, i.e. $\sigma_0 = \varphi[\mathscr{M}(\mathscr{H}_T^\infty)]$ by definition.

We will prove that

$$\sigma[u(T)] = \hat{u}(\sigma_0) \quad \text{for every } u \in H^\infty. \tag{2.2}$$

Let $\tilde{\sigma}[u(T)]$ stand for the spectrum of $u(T)$ considered as an element of $\mathscr{H}_T^\infty$. It follows

$$\sigma[u(T)] \subset \tilde{\sigma}[u(T)]. \tag{2.3}$$

On the other hand, if $z \notin \sigma[u(T)]$, then $(zI - u(T))^{-1} \in \mathscr{H}_T^\infty$ which implies $z \notin \tilde{\sigma}[u(T)]$. Using (2.3) we get therefore that

$$\sigma[u(T)] = \tilde{\sigma}[u(T)]. \tag{2.4}$$

By Gelfand formula

$$\tilde{\sigma}[u(T)] = \widehat{u(T)}[\mathscr{M}(\mathscr{H}_T^\infty)]. \tag{2.5}$$

It follows from the definition (2.0) of φ and from (2.4) that $\sigma[u(T)] = \widehat{u(T)}[\mathscr{M}(\mathscr{H}_T^\infty)] = \hat{u}(\sigma_0)$ which proves (2.2).

We put

$$\sigma_T(u) = \{m: m \in \mathscr{M}(H^\infty),\ \hat{u}(m) \in \sigma[u(T)]\}$$

for $u \in H^\infty$. Equality (2.2) implies

$$\sigma_0 \subset \sigma_T(u) \quad \text{and} \quad \hat{u}[\sigma_T(u)] \subset \sigma[u(T)] = \hat{u}(\sigma_0) \tag{2.6}$$

for every $u \in H^\infty$. Consequently, the intersection $\bigcap_{u \in H^\infty} \sigma_T(u)$ is non-empty. We call this intersection the *extended spectrum of* T and denote it by $\sigma_{\text{ext}}(T)$. Hence, by definition

$$\sigma_{\text{ext}}(T) = \bigcap_{u \in H^\infty} \sigma_T(u).$$

Our basic theorem is the following one:

THEOREM. *Suppose that T is a completely non-unitary contraction in a complex Hilbert space. Then:*

(a) *For every $u \in H^\infty$ the equality $\hat{u}(\sigma_{\text{ext}}(T)) = \sigma[u(T)]$ holds true.*

(b) *If γ is a closed subset of $\mathscr{M}(H^\infty)$ such that $\hat{u}(\gamma) = \sigma[u(T)]$ for every $u \in H^\infty$, then $\gamma \subset \sigma_{\text{ext}}(T)$.*

Proof. It follows from (2.6) that $\sigma_0 \subset \sigma_{\text{ext}}(T)$. Hence, using again (2.6), we obtain $\hat{u}(\sigma_0) \subset \hat{u}(\sigma_{\text{ext}}(T)) \subset \hat{u}(\sigma_T(u)) \subset \sigma[u(T)] = \hat{u}(\sigma_0)$ for $u \in H^\infty$. This proves (a).

Suppose now that $m \in \mathscr{M}(H^\infty)$ and $\hat{u}(m) \in \sigma[u(T)]$ for every $u \in H^\infty$. Then

$$m \in \bigcap_{u \in H^\infty} \sigma_T(u) = \sigma_{\text{ext}}(T)$$

which proves (b).

The assertion (a) of the theorem is just the analogy of the spectral mapping theorem. We could use overthere only σ_0 in place of the extended spectrum. However, (b) shows that the extended spectrum is a maximal subset of $\mathscr{M}(H^\infty)$ among all closed sets γ for which $\sigma[u(T)] = \hat{u}(\gamma)$ for each $u \in H^\infty$.

3. Several comments are now in order. We first remind some properties of $\mathscr{M}(H^\infty)$. For references see [3].

Let $u_1(z) = z$ and $m \in \mathscr{M}(H^\infty)$. Define the map $\pi: \mathscr{M}(H^\infty) \to \bar{\varDelta}$ by formula

$$\pi(m) = m(u_1). \tag{3.0}$$

Then

$$\pi(\mathscr{M}(H^\infty)) = \bar{\Delta} \tag{3.1}$$

and π restricted to $\pi^{-1}(\Delta)$ is a homeomorphism. Let $D = \pi^{-1}(\Delta)$.

The fiber $\mathscr{F}_a$ of $a \in C$ ($|a| = 1$) is defined by

$$\mathscr{F}_a = \{m: m \in \mathscr{M}(H^\infty), \pi(m) = a\}.$$

It is known that if u is continuously extendable to a and $u(a)$ is the corresponding limit value of u, then $\hat{u}(m) = u(a)$ for all $m \in \mathscr{F}_a$.

COROLLARY 3.0. *We always have*

$$\pi(\sigma_{\text{ext}}(T)) = \sigma(T). \tag{3.2}$$

Indeed, $\pi(\sigma_{\text{ext}}(T)) = \{z: z = u_1(m) \text{ where } m \in \sigma_{\text{ext}}(T)\}$ and $\hat{u}_1(\sigma_{\text{ext}}(T)) = \sigma(T)$ by definition of $u_1(z) \equiv z$ and (a). If $\sigma(T)$ reduces to a single point set $\{a\}$ and $|a| = 1$, then, by (3.2), $\sigma_{\text{ext}}(T) \subset \mathscr{F}_a$.

Let us remark also that since $\pi^{-1}(\Delta) = D$, (3.2) implies

$$\sigma_{\text{ext}}(T) \cap D = \pi^{-1}(\sigma(T) \cap \Delta). \tag{3.3}$$

It was proved by M. Schreiber in [4] that for T which unitary dilation has an absolutely spectrum

$$u(\sigma(T) \cap \Delta) \subset \sigma[u(T)] \quad \text{for} \quad u \in H^\infty. \tag{3.4}$$

For c.n.u. T we can prove this as follows. Let $z \in \sigma(T) \cap \Delta$. Then there is a unique $m \in \mathscr{M}(H^\infty)$ such that $u(z) = \hat{u}(m)$ for $u \in H^\infty$. For $u_1(z) \equiv z$ we get then $z = \pi(m)$ which implies $\pi^{-1}(z) = m$ because π is one-to-one on D. It follows then from (3.3) that $m \in \sigma_{\text{ext}}(T)$ and consequently $u(z) = \hat{u}(m) \in \hat{u}(\sigma_{\text{ext}}(T)) = \sigma[u(T)]$, q.e.d.

Since $\hat{u}(\sigma_{\text{ext}}(T)) \subset \hat{u}(\mathscr{M}(H^\infty)) = \overline{u(\Delta)}$, we may complete (3.4) by the inclusion $\sigma[u(T)] \subset \overline{u(\Delta)}$. Finally we obtain the following

COROLLARY 3.1 *For every completely non-unitary contraction T*

$$u(\sigma(T) \cap \Delta) \subset \sigma(u(T)) \subset \overline{u(\Delta)}.$$

COROLLARY 3.2. *Suppose now that the set of boundary points of Δ to which the given $u \in H^\infty$ is continuously extendable includes the set $\sigma(T) \cap C$ (C is the unit circle). Then*

$$\sigma[u(T)] = u[\sigma(T)], \tag{3.5}$$

that is, the classical spectral mapping theorem holds true.

In the proof we argue as follows:

$$\begin{aligned}\sigma_{\text{ext}}(T) &= (\sigma_{\text{ext}}(T) \cap D) \cup \bigcup_{|a|=1} \mathscr{F}_a \cap \sigma_{\text{ext}}(T) \\ &= \pi^{-1}(\sigma(T) \cap \Delta) \cup \bigcup_{|a|=1} (\mathscr{F}_a \cap \sigma_{\text{ext}}(T)).\end{aligned}$$

Since $\pi(\sigma_{\text{ext}}(T)) = \sigma(T)$, the intersection $\mathscr{F}_a \cap \sigma_{\text{ext}}(T)$ is non-empty iff $a \in \sigma(T)$. But in this case $\hat{u}(\mathscr{F}_a) = \{u(a)\}$ because u is continuously extendable to $\sigma(T) \cap C$. Hence

$$\begin{aligned}\sigma[u(T)] &= \hat{u}[\pi^{-1}(\sigma(T) \cap \Delta)] \cup \bigcup_{a \in \sigma(T) \cap C} \{u(a)\} \\ &= u[\sigma(T) \cap \Delta] \cup \bigcup_{a \in \sigma(T) \cap C} \{u(a)\} = u[\sigma(T)],\end{aligned}$$

which proves (3.5).

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