STUDIA MATHEMATICA, T. XXVI. (1966)

Pointwise convergence for parabolic singular integrals

by

E. B. FABES and C. SADOSKY (Chicago)

1. B. F. Jones Jr. introduced in [1] a class of convolution singular integrals of the form

$$\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(x-y,t-s)f(y,s)\,dyds, \qquad 0<t<\infty, \tag{1}$$

where $f(y,s)$ belongs to $L^p\big(E^n\times(0,\infty)\big)$, $1<p<\infty$, and $K(x,t)$ satisfies:

(i) $K(x,t)=0$ for $t<0$,

(ii) $K(\lambda x,\lambda^m t)=\lambda^{-n-m}K(x,t)$, λ any positive number and m a fixed positive number,

(iii) $\int_{E^n} K(x,1)\,dx=0$.

Under additional "smoothness" conditions on $K(x,1)$, Jones shows the existence of the above limit in the L^p-sense. At the conclusion he raises two questions, namely:

a) Under what conditions does the limit (1) exist pointwise almost everywhere in $E^n\times(0,\infty)$ and

b) If

$$f^*(x,t)=\operatorname*{Sup}_{\varepsilon>0}\left|\int_0^{t-\varepsilon}\int_{E^n} K(x-y,t-s)f(y,s)\,dyds\right|,$$

under what conditions do we have $\|f^*\|_p\leqslant C_p\|f\|_p$, $1<p<\infty$ (p^{th}-norms here are taken over $E^n\times(0,\infty)$).

This paper is devoted to the study and answering of these two questions. As will be seen, an affirmative answer to question (b) will immediately imply the same for (a).

2. Throughout this paper we will assume that $K(x,t)$ satisfies (i), (ii), and (iii) of the introduction, and in addition that

(iv) $|K(x,\ 1)|+|(\partial/\partial x_i)K(x,\ 1)| \leqslant \frac{C}{1+|x|^{n+2}}$, $i = 1, \ldots, n$, C an absolute constant.

From the "homogeneity" condition, (ii), we can write

$$K(x,t) = \Omega(x/t^{1/m})t^{-n/m-1}, \quad t > 0,$$

where $\Omega(x) = K(x,1)$.

The above conditions are sufficient to guarantee the L^p-convergence of (1), $1 < p < \infty$.

We will consider functions $f(y,s)$ defined on $E^n x(0,\infty)$ and extended to all of E^{n+1} by making $f(y,s) = 0$ for $s < 0$.

Set

$$K_\varepsilon(x,t) = \begin{cases} K(x,t) & \text{for} \quad t > \varepsilon > 0, \\ 0 & \text{for} \quad t \leqslant \varepsilon, \end{cases}$$

$$\tilde{f}_\varepsilon(x,t) = \int_0^{t-\varepsilon} \int_{E^n} K(x-y, t-s) f(y,s)\, dy\, ds = f * K_\varepsilon(x,t),$$

$$\tilde{f}(x,t) = \lim_{\varepsilon\to 0} \tilde{f}_\varepsilon(x,t)$$

(the convergence in L^p), and finally let

$$\psi(s) = \frac{1}{1+s^{n+2}}, \quad 0 \leqslant s < \infty.$$

We shall adopt the custom of letting C stand for an absolute positive constant, not necessarily the same at each occurrence, depending only on the dimension n, of E^n and p.

THEOREM 1. *Suppose* $f \in L^p(E^n \times (0,\infty))$, $1 < p < \infty$, $\tilde{f}^*(x,t) = \operatorname{Sup}_{\varepsilon>0} |\tilde{f}_\varepsilon(x,t)|$.

Then $\|\tilde{f}^*\|_p \leqslant C\|f\|_p$.

Proof. Let $H(y,s)$ be any function infinitely differentiable, with compact support in the cylinder, $\{(y,s)\,|y| < 1,\ s \in (\frac{1}{2}, 1)\}$, and such that $\int_{E^{n+1}} H(y,s)\, dy\, ds = 1$.

Set

$$\tilde{f}^\varepsilon(x,t) = \frac{1}{\varepsilon^{n+m}} \int_{E^{n+1}} H\big((x-y)\varepsilon, (t-s)/\varepsilon^m\big)\tilde{f}(y,s)\, dy\, ds.$$

$$\tilde{f}_{\varepsilon^m}(x,t) = \int_{E^{n+1}} K_{\varepsilon^m}(x-y, t-s) f(y,s)\, dy\, ds.$$

(2) $$|\tilde{f}^\varepsilon(x,t)| \leqslant \frac{C}{\varepsilon^{n+m}} \int_{t-s<\varepsilon^m} \int_{x-y<\varepsilon} |\tilde{f}(y,s)|\, dy\, ds.$$

Let I denote any rectangle in E^{n+1} with center (x,t) and set

$$\tilde{f}_*(x,t) = \operatorname{Sup}_I \frac{1}{|I|} \int_I |\tilde{f}(y,s)|\, dy\, ds.$$

From (2) it is clear that

$$\operatorname{Sup}_{\varepsilon>0} |\tilde{f}^\varepsilon(x,t)| \leqslant C\tilde{f}_*(x,t).$$

It is known that $\|\tilde{f}_*\|_p \leqslant C\|\tilde{f}\|_p$ (see [3]). Using the fact that the operation, $f \to \tilde{f}$, is a continuous operation on $L^p(E^n \times (0,\infty))$ we have

$$\|\operatorname{Sup}_{\varepsilon>0} |\tilde{f}^\varepsilon(x,t)|\|_p \leqslant C\|f\|_p.$$

So, to prove theorem 1 it will be sufficient to show that

(3) $$\|\operatorname{Sup}_{\varepsilon>0} |\tilde{f}^\varepsilon - \tilde{f}_{\varepsilon^m}|\|_p \leqslant C\|f\|_p.$$

$$\tilde{f}^\varepsilon(x,t) = \varepsilon^{-(n+m)} \int_{E^{n+1}} \big(H(x-y)/\varepsilon, (t-s)/\varepsilon^m\big)\big(\lim_{\mu\to 0} f * K_\mu\big)(y,s)\, dy\, ds$$

$$= \varepsilon^{-(n+m)} \lim_{\mu\to 0} \int_{E^{n+1}} \big(H(x-y)/\varepsilon, (t-s)/\varepsilon^m\big) \times$$

$$\times \Big\{ \int_{E^{n+1}} f(z,r) K_\mu(y-z, s-r)\, dz\, dr \Big\}\, dy\, ds.$$

Interchanging the order of integration,

$$\tilde{f}^\varepsilon(s,t)$$

$$= \varepsilon^{-(n+m)} \lim_{\mu\to 0} \int_{E^{n+1}} f(z,r) \Big\{ \int_{E^{n+1}} H\big((x-y)/\varepsilon, (t-s)/\varepsilon\big)(y-z, s-r)\, dy\, ds \Big\}\, dz\, dr.$$

Setting $W = (x-y)/\varepsilon$, $\omega = (t-s)/\varepsilon^m$, and noting that $K_\mu(\varepsilon x, \varepsilon^m t) = \varepsilon^{-(n+m)} K_{\mu/\varepsilon^m}(x,t)$, we have

$$\tilde{f}^\varepsilon(x,t) = \varepsilon^{-(n+m)} \lim_{\mu\to 0} \int_{E^{n+1}} f(z,r) \times$$

$$\times \Big\{ \int_{E^{n+1}} H(W,\omega) K_{\mu/\varepsilon^m}\big((x-y)/\varepsilon - W, (t-s)/\varepsilon^m - \omega\big)\, dW\, d\omega \Big\}\, dz\, dr$$

and finally,

$$\tilde{f}^\varepsilon(x,t) = \varepsilon^{-(n+m)} \int_{E^{n+1}} f(z,r) \tilde{H}\big((x-z)/\varepsilon, (t-s)/\varepsilon\big)\, dz\, dr.$$

Hence,

$$\tilde{f}^\varepsilon(x,t) - \tilde{f}_{\varepsilon^m}(x,t)$$

$$= \varepsilon^{-(n+m)} \int_{E^{n+1}} f(z,r) \Big\{ \tilde{H}\big((x-z)/\varepsilon, (t-s)/\varepsilon^m\big) - K_1\big((x-z)/\varepsilon, (t-r)/\varepsilon^m\big) \Big\}\, dz\, dr.$$

We now consider the properties of the kernel,

$$N(x,t) = \tilde{H}(x,t) - K_1(x,t).$$

(a) If $t \geqslant 2$, then

$$|N(x,t)| \leqslant C(1+t^{n/m+2})^{-1}\{\psi(|x|/2t^{1/m}) + |x|/t^{1/m}\psi(|x|/2t^{1/m})\}.$$

If $t \geqslant 2$,

$$N(x,t) = \int_{E^{n+1}} H(y,s)\{K(x-y,t-s)-K(x,t)\}\,dyds,$$

$$\begin{aligned} K(x-y,t-s)-K(x,t) &= [K(x-y,t-s)-K(x,t-s)]+[K(x,t-s)-K(x,t)] \\ &= \Omega\big((x-y)/(t-s)^{1/m}\big) - \Omega\big(x/(t-s)^{1/m}\big)(t-s)^{-(n/m+1)} + \\ &\quad + \Omega\big(x/(t-s)^{1/m}\big)(t-s)^{-(n/m+1)} - \Omega(x/t^{1/m})t^{-(n/m+1)}. \end{aligned}$$

Applying the mean-value theorem to each term in brackets and using the properties of Ω, we have

$$|N(x,t)| \leqslant C \int_{E^{n+1}} |H(y,s)|\psi\big(|x-\theta_1 y|/(t-s)^{1/m}\big)(t-s)^{-(n/m+2)} +$$
$$+\psi\big(|x|/(t-\theta_2 s)^{1/m}\big)(t-\theta_2 s)^{-(n/m+2)} + |x|\,\psi\big(|x|/(t-\theta_2 s)^{1/m}\big)(t-\theta_2 s)^{-n(1/m+1)-2},$$

$0<\theta_1<1$, $0<\theta_2<1$. Since $(t-s)>1$ and $|\theta_1 y|<1$ in the last integral, we can conclude that for $t \geqslant 2$

$$|N(x,t)| \leqslant C(1+t^{n/m+2})^{-1}\{\psi(|x|/2t^{1/m}) + |x|/t^{1/m}\psi(|x|/2t^{1/m})\}.$$

(b) If $0<t\leqslant 2$ and $|x|\geqslant 2$, then

$$|N(x,t)| \leqslant C\psi(|x|/t^{1/m})(1+t^{n/m+2})^{-1}.$$

If $0<t\leqslant \frac{1}{2}$, then $K_1(x,t)=0$ and similarly

$$\tilde{H}(x,t) = \lim_{\varepsilon\to 0}\tilde{H}_\varepsilon(x,t) = \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(x-y,t-s)H(y,s)\,dyds = 0$$

since $H(y,s)=0$ for $s<\frac{1}{2}$.

Assume, then, that $\frac{1}{2}<t\leqslant 2$. We get

$$\begin{aligned} |\tilde{H}_\varepsilon(x,t)| &\leqslant C\int_0^{t-\varepsilon}\int_{|y|\leqslant 1}\psi\big(|x-y|/(t-s)^{1/m}\big)(t-s)^{-(n/m+1)}\,dyds \\ &\leqslant C\int_0^t \psi(|x|/2s^{1/m})s^{-(n/m+1)}\,ds \\ &\leqslant C\psi(|x|/2t^{1/m}) \leqslant C\psi(|x|/t^{1/m}). \end{aligned}$$

Since $\tilde{H}_\varepsilon(x,t)$ tends point-wise to $\tilde{H}(x,t)$ for every $(x,t)f$ the same inequality holds for $\tilde{H}(x,t)$ and so (a) follows. With $0<t\leqslant 2$ we can conclude that for

$$|N(x,t)| \leqslant C\psi(|x|/t^{1/m})(1+t^{n/m+2})^{-1}, \qquad |x|\geqslant 2.$$

(c) $N(x,t)$ is bounded for $0<t\leqslant 2$ and $|x|<2$. $K_1(x,t)$ is clearly bounded,

$$\tilde{H}_\varepsilon(x,t) = \int_0^{t-\varepsilon}\int_{E^n} K(x-y,t-s)\{H(y,s)-H(x,t)\}\,dyds.$$

Using the mean-value theorem once again we have

$$\begin{aligned} |\tilde{H}_\varepsilon(x,t)| &\leqslant C\int_0^{t-\varepsilon}\int_{E^n}\Omega\big((x-y)/(t-s)^{1/m}(t-s)^{-(n/m+1)}\big)[|x-y|+(t-s)]\,dyds \\ &\leqslant C\int_0^2\int_{E^n}|\Omega(y)|\{s^{1/m}|y|+s\}s^{-1}\,dyds < \infty. \end{aligned}$$

Hence $|\tilde{H}(x,t)| \leqslant C$.

From (a), (b), (c), we have the following majorization of $N(x,t)$,

$$|N(x,t)| \leqslant C\{\psi(|x|/t^{1/m}) + |x|/t^{1/m}\psi(|x|/t^{1/m})\}(1+t^{n/m+2})^{-1} + B(x,t),$$

where $B(x,t)$ is a bounded function with compact support.

Recall that

$$\tilde{f}^\varepsilon(x,t) - \tilde{f}_{\varepsilon^m}(x,t) = \frac{1}{\varepsilon^{n+m}}\int_{E^{n+1}} f(x-z,t-r)N(z/\varepsilon, r/\varepsilon^m)\,dzdr.$$

Set

$$f_*(x,t) = \operatorname*{Sup}_{I_\varepsilon(x,t)}\frac{1}{I}\int_I |f(z,r)|\,dzdr.$$

To prove that

$$\|\operatorname*{Sup}_{\varepsilon>0}|\tilde{f}^\varepsilon - \tilde{f}_{\varepsilon^m}|\|_p \leqslant C\|f\|_p$$

it will be sufficient to show that each of the following functions are majorized by $Cf_*(x,t)$:

(i) $\operatorname*{Sup}_{\varepsilon>0}\varepsilon^m\int_0^\infty\int_{E^n}\psi(|z|/r^{1/m})(\varepsilon^{n+2m}+r^{n/m+2})^{-1}|f(x-z,t-r)|\,dzdr,$

(ii) $\operatorname*{Sup}_{\varepsilon>0}\varepsilon^{m-1}\int_0^\infty\int_{E^n}|z|\,\psi(|z|/r^{1/m})(\varepsilon^{n+2m}+r^{n/m+2})^{-1}|f(x-z,t-r)|\,dzdr,$

(iii) $\operatorname*{Sup}_{\varepsilon>0}\varepsilon^{-(n+m)}\int_0^\infty\int_{E^n}B(z/\varepsilon, r/\varepsilon^m)|f(x-z,t-r)|\,dzdr.$

Since $B(z, r)$ has finite support it is clear that (iii) $\leqslant Cf_*(x, t)$. This inequality for (i) and (ii) will result from application of the following lemma:

LEMMA. *Suppose* $\varphi(s) = (1+s^{n+\lambda})^{-1}$, $\lambda > 0$, *and set*

$$N(x, t) = \varphi(|x|/t^{1/m})(1+|t|^{n/m+2})^{-1}.$$

Then

$$\sup_{\varepsilon>0} \varepsilon^{-(n+m)} \int_{E^{n+1}} N(z/\varepsilon, r/\varepsilon^m)|f(x-z, t-r)|\,dzdr \leqslant Cf_*(x, t).$$

Proof. Set

$$I(\varrho) = \int_{|r|<\varrho}\int_{E^n} \varphi(|z|/|r|^{1/m})|f(x-z, t-r)|\,dzdr,$$

$$\varepsilon^{-(n+m)} \int_{E^{n+1}} N(z/\varepsilon, r/\varepsilon^m)|f(x-z, t-r)|\,dzdr$$

$$= \varepsilon^m \int_{E^{n+1}} \varphi(|z|/|r|^{1/m})(\varepsilon^{n+2m}+r^{n/m+2})^{-1}|f(x-z, t-r)|\,dzdr$$

$$= \varepsilon^m \int_0^\infty (\varepsilon^{n+2m}+\varrho^{n/m+2})^{-1} dI(\varrho).$$

Assume for the moment that

$$I(\varrho)\varrho^{-n/m-1} \leqslant Cf_*(x, t).$$

Then

$$\varepsilon^m \int_0^\infty (\varepsilon^{n+2m}+\varrho^{n/m+2}) dI(\varrho) \leqslant Cf_*(x, t)\int_0^\infty \varrho^{2n/m+2}(\varepsilon^{n+2m}+\varrho^{n/m+2})^{-2} d\varrho.$$

Setting $u = \varrho/\varepsilon^m$ we have

$$\varepsilon^m \int_0^\infty (\varepsilon^{n+2m}+\varrho^{n/m+2})^{-1} dI(\varrho) \leqslant Cf_*(x, t)\int_0^\infty \frac{u^{2n/m+2}}{(1+u^{n/m+2})^2}\,du \leqslant Cf_*(x, t).$$

Hence our proof will be complete once we show that

$$I(\varrho)\varrho^{-n/m-1} \leqslant Cf_*(x, t).$$

Set

$$G(\varrho, v) = \int_{|r|<\varrho}\int_{|z|<v} f(x-z, t-r)\,dzdr,$$

$$G(\varrho, v) \leqslant C\varrho|v|^n f_*(x, t),$$

$$I(\varrho)\varrho^{-n/m-1}$$

$$= \varrho^{-n/m-1}\Big\{\int_{|r|<\varrho}\int_{|z|<\varrho^{1/m}} + \int_{|r|<\varrho}\int_{|z|\geqslant\varrho^{1/m}}\Big\}\varphi(|z|/|r|^{1/m})|f(x-z, t-r)|\,dzdr.$$

Since φ is bounded,

$$\varrho^{-n/m-1}\int_{|r|<\varrho}\int_{|z|<\varrho^{1/m}} \varphi(|z|/|r|^{1/m}) f(x-z, t-r)\,dzdr$$

is majorized by a constant times $f_*(x, t)$.

$$\varrho^{-n/m-1}\int_{|r|<\varrho}\int_{|z|>\varrho^{1/m}} \varphi(|z|/|r|^{1/m})|f(x-z, t-r)|\,dzdr$$

$$\leqslant \varrho^{-n/m-1}\int_{|r|<\varrho}\int_{|z|>\varrho^{1/m}} \varphi(|z|/\varrho^{1/m})|f(x-z, t-r)|\,dzdr$$

$$= \varrho^{-n/m-1}\int_{\varrho^{1/m}}^\infty \varphi(v/\varrho^{1/m})\,dG(\varrho, v).$$

Integrating by parts we see that the last expression is equal to

$$-G(\varrho, \varrho^{1/m})\varrho^{-n/m-1} - \varrho^{-(n+1)/m-1}\int_{\varrho^{1/m}}^\infty G(\varrho, v)\varphi'(v/\varrho^{1/m})\,dv$$

$$\leqslant Cf_*(x, t)\int_{\varrho^{1/m}}^\infty \varrho^{\lambda/m} v^{2n+\lambda-1}(\varrho^{(n+\lambda)/m}+v^{n+\lambda})^{-2}\,dv$$

$$\leqslant Cf_*(x, t)\int_1^\infty v^{2n+\lambda-1}(1+v^{n+\lambda})^{-2}\,dv$$

$$\leqslant Cf_*(x, t).$$

Hence the proof of the lemma and therefore of Theorem 1 is complete.

THEOREM 2. *For* $f \in L^p(E^n\times(0, \infty))$, $1 < p < \infty$, *the*

$$\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{E^n} K(x-y, t-s)f(y, s)\,dyds$$

exists pointwise for almost every (x, t).

Proof. Set

$$\delta(x, t; f) = \limsup_{\varepsilon\to 0}\tilde f_\varepsilon(x, t) - \liminf_{\varepsilon\to 0}\tilde f_\varepsilon(x, t).$$

By theorem 1, $\|\delta(x, t; f)\|_p \leqslant C\|f\|_p$.

If $g(x, t)$ is any function infinitely differentiable and with compact support in $E^n \times (0, \infty)$, then $\delta(x, t; g) = 0$.

Therefore $\delta(x, t; f) = \delta(x, t; f-g)$. Selecting $g(x, t)$ so that $\|f-g\|_p$ is as small as we wish, it follows that $\|\delta(x, t; f)\|_p = 0$, and theorem 2 follows.

References

[1] B. F. Jones, Jr., *A class of singular integrals*, Amer. J. Math. 86, no. 2 (1964), p. 441-462.

[2] E. B. Fabes, *Parabolic singular integrals with functions in L^1*, Abstract 65T-55, Notices of the Amer. Math. Soc., 12 (1965), p. 142.

[3] A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge 1959.

THE UNIVERSITY OF CHICAGO

Reçu par la Rédaction le 1. 7. 1965

STUDIA MATHEMATICA, T. XXVI. (1966)

(F)-Räume mit absoluter Basis

von

A. PIETSCH (Berlin)

Eine Folge von Elementen $e_1, e_2, \ldots$ aus einem lokalkonvexen Raum E wird als *Basis* bezeichnet wenn sich jedes Element $x \in E$ mit einer eindeutig bestimmten Folge von Zahlen $\xi_1, \xi_2, \ldots$ in der Form

$$x = \sum_{n=1}^{\infty} \xi_n e_n$$

darstellen läßt, so daß die durch den Ansatz

$$\langle x, f_n \rangle = \xi_n$$

definierten Linearformen f_n stetig sind (1).

Eine Basis $\{e_n\}$ heißt *absolut*, wenn für jede stetige Halbnorm p und alle $x \in E$ die Ungleichung

$$\sum_{n=1}^{\infty} |\langle x, f_n \rangle| p(e_n) < +\infty$$

besteht. Die Existenz einer absoluten Basis hat für (F)-Räume weitreichende Konsequenzen, weil man den linearen Raum der zu den Elementen gehörigen Koeffizientenfolgen sehr einfach beschreiben und topologisieren kann. Man darf sich deshalb auf die Betrachtung von gewissen Folgenräumen beschränken. Insbesondere zeigt sich, daß alle Banachräume mit absoluter Basis zu dem Folgenraum l^1 isomorph sind.

Es erhebt sich folglich die Frage, ob es überhaupt (F)-Räume mit absoluter Basis gibt, deren starker topologischer Dual ebenfalls eine absolute Basis besitzt. Als Hauptergebnis der vorliegenden Arbeit werden wir beweisen, daß dieser Sachverhalt gerade für nukleare Räume eintritt.

Eine Einführung in die von A. Grothendieck [4] begründete Theorie der nuklearen lokalkonvexen Räume findet man in meinem Buch [7], aus dem auch die Bezeichnungen und Definitionen übernommen werden.

(1) In (F)-Räumen sind die Linearformen f_n automatisch stetig. Vgl. [2].