[9] — *On modular spaces of strongly summable sequences*, Studia Math. 22 (1962), p. 127-146.

[10] W. Orlicz, *A note on modular spaces I*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 9 (1961), p. 157-162.

[11] — *On some spaces of strongly summable sequences*, Studia Math. 22 (1963), p. 331-336.

[12] — *A note on modular spaces VII*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 12 (1964), p. 305-309.

[13] — *On the convergence of norms in spaces of φ-integrable functions*, ibidem 13 (1965), p. 205-210.

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Extensions of sequentially continuous linear functionals in inductive sequences of $(\mathscr{F})$-spaces

par

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1. Introduction [1]. These investigations was inspired a long time ago by a problem communicated to the author by L. Ehrenpreis. The problem concerned extensibility of sequentially continuous linear functionals defined on subspaces of Schwartz's spaces $\mathscr{D}(\Omega)$ of infinitely differentiable functions with compact carriers contained in a fixed domain Ω (cf. [3]). It can be easily verified that distributions from the domain of a partial differential operator on $\mathscr{D}'(\Omega)$ can always be considered as extensions of sequentially continuous functionals defined on the range of the adjoint differential operator acting on $\mathscr{D}(\Omega)$. Hence, it becomes apparent that a necessary and sufficient condition for existence of such extensions must be closely connected with any set of conditions that are necessary and sufficient for the operator to map onto $\mathscr{D}'(\Omega)$. For convolution operators, including as a particular case differential operators with constant coefficients, such a set of conditions was given by Hörmander in [2].

Going one step further in generality, call $(\mathscr{IF})$-sequence any sequence $\mathfrak{X}$ of $(\mathscr{F})$-spaces such that every linear space from $\mathfrak{X}$ is a subspace of the subsequent linear space from the sequence and that the identical injection of every $(\mathscr{F})$-space from $\mathfrak{X}$ into the following one is continuous (cf. [12]).

Situation that necessitates using such a notion arises, for instance, when we discuss factor spaces of the Schwartz's $(\mathscr{D}, \tau_{\mathscr{D}})$ space. Such factor spaces need not be $(\mathscr{LF})$-spaces any more though they always naturally decompose into $(\mathscr{IF})$-sequences.

Let X denote the union of linear spaces from an $(\mathscr{IF})$-sequence $\mathfrak{X}$. A linear functional defined on a linear subspace of X is called *sequentially continuous* if it is continuous in every $(\mathscr{F})$-space from $\mathfrak{X}$. We formulate the general problem of extension as follows.

Given an $(\mathscr{IF})$-sequence $\mathfrak{X}$ find a natural condition for a linear subspace X_0 of X defined above which is necessary and sufficient

[1] A substantial part of the results presented here was obtained when the author was at the Institute for Advanced Study in Princeton on the NSF Grant G-14600.

for every linear sequentially continuous functional on X_0 to admit an extension to a sequentially continuous functional defined on the whole X.

First results of such sort were published by Ehrenpreis [1] who proved bornologicity of the relative topologies of some special subspaces of $\mathcal{D}$. This is very close to the way in which Theorem 2.1′ of this paper treats the problem of extension.

Conditions presented in this paper use some topological properties to describe subspaces with the extension property by means of so called *good location* in the whole space. Some preliminary results concerning good location (though the name was not yet introduced) was already reported in [4, 5, 6].

This paper does not use the standard notions and methods of the general theory of linear locally convex topological spaces and this is because the author does not know about any way of expressing the condition for good location given here in terms of that theory.

Hovever, there are some statements in this paper, suitable for translation on the language of locally convex spaces. Such statements are Theorems 2.1 and 4.1 and their translations, mere technicalities, are Theorems 2.1′ and 4.1′ respectively. It seems that not much more can be done in this direction.

To make the paper more comprehensible we start with the particular case describing the general results in the case of Schwartz's spaces $\mathcal{D}$. These are Theorems 2.1, 2.2 and 2.1′ of this paper which has been announced in [1]. As the next comes an example which provides the reader with an intuition in regard to the good location and question of extensibility of linear functionals. This are Propositions 3.1 and 3.2 of this paper which have been announced in [8].

The example given here is not the first one to that effect. The already mentioned paper of Hörmander [2] yields a full spectrum of functionals not extensible to distributions. To produce such functionals, it is sufficient to take a distribution S with compact carrier and any S-convex pair of domains (Ω_1, Ω_2) which is not strongly S-convex. Then, the image $S^*\mathcal{D}(\Omega_1)$ admits sequentially continuous functionals with no extensions to the whole $\mathcal{D}(\Omega_2)$.

In the next step, Theorems 2.1, 2.2 and 2.1′ are expressed for arbitrary strict $(\mathcal{IF})$-sequences as Theorems 4.1, 4.2 and 4.1′. The announcement of this stage of the theory was given in [9]. The final step gives the most general form of the extension theorem in the case of arbitrary $(\mathcal{IF})$-sequences. This is Theorem 5.3 in this paper. To provide the missing links between Theorems 4.1 and 4.2 and Theorem 5.3 there is proved Theorem 5.1 which brings some facts verifiable probably only in the $(\mathcal{LF})$ case as they were proved. All of it was announced in [10]. It is [13] where some of results obtained here will find important applications.

Finally, the importance of the out-of-theory facts used here should not be neglected. These are some statements taken from [11] and [12]. Actually, the temptation for independent discussion of these facts in [11,12] caused a considerable delay in the final publication of the results exposed in this paper. Being more precise, Proposition 5.6 of this paper supporting all kind of necessery conditions for existence of extensions is verified here by use of a version of the open-mapping theorem introduced in [12]. It would be interesting to find whether a parallel necessary condition follows from the Pták's version of the open-mapping theorem [14]. In the following, the proof of Proposition 5.2 of this paper is derived from Proposition 12 of [11] which is nothing but another version of the well-known Banach reasoning that leads from nearly open to open mappings.

I wish to express my sincere gratitude to Dr A. Wiweger who was so kind as to read the manuscript and point out several inaccuracies to the author.

2. Extensions of sequentially continuous linear functionals to distributions. Consider the N-dimensional Euclidean space R_N. To any subset G of R_N assign the linear space $\mathscr{C}(G)$ of all continuous complex-valued functions defined on G. By D_i, $i = 1, 2, \ldots, N$, we denote the operations of partial derivation and for $p = (p_1, \ldots, p_N)$, where p_i are non-negative integers, we write D^p for $D_1^{p_1} D_2^{p_2} \ldots D_N^{p_N}$, i. e. the superposition of the partial differentiation. The number $|p| = p_1 + \ldots + p_N$ denotes as usually the rank of differentiation. For open G we write

$$\mathscr{E}(G) \overset{\mathrm{df}}{=} \{f \in \mathscr{C}(G) : D^p f \text{ exists and is in } \mathscr{C}(G) \text{ for every } p\}.$$

Compact subsets of R_N with the closure of the interior identical with the original subset will be called *compact domains*. For a compact domain $K \subset R_N$ we write

$$\mathscr{C}^n(K) = \{f \in \mathscr{C}(K) : (D^p f)(t) \text{ exists and is uniformly continuous for } t \in \operatorname{Int} K \text{ and every } p \text{ with } |p| \leqslant n\}.$$

Further, let

$$\|f\|_K \overset{\mathrm{df}}{=} \sup\{|f(t)| : t \in K\},$$

for $f \in \mathscr{C}(K)$ and

$$\|f\|_K^n \overset{\mathrm{df}}{=} \sum_{|p| \leqslant n} \|D^p f\|_K$$

for $f \in \mathscr{C}^n(K)$, where $D^p f$ denotes the extension of $(D^p f)(t)$, $t \in \operatorname{Int} K$, over the whole K.

Every $(\mathscr{C}^n(K), \|\cdot\|_K^n)$ is a Banach space.

Fix an open subset Ω of R_N. A sequence $\mathfrak{K} = \{K_n\}$ of subsets of Ω is said to be a *special cover* of Ω iff $\{K_n\}$ is locally finite and the sequence $\{\mathrm{Int}\, K_n\}$ is a cover of Ω.

Denote by $\mathrm{supp}\, f$ the carrier of the function f. Let

$$\mathscr{D} \overset{\mathrm{df}}{=} \{f \in \mathscr{E}(\Omega) \colon \mathrm{supp}\, f \text{ is compact}\}.$$

Further, for a fixed special cover $\mathfrak{K} = \{K_n\}$ of Ω, where all K_n are compact domains, we write

$$\mathscr{D}_n = \{f \in \mathscr{D} \colon \ \mathrm{supp}\, f \subset \bigcup_{i=1}^{n} K_i\}.$$

Clearly,

$$\mathscr{D} = \bigcup_{n=1}^{\infty} \mathscr{D}_n.$$

Denote by τ_n the topology of $\mathscr{D}_n$ (topology of uniform convergence with all derivatives). This convergence is induced by pseudonorms $\{\|\cdot\|_{K_i}^{h} \colon 1 \leqslant h < \infty,\ 1 \leqslant i \leqslant n\}$.

It is obvious that every $(\mathscr{D}_n, \tau_n)$ is an $(\mathscr{F})$-space. The sequence

$$\mathfrak{D} = \{(\mathscr{D}_n, \tau_n)\}$$

will be called a *decomposing sequence* of $(\mathscr{D}, \tau_{\mathscr{D}})$, where $\tau_{\mathscr{D}}$ denotes the usual $(\mathscr{L}\mathscr{F})$-topology of $\mathscr{D}$ (cf. [3]).

Denote by $\mathscr{N}$ the family of all increasing sequences of natural numbers. For $\mathfrak{k}, \mathfrak{h} \in \mathscr{N}$, $\mathfrak{k} = \{k_n\}$, $\mathfrak{h} = \{h_n\}$, we write $\mathfrak{h} \leqslant \mathfrak{k}$ whenever $h_n \leqslant k_n$ for every n. Let for $\mathfrak{h} = \{h_n\} \in \mathscr{N}$ (2)

$$\mathscr{D}^{\mathfrak{h}} = \{f \in C(\Omega) \colon \ \mathrm{supp}\, f \text{ is compact},\ f_{|K_n} \in \mathscr{C}^{h_n}(K_n) \text{ for every } n\}.$$

Here $f_{|K}$ denotes the restriction of f to K. Further

$$\|f\|^{\mathfrak{h}} \overset{\mathrm{df}}{=} \sum_{n=1}^{\infty} \|f\|_{K_n}^{h_n}$$

for $f \in \mathscr{D}^{\mathfrak{h}}$. In view of the local finiteness of $\mathfrak{K}$ the sum on the right-hand side is always finite.

We define

$$\mathscr{D}_n^{\mathfrak{h}} = \{f \in \mathscr{D}^{\mathfrak{h}} \colon \ \mathrm{supp} f \subset \bigcup_{i=1}^{n} K_i\} \quad \text{and} \quad \|f\|_n^{\mathfrak{h}} = \sum_{i=1}^{n} \|f\|_{K_i}^{h_i} = \|f\|^{\mathfrak{h}}$$

for $f \in \mathscr{D}_n^{\mathfrak{h}}$.

(2) Small gotic letters are used here to denote elements of $\mathscr{N}$ which in [4]-[11] were bold faced.

This way, with every $\mathfrak{h} \in \mathscr{N}$ there is associated a sequence of Banach spaces

$$\mathfrak{D}^{\mathfrak{h}} = \{(\mathscr{D}_n^{\mathfrak{h}}, \|\cdot\|_n^{\mathfrak{h}})\}.$$

To complete the list of objects necessary for presentation of the results concerning existence and behaviour of extensions to the whole $\mathscr{D}$ of linear functionals defined on subspaces of $\mathscr{D}$, it is necessary to add one more type of space to the already defined collection.

Put

$$\mathscr{D}^{\mathfrak{h},m} = \Big\{f \in \mathscr{D}^{\mathfrak{h}} \colon f_{|\Omega - \bigcup_{i=1}^{m} K_i} \in \mathscr{E}\big(\Omega - \bigcup_{i=1}^{m} K_i\big)\Big\}$$

and

$$\mathscr{D}_n^{\mathfrak{h},m} = \Big\{f \in \mathscr{D}^{\mathfrak{h},m} \colon \ \mathrm{supp} f \subset \bigcup_{i=1}^{n} K_i\Big\}.$$

We provide each $\mathscr{D}_n^{\mathfrak{h},m}$ with the topology $\tau_n^{\mathfrak{h},m}$ of the uniform convergence with all derivatives on $\Omega - \bigcup_{i=1}^{m} K_i$ and the convergence with the norm $\|\cdot\|^{\mathfrak{h}}$ simultaneously.

The sequences of $(\mathscr{F})$-spaces

$$\mathfrak{D}^{\mathfrak{h},m} \overset{\mathrm{df}}{=} \{(\mathscr{D}_n^{\mathfrak{h},m}, \tau_n^{\mathfrak{h},m})\}, \qquad m = 1, 2, \ldots,$$

are called *moderations* of $\mathfrak{D}$ by $\mathfrak{D}^{\mathfrak{h}}$.

Consider a linear subspace $\mathscr{D}_0$ of $\mathscr{D}$. We say that $\mathscr{D}_0$ is *good located* in $\mathscr{D}$ iff the following condition holds:

To every $\mathfrak{k} \in \mathscr{N}$ there correspond $\mathfrak{h} \in \mathscr{N}$ with $\mathfrak{k} \leqslant \mathfrak{h}$ such that to every k there correspond m such that

$$\mathscr{D}_k^{\mathfrak{h}} \cap \mathrm{Closure}_1(\mathscr{D}_0 \cap \mathscr{D}_p) \subset \mathrm{Closure}_2(\mathscr{D}_0 \cap \mathscr{D}_m)$$

for every p, where the first closure is taken in $(\mathscr{D}_p^{\mathfrak{h},k}, \tau_p^{\mathfrak{h},k})$ and the second in $(\mathscr{D}_m^{\mathfrak{k}}, \|\cdot\|_m^{\mathfrak{k}})$.

PROPOSITION 2.1. *The definition of the good location of subspace in $\mathscr{D}$ does not depend on the choice of the special cover $\mathfrak{K}$ of Ω.*

This Proposition is an immediate consequence of the coming Theorem 2.1.

Consider the sequence $\mathfrak{D}$. A sequence $\{f_n\} \subset \mathscr{D}$ is said to be *convergent* to $f \in \mathscr{D}$ in $\mathfrak{D}$ iff $\{f_n - f\}$ is contained in some $\mathscr{D}_m$ and tends to zero in $(\mathscr{D}_m, \tau_m)$.

A linear functional F defined on a subspace $\mathscr{D}_0$ of $\mathscr{D}$ is said to be *sequentially continuous* iff $\{Ff_n\}$ tends to zero for every $\{f_n\} \subset \mathscr{D}_0$ which is convergent to zero in $\mathfrak{D}$.

According to the generally accepted terminology every sequentially continuous functional defined on the whole $\mathscr{D}$ is called *a distribution*.

The space $\mathscr{D}'$ of such functionals coincide with the space of distributions over Ω in the sense of Schwartz.

THEOREM 2.1[3]. *Consider a subspace $\mathscr{D}_0$ of $\mathscr{D}$. Every sequentially continuous linear functional defined on $\mathscr{D}_0$ admits an extension to a distribution if and only if $\mathscr{D}_0$ is good located in $\mathscr{D}$.*

THEOREM 2.2. *If every sequentially continuous functional defined on a subspace $\mathscr{D}_0$ of $\mathscr{D}$ admits an extension to a distribution, then to every $\mathfrak{k}\in\mathscr{N}$ there correspond $\mathfrak{h}\in\mathscr{N}$, $\mathfrak{h}\geqslant\mathfrak{k}$, and to every n there correspond m and $\eta>0$ (depending on k) such that every sequentially continuous functional defined on $\mathscr{D}_0$ with the restriction to $\mathscr{D}_0\cap\mathscr{D}_m$ having the $\|\cdot\|_m^{\mathfrak{k}}$-norm smaller than η, admits an extension to a distribution which has the restriction to $\mathscr{D}_n$ having the $\|\cdot\|_n^{\mathfrak{h}}$-norm smaller than one.*

Proof of Proposition 2.1. The good location of a $\mathscr{D}_0\subset\mathscr{D}$ is, by virtue of Theorem 2.1, equivalent to the fact that every sequentially continuous functional on $\mathscr{D}_0$ admits an extension to a distribution. Since this last fact does not depend on the choice of the cover $\mathfrak{R}$, the same must be true about the good location which proves the Proposition.

The proofs of Theorems 2.1 and 2.2 will be derived later from more general Theorems 4.1 and 4.2.

We can provide Theorem 2.1 with a certain equivalent formulation based on the notion of relative topology.

THEOREM 2.1'. *A subspace $\mathscr{D}_0$ of $\mathscr{D}$ is good located in $\mathscr{D}$ if and only if every sequentially continuous functional defined on $\mathscr{D}_0$ is continuous in the relative topology of $\mathscr{D}$ induced by the usual $(\mathscr{LF})$-topology $\tau_{\mathscr{D}}$ of $\mathscr{D}$.*

To establish the equivalence of Theorems 2.1 and 2.1' we notice two following facts. First, if a functional defined on $\mathscr{D}_0$ is continuous in the relative topology of $\mathscr{D}_0$, the existence of the extension of the functional to a distribution is merely an application of the Hahn-Banach Theorem and, secondly, if it is known that a functional admits an extension to a distribution, it must be necessarily continuous in the relative topology of $\mathscr{D}_0$.

Once these facts are noticed, it becomes apparent that Theorem 2.1 and Theorem 2.1' are just only different formulations of the same fact.

3. An example. Before starting with a general theory of extensions consider a simple but iluminating example.

Let Ω be the open interval $(-1,1)$. To every k we assign a function $g_k^0\in\mathscr{C}^k(R)$ with $\operatorname{supp} g_k^0\subset(2^{-k-1},2^{-k})$ such that the $k+1$-th derivative of g_k^0 does not exist.

Take $\varphi\in\mathscr{D}(R)$, $\varphi\geqslant 0$, $\operatorname{supp}\varphi=[-1,1]$ and $\int\varphi(t)dt=1$ and put $\varphi_\varepsilon(t)=\varepsilon^{-1}\varphi(t/\varepsilon)$. Then, for $\varepsilon\to 0$, $\varphi_\varepsilon * g_k^0$ tends to g_k^0 uniformly with derivatives up to the order k. Let $\varepsilon_n\to 0$, monotone, be such that $\operatorname{supp} g_{k,m}^0\subset(2^{-k-1},2^{-k})$, where $g_{k,m}^0\overset{\mathrm{df}}{=}\varphi_{\varepsilon_m}*g_k^0\in\mathscr{D}(R)$. For any fixed k we define the scalar product

$$(g,h)_k=\sum_{i=1}^{k}\int D^i g(t)\overline{D^i h(t)}\,dt\quad\text{for}\quad g,h\in\mathscr{C}^k(R)$$

with compact carriers.

Let $\{h_{k,m}^0\}$ be the Hilbert-Schmidt orthonormalization of $\{g_{k,m}^0\colon m=1,2,\ldots\}$. Since the uniform convergence with derivatives up to the order k is stronger than the $\|\cdot\|_k$-convergence, where $\|g\|_k\overset{\mathrm{df}}{=}(g,g)_k^{1/2}$, we find that there exists a sequence $\{t_{k,m}^0\}$ such that

$$\sum_{m=1}^{\infty}|t_{k,m}^0|^2<\infty$$

and that

$$g_k^0=\sum_{m=1}^{\infty}t_{k,m}^0h_{k,m}^0$$

in the topology of $\|\cdot\|_k$. Dropping some $g_{k,m}^0$ if necessary, we can have $t_{k,m}^0\neq 0$ for every k and m.

Now, to every (k,m) we assign a function $f_{k,m}^0\in\mathscr{D}(R)$ such that $\operatorname{supp} f_{k,m}^0\subset(1-2^{-k}+2^{-k-m-1},1-2^{-k}+2^{-k-m})$ and that the carrier of $f_{k,m}^0$ is non-void.

There always exists a sequence $\{s_{k,m}\}$, $s_{k,m}>0$, such that for every p, $\sup\{\|f_{k,m}^0\|_p/s_{k,m}\colon m=1,2,\ldots\}$ is finite. Hence, substituting if necessary $\min(t_{k,m}^0,t_{k,m+1}^0)f_{k,m}^0/s_{k,m}$ for $f_{k,m}^0$, it can always be made

$$r_{p,k}\overset{\mathrm{df}}{=}\sup\{\|f_{k,m}^0\|_p/t_{k,m+i}^0\colon i=0,1;\ m=1,2,\ldots\}<\infty$$

for every p.

Let $\mathscr{H}_k$ consists of all elements of the completion of $(\mathscr{C}^k(R),\|\cdot\|_k)$ which have carriers contained in $(-1,1)$. Define

$$\mathscr{H}_{1,k}=\{h\in\mathscr{H}_k\colon h=\sum_{m=1}^{\infty}t_{k,m}h_{k,m}^0,\ \sum_{m=1}^{\infty}|t_{k,m}|^2<\infty\}$$

and

$$\mathscr{H}_{2,k}=\{h\in\mathscr{H}_k\colon h=\sum_{m=1}^{\infty}t_{k,m}f_{k,m},\ \sum_{m=1}^{\infty}|t_{k,m}|^2<\infty\},$$

where $f_{k,m}=(f_{k,m}^0-f_{k,m-1}^0)/t_{k,m}^0$ for $m=1,2,\ldots$ and $f_{k,0}^0=0$.

The carriers of the elements of $\mathscr{H}_{1,k}$ are all contained in $(2^{-k-1},2^{-k})$ and those of the elements of $\mathscr{H}_{2,k}$ are contained in $(1-2^{-k},1-2^{-k-1})$. Hence,

(3) This is an improved version of Proposition 1 of [6].

$\mathscr{H}_{1,k}$ and $\mathscr{H}_{2,k}$ are orthogonal in $(\mathscr{H}_k, \|\cdot\|_k)$. We have

$$\sum_{m=1}^{\infty} t_{k,m} f_{k,m} = \sum_{m=1}^{\infty} (t_{k,m}/t^0_{k,m} - t_{k,m+1}/t^0_{k,m+1}) f^0_{k,m}$$

and then

$$\Big\|\sum_{m=1}^{\infty} t_{k,m} f_{k,m}\Big\|_p^2 = \sum_{m=1}^{\infty} |t_{k,m}/t^0_{k,m} - t_{k,m+1}/t^0_{k,m+1}|^2 \|f^0_{k,m}\|_p^2$$

$$\leqslant r^2_{p,k} \sum_{m=1}^{\infty} (|t_{k,m}| + |t_{k,m+1}|)^2$$

$$\leqslant 2r^2_{p,k} \sum_{m=1}^{\infty} (|t_{k,m}|^2 + 2r^2_{p,k} \sum_{m=1}^{\infty} |t_{k,m}||t_{k,m+1}| \leqslant 4r^2_{p,k} \sum_{m=1}^{\infty} |t_{k,m}|^2.$$

Hence, we have proved the following statement:

LEMMA 3.1. *For every* $f = \sum_{m=1}^{\infty} t_{k,m} f_{k,m} \in \mathscr{H}_{2,k}$ *we have*

$$\|f\|_p^2 \leqslant 4r^2_{p,k} \sum_{m=1}^{\infty} |t_{k,m}|^2.$$

From Lemma 3.1 it follows that every $f \in \mathscr{H}_{2,k}$ is infinitely differentiable (4).

Consider an operator F_k from $\mathscr{H}_{1,k}$ into $\mathscr{H}_{2,k}$ defined as follows

$$F_k h \overset{\text{df}}{=} \sum_{m=1}^{\infty} t_{k,m} f_{k,m},$$

where $h = \sum_{m=1}^{\infty} t_{k,m} h^0_{k,m} \in \mathscr{H}_{1,k}$.

LEMMA 3.2. *The mapping* F_k *is continuous from* $(\mathscr{H}_{1,k}, \|\cdot\|_k)$ *into* $(\mathscr{H}_{2,k}, \tau_{2,k})$, *where* $\tau_{2,k}$ *denotes the topology of uniform convergence with all derivatives.*

Proof. If $\|h_n - h\|_k$ tends to zero, then $\sum_{m=1}^{\infty} |t^n_{k,m} - t_{k,m}|^2$ tends to zero for n tending to infinity, where $h_n = \sum_{m=1}^{\infty} t^n_{k,m} h^0_{k,m}$. The estimation of Lemma 3.1 gives $\|F_k h_n - F_k h\|_p \to 0$ for $n \to \infty$, for every p, and the Lemma follows.

(4) It is well known that in spaces $\mathscr{D}(K)$, K being a compact domain, simultaneous convergence with pseudonorms $\sum_{|p|\leqslant n} \|D^p f\|_r$, $n = 1, 2, \ldots$, is the same for every L^r-norm $\|\cdot\|_r$ for $1 < r < \infty$.

Define

$$\mathscr{H}_{0,k} = \{h + F_k h \in \mathscr{H}_k:\ h \in \mathscr{H}_{1,k}\}.$$

It is clear that the carriers of elements of $\mathscr{H}_{0,k}$ are all contained in $I_k = (2^{-k-1}, 2^{-k}) \cup (1-2^{-k}, 1-2^{-k-1})$.

LEMMA 3.3. *The space* $(\mathscr{H}_{0,k}, \|\cdot\|_k)$ *is complete, i. e. is a Hilbert space.*

Proof. Indeed, if $\{h_n + F_k h_n\}$ satisfies the Cauchy condition, then, in view of Lemmas 3.1 and 3.2 the sequence $\{F_k h_n\}$ must tend to $F_k h$ for $h = \lim h_n$ in $(\mathscr{H}_{1,k}, \|\cdot\|_k)$ which concludes the proof.

LEMMA 3.4. *If for some* $g \in \mathscr{H}_{0,k}$ *the intersection* $(\operatorname{supp} g) \cap (1-2^{-k}, 1-2^{-k-1})$ *is void, then for some* t_0 *there is* $t_{k,m} = t_0 t^0_{k,m}$, *where* $g = h + F_k h$ *and*

$$h = \sum_{m=1}^{\infty} t_{k,m} h^0_{k,m} \in \mathscr{H}_{1,k}.$$

Proof. The intersection $(\operatorname{supp} g) \cap (1-2^{-k}, 1-2^{-k-}$ is void iff $h = g \in \mathscr{H}_{1,k}$, i. e. iff $F_k h = F_k g = 0$. Then, it must be

$$\sum_{m=1}^{\infty} t_{k,m} f_{k,m} = 0$$

which gives

$$\sum_{m=1}^{\infty} (t_{k,m}/t^0_{k,m} - t_{k,m+1}/t^0_{k,m+1}) f^0_{k,m} = 0$$

and this holds only for $t_{k,m}/t^0_{k,m} = t_{k,m+1}/t^0_{k,m+1} = t_0$ for $m = 1, 2, \ldots$ The proof is finished.

For a subset U of a linear space X, $[U]$ denotes the subspace of X spanned by U. Elements of $\mathscr{D}(\Omega)$ will be identified here with elements of $\mathscr{D}(R)$ with carriers in Ω.

Define

$$\mathscr{H} = \Big[\bigcup_{k=1}^{\infty} \mathscr{H}_{0,k}\Big], \qquad \mathscr{D}_0 = \mathscr{H} \cap \mathscr{D}(\Omega).$$

PROPOSITION 3.1. *The subspace* $\mathscr{D}_0$ *of* $\mathscr{D}(\Omega)$ *is not good located in* $\mathscr{D}(\Omega)$.

Proof. Consider an interval $K_1 = [-1/2, 2/3]$ and let $\{K_n : n = 2, 3, \ldots\}$ be a sequence of compact subsets of $(-1, 1)$ such that $\mathfrak{K} = \{K_n\}$ is a special cover of $(-1, 1)$. Fix any $\mathfrak{k} \in \mathscr{N}$. To contradict the good location of $\mathscr{D}_0$ in $\mathscr{D}$ it is sufficient to show that no matter how great is a natural number m and no matter what an $\mathfrak{h} \geqslant \mathfrak{k}$, $\mathfrak{h} = \{h_n\} \in \mathscr{N}$, is taken, there always exists $g_m \in \mathscr{D}_1^{\mathfrak{h}}$ which is the limit in $\tau_p^{\mathfrak{h},1}$-topology with properly adjusted p of a sequence of elements of $\mathscr{D}_0$, such that there is no sequence in $\mathscr{D}_0$ that tends to g_m in $(\mathscr{D}^{\mathfrak{k}}_m, \|\cdot\|^{\mathfrak{k}}_m)$.

Take k_m such that $(1-2^{-k_m}, 1-2^{-k_m-1}) \cap \bigcup_{i=1}^{m} K_i$ is void and that the $\|\cdot\|_{k_m}$-convergence implies the uniform convergence with derivatives up to the order h_1. Put $g_m = g^0_{k_m}$. Then,

$$g_m = \sum_{n=1}^{\infty} t^0_{k_m,n}(h^0_{k_m,n}+f_{k_m,n})$$

and the convergence on K_1 must be uniform with derivatives up to the order h_1.

It remains to prove that no sequence from $\mathscr{D}_0$ with carriers in $\bigcup_{i=1}^{m} K_i$ can be convergent to g_m in $(\mathscr{D}^{\mathfrak{k}}_m, \|\cdot\|^{\mathfrak{k}}_m)$. Indeed, if $\{g_{m,n}: n=1,2,\ldots\} \subset \subset \mathscr{D}_0 \cap \mathscr{D}^{\mathfrak{k}}_m$ converges in $(\mathscr{D}^{\mathfrak{k}}_m, \|\cdot\|^{\mathfrak{k}}_m)$ to g_m, then it must tend to g_m in the pointwise sense on $(2^{-k_m-1}, 2^{-k_m})$. Hence, almost all $g_{m,n}$, $n=1,2,\ldots$, do not vanish identically on $(2^{-k_m-1}, 2^{-k_m})$ and by virtue of Lemma 3.4 the set $(\operatorname{supp} g_{m,n}) \cap (1-2^{-k_m}, 1-2^{-k_m-1})$ must be non-void. Hence, having $(1-2^{-k_m}, 1-2^{k_m-1})$ disjoint with $\bigcup_{i=1}^{m} K_i$, we find that almost all $g_{m,n}$, $n=1,2,\ldots$, have carriers not contained in $\bigcup_{i=1}^{m} K_i$ which finishes the proof.

Proposition 3.2. *The subspace $\mathscr{D}_0$ is closed in the usual $(\mathscr{LF})$-topology of $\mathscr{D}(-1,1)$.*

Proof. Take f_0 from the closure of $\mathscr{D}_0$ in the $(\mathscr{LF})$-topology of $\mathscr{D}(-1,1)$. Since f_0 admits a compact carrier in $(-1,1)$, there must be k_0 such that $\operatorname{supp} f_0 \subset (-1+2^{-k_0}, 1-2^{-k_0})$. Furthermore, f_0 must be the pointwise limit of a net of elements of $\mathscr{D}_0$. Since every element of $\mathscr{D}_0$ vanishes on $(-1,0)$, so must do f_0 and the inclusion for the carrier of f_0 strengthens as follows: $\operatorname{supp} f_0 \subset [0, 1-2^{-k_0}]$.

As the element of the closure of $\mathscr{D}_0$, f_0 must be the $(\mathscr{LF})$-topology limit of a net of elements of $\mathscr{D}_0$. Then, having every $\|\cdot\|_k$ continuous in the $(\mathscr{LF})$-topology of $\mathscr{D}(-1,1)$, we can pick a subsequence $\{f_n\}$ of the net, tending to f_0 with respect to all norms $\|\cdot\|_k$, $k=1,2,\ldots$

As it was done before, put

$$I_k = (2^{-k-1}, 2^{-k}) \cup (1-2^{-k}, 1-2^{-k-1})$$

and write for any h defined on R

$$h_{(k)}(t) = \begin{cases} h(t) & \text{for} \quad t \in I_k, \\ 0 & \text{for} \quad t \in R-I_k. \end{cases}$$

Since $\{f_n\}$ tends to f_0 with respect to every $\|\cdot\|_k$ and $f_{n(k)} \in \mathscr{H}_{0,k}$ for every k, we have $f_{0(k)} \in \mathscr{H}_{0,k}$ for $k=1,2,\ldots$ Since the simultaneous $\|\cdot\|_k$-convergence is uniform with all derivatives, every $f_{0(k)} \in \mathscr{D}(-1,1)$ and

$$\sum_{i=1}^{k_0} f_{0(i)} \in \mathscr{D}_0.$$

Consider

$$f_1 = f_0 - \sum_{i=1}^{k_0} f_{0(i)}.$$

Having

$$\sum_{i=1}^{k_0} f_{0(i)} \in \mathscr{D}_0,$$

it is sufficient to show that $f_1 = 0$. The carrier of f_1 is disjoint with $(1/2,1)$ and $\|f_1 - g_n\|_k \to 0$ for every k, where

$$g_n = f_n - \sum_{i=1}^{k_0} f_{0(i)}$$

and then $f_{1(k)} \in \mathscr{H}_{0,k}$ for $k=1,2,\ldots$ Thus, by virtue of Lemma 3.4 and the fact that $F_k f_{1(k)} = 0$ and $f_{1(k)} \neq g^0_k$, we obtain $f_{1(k)} = 0$ for every k. Hence, finally, $f_1 = 0$ and the Proposition follows.

In the case of the presented example of not good located subspace $\mathscr{D}_0$ of $\mathscr{D}(\Omega)$ it is easy to construct directly a sequentially continuous functional on $\mathscr{D}_0$ which does not admit any extension to a distribution.

The identical imbeddings of $(\mathscr{H}_{1,k}, \|\cdot\|_{k+1})$ into $(\mathscr{H}_{1,k}, \|\cdot\|_k)$ are continuous but not bicontinuous so we can produce functionals φ_k defined on $\mathscr{D}(R) \cap \mathscr{H}_{1,k}$, continuous in $(\mathscr{D}(R) \cap \mathscr{H}_{1,k}, \|\cdot\|_{k+1})$ and not continuous in $(\mathscr{D}(R) \cap \mathscr{H}_{1,k}, \|\cdot\|_k)$ respectively. Define φ over $\mathscr{D}_0$ setting

$$\varphi(h) = \sum_{k=1}^{\infty} \varphi_k(h_k),$$

where

$$h = \sum_{k=1}^{\infty} (h_k + F_k h_k), \qquad h_k \in \mathscr{H}_{1,k} \quad \text{for} \quad k=1,2,\ldots$$

and almost all h_k vanish identically. If $\{h_n\} \subset \mathscr{D}_0$ tends to some limit in $\mathscr{D}_0$ in the sense of the convergence in $\mathscr{D}(\Omega)$, then there exists k_0 such that $\operatorname{supp} h_n \subset (2^{-k_0}, 1-2^{-k_0})$ for all n and then $\lim \varphi(h_n) = \varphi(\lim h_n)$ which means that φ is sequentially continuous.

Suppose that φ admits a continuous extension $\tilde{\varphi}$ to the whole $\mathscr{D}(\Omega)$. In $[0,1/2]$ the distribution $\tilde{\varphi}$ must be of the finite order, i. e. there exists k_0 such that $\tilde{\varphi}$ is continuous with respect to $\|\cdot\|_{k_0}$ on

$$\Big[\bigcup_{k=1}^{\infty} \mathscr{H}_{1,k}\Big] \cap \mathscr{D}(\Omega).$$

Suppose that $\{h_n\} \subset \mathscr{H}_{1,k_0} \frown \mathscr{D}(\Omega)$ is $\|\cdot\|_{k_0}$-convergent to zero and that simultaneously $\{\varphi_{k_0}(h_n)\}$ is not bounded. We have

$$|\varphi_{k_0}(h_n)| = |\varphi(h_n + F_{k_0}h_n)| \leqslant |\tilde{\varphi}(h_n)| + |\tilde{\varphi}(F_{k_0}h_n)| \leqslant M\|h_n\|_{k_0} + |\tilde{\varphi}(F_{k_0}h_n)|.$$

Now, since $\{h_n\}$ is $\|\cdot\|_{k_0}$-convergent to zero, it follows from Lemma 3.2 that $\{F_{k_0}h_n\}$ tends to zero uniformly with all derivatives and then $\lim \varphi_{k_0}(h_n) = 0$ in spite of the unboundedness of this sequence. Hence the functional φ can not be extended over the whole $\mathscr{D}(\Omega)$ to a distribution.

4. Extensions in $(\mathscr{L}\mathscr{F})$-spaces. In this paper it is often written briefly space, subspace, functional, while it always means linear space, linear subspace and linear functional. In particular, spaces of functionals are always considered provided with pointwise operations. Linear topological spaces are written as pairs (U, τ), where U is the space and τ is the topology. If U is provided with a pseudonorm $\|\cdot\|$ that induces the topology of U, then we write $(U, \|\cdot\|)$ for this linear topological space calling it a *pseudonormed space*.

If (U, τ) is a linear topological space, in particular a pseudonormed space, $|(U, \tau)'|$ denotes the linear space adjoint to (U, τ), i. e. the space of all continuous linear functionals defined on (U, τ).

Following [11] we write $(U_1, \tau_1) \geqslant (U_2, \tau_2)$ iff U_1 is a subspace of U_2 and the identical injection of (U_1, τ_1) into (U_2, τ_2) is continuous.

We shall accept a certain special notation for sequences of linear topological spaces. If $\mathfrak{U} = \{(U_n, \tau_n)\}$ is a sequence of linear topological spaces, then we write for every n

$$|\mathfrak{U}|_n \overset{\text{df}}{=} U_n, \quad \tau_{\mathfrak{U},n} \overset{\text{df}}{=} \tau_n.$$

If in particular (U_n, τ_n) are pseudonormed spaces, the pseudonorm inducing the topology τ_n is denoted by $\|\cdot\|_{\mathfrak{U},n}$ for $n = 1, 2, \ldots$ respectively. Hence, for sequences of pseudonormed spaces $\mathfrak{U}$ we have the double notation — $\tau_{\mathfrak{U},n}$ for the topologies and $\|\cdot\|_{\mathfrak{U},n}$ for the pseudonorms that induce the topologies $\tau_{\mathfrak{U},n}$ respectively.

As in [12] an *inductive sequence* $\mathfrak{U}$ is a sequence of linear topological spaces such that $(|\mathfrak{U}|, \tau_{\mathfrak{U},n}) \geqslant (|\mathfrak{U}|_{n+1}, \tau_{\mathfrak{U},n+1})$ for every n. We write

$$|\mathfrak{U}| \overset{\text{df}}{=} \bigcup_n |\mathfrak{U}|_n.$$

An inductive sequence $\mathfrak{U}$ is called *strict* iff for every n the identical mapping of $(|\mathfrak{U}|_n, \tau_{\mathfrak{U},n})$ into $(|\mathfrak{U}|_{n+1}, \tau_{\mathfrak{U},n+1})$ is bicontinuous; $\mathfrak{U}$ is called complete (Banach, reflexive Banach) sequence iff all $(|\mathfrak{U}|_n, \tau_{\mathfrak{U},n})$ are complete (Banach, reflexive Banach) spaces. The relation

$$\mathfrak{U} \leqslant (U, \tau)$$

between an inductive sequence $\mathfrak{U}$ and a linear topological space (U, τ) means that $(|\mathfrak{U}|_n, \tau_{\mathfrak{U},n}) \leqslant (U, \tau)$ for at least one n. For inductive sequences we write

$$\mathfrak{U}_1 \leqslant \mathfrak{U}_2$$

iff $\mathfrak{U}_1 \leqslant (|\mathfrak{U}_2|_n, \tau_{\mathfrak{U}_2,n})$ for every n.

If $\mathfrak{U}_1 \leqslant \mathfrak{U}_2$ and $\mathfrak{U}_2 \leqslant \mathfrak{U}_1$, then sequences $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are called *equivalent*.

Consider an inductive sequence $\mathfrak{X}$. A sequence $\{x_n\} \subset |\mathfrak{X}|$ is said to be *convergent* to $x \in |\mathfrak{X}|$ iff $\{x_n - x\} \subset |\mathfrak{X}|_m$ for some fixed m and $\{x_n - x\}$ tends to zero in $(|\mathfrak{X}|_m, \tau_{\mathfrak{X},m})$. A functional z' defined on a subspace X_0 of $|\mathfrak{X}|$ is said to be *sequentially continuous* on X_0 iff for every $\{z_n\} \subset X_0$ which is convergent to zero, $\{z'z_n\}$ tends to zero too.

The *adjoint space* to $\mathfrak{X}$ is the space of all sequentially continuous functionals defined on the whole $|\mathfrak{X}|$.

An inductive sequence of $(\mathscr{F})$-spaces is called an $(\mathscr{I}\mathscr{F})$-*sequence*; a strict $(\mathscr{I}\mathscr{F})$-sequence is called an $(\mathscr{L}\mathscr{F})$-*sequence*.

If $\mathfrak{X}$ is an inductive sequence and X_0 is a subspace of $|\mathfrak{X}|$, then we define the inductive sequence $X_0 \frown \mathfrak{X}$ setting:

$|X_0 \frown \mathfrak{X}|_n \overset{\text{df}}{=} X_0 \frown |\mathfrak{X}|_n$;

$\tau_{X_0 \frown \mathfrak{X},n} \overset{\text{df}}{=}$ the topology induced on $|X_0 \frown \mathfrak{X}|_n$ by $\tau_{\mathfrak{X},n}$.

Similarly, we define the inductive sequence $X_0 \overline{\frown} \mathfrak{X}$ setting:

$|X_0 \overline{\frown} \mathfrak{X}|_n \overset{\text{df}}{=}$ the closure of $X_0 \frown |\mathfrak{X}|_n$ in $(|\mathfrak{X}|_n, \tau_{\mathfrak{X},n})$;

$\tau_{X_0 \overline{\frown} \mathfrak{X},n} \overset{\text{df}}{=}$ the topology induced on $|X_0 \overline{\frown} \mathfrak{X}|_n$ by $\tau_{\mathfrak{X},n}$.

Consider a linear space X. An inductive sequence of pseudonormed spaces $\mathfrak{Z}$ is called a *covering* of X iff X is a subspace of $\mathfrak{Z}$ and $X \overline{\frown} \mathfrak{Z} \leqslant \mathfrak{Z}$. Since always $\mathfrak{Z} \leqslant X \overline{\frown} \mathfrak{Z}$, the sequences $X \overline{\frown} \mathfrak{Z}$ and $\mathfrak{Z}$ are equivalent. If $\mathfrak{X}$ is an $(\mathscr{I}\mathscr{F})$-sequence, then $\mathfrak{Z}$ is called a *covering of* $\mathfrak{X}$ iff $\mathfrak{Z}$ is a covering of $|\mathfrak{X}|$ and $\mathfrak{X} \geqslant \mathfrak{Z}$.

Consider a pseudonormed space $(U, \|\cdot\|)$. To every functional v' defined on a linear space V for which the intersection $U \frown V$ is uniquely defined we assign a number $\|v'\|^*$ called the *polar pseudonorm* of v' induced by $(U, \|\cdot\|)$, setting

$$\|v'\|^* = \sup\{|v'x| \colon x \in U \frown V, \|x\| < 1\}.$$

Here $\|\cdot\|^*$ may assume the value $+\infty$.

Though $\|\cdot\|^*$ is a function of the pseudonormed space $(U, \|\cdot\|)$ as variable, its value depends on the way of forming the intersection $U \frown V$. Hence, before $\|v'\|^*$ is considered, both U and V must be placed within the same linear space as subspaces. If $\mathfrak{Z}$ is a sequence of pseudonormed spaces, then $\|\cdot\|^*_{\mathfrak{Z},n}$ denotes the polar pseudonorm induced by $(\|\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n})$ respectively. If $\mathfrak{Z}$ is a covering of an $(\mathscr{I}\mathscr{F})$-sequence $\mathfrak{X}$, we produce a *polar* $\mathfrak{Z}^*$ of $\mathfrak{Z}$ setting

$$|\mathfrak{Z}^*|_n \overset{\text{df}}{=} \{x' \in X' \colon \|x'\|^*_{\mathfrak{Z},n} < \infty\},$$

where X' denotes the adjoint to $\mathfrak{X}$. Clearly, $\mathfrak{Z}^*$ is an $(\mathscr{F})$-sequence (cf. [11], Example III).

Consider an $(\mathscr{I}\mathscr{F})$-sequence $\mathfrak{X}$. A family Ξ of coverings of $\mathfrak{X}$ is said to be a *basis* of coverings of $\mathfrak{X}$ iff the following conditions hold:

1. To every covering $\mathfrak{Z}$ of $\mathfrak{X}$ there correspond a covering $\mathfrak{S} \in \Xi$ such that $|\mathfrak{X}| \frown \mathfrak{S} \geqslant \mathfrak{Z}$.

2. To every $\mathfrak{Z}_1, \mathfrak{Z}_2 \in \Xi$ there corresponds $\mathfrak{Z}_3 \in \Xi$ such that $\mathfrak{Z}_i \leqslant \mathfrak{Z}_3$ for $i = 1, 2$.

The definition of basis as given above is a little too strong to cover some interesting situations that can arise in the theory outlined in this paper. However, such situations can be met only in the non-$(\mathscr{L}\mathscr{F})$-case. Since discussion of $(\mathscr{L}\mathscr{F})$-sequences is what we are here mainly interested in, the subtlety of the alternate definition of basis is left over to another publication.

There is a simple construction that will be of use in the further investigations. Consider a linear space V and linear topological spaces (U_i, τ_i), $i = 1, 2, \ldots, n$, where all U_i are subspaces of V. Define the linear topological space

$$(U, \tau) = (U_1, \tau_1) \wedge \ldots \wedge (U_n, \tau_n)$$

as follows. Let $U \overset{\mathrm{df}}{=} U_1 + \ldots + U_n$. Further, consider the product space $(U_1 \times \ldots \times U_n, \tau_1 \times \ldots \times \tau_n)$ and a subspace

$$\boldsymbol{L} = \{(u_1, \ldots, u_n) \in U_1 \times \ldots \times U_n \colon u_1 + \ldots + u_n = 0\}.$$

Identify $U_1 \times \ldots \times U_n / \boldsymbol{L}$ with $U_1 + \ldots + U_n$ by the one-to-one isomorphism I, where $I\big((u_1, \ldots, u_n)/\boldsymbol{L}\big) \overset{\mathrm{df}}{=} u_1 + \ldots + u_n$. Denote by $\tau_1 \wedge \ldots \wedge \tau_n$ the topology of $U_1 + \ldots + U_n$ induced by the topology $\tau_1 \times \ldots \times \tau_n / \boldsymbol{L}$ of $U_1 \times \ldots \times U_n / \boldsymbol{L}$ using the above defined identification. We set $\tau \overset{\mathrm{df}}{=} \tau_1 \wedge \ldots \wedge \tau_n$ which completes the definition of (U, τ). In the case, when topologies τ_1 up to τ_n are all given by means of pseudonorms $\|\cdot\|_1, \ldots, \|\cdot\|_n$ respectively, it is convenient to have $\tau_1 \wedge \ldots \wedge \tau_n$ induced by a pseudonorm. Here we shall always take as such pseudonorm the pseudonorm $\|\cdot\| = \|\cdot\|_1 \wedge \ldots \wedge \|\cdot\|_n$ equal to the quotient pseudonorm $\|\cdot\| \overset{\mathrm{df}}{=} |||\cdot|||/\boldsymbol{L}$ with $|||(u_1, \ldots, u_n)||| \overset{\mathrm{df}}{=} \|u_1\|_1 + \ldots + \|u_n\|_n$.

PROPOSITION 4.1. *If (U_0, τ_0) is a linear topological Hausdorff space and (U_i, τ_i), $i = 1, 2, \ldots, n$, are $(\mathscr{F})$-spaces such that $(U_0, \tau_0) \leqslant (U_i, \tau_i)$ for $i = 1, \ldots, n$, then $(U, \tau) \overset{\mathrm{df}}{=} (U_1, \tau_1) \wedge \ldots \wedge (U_n, \tau_n)$ is an $(\mathscr{F})$-space and for any locally convex space (X, ϱ), if $(X, \varrho) \leqslant (U_i, \tau_i)$ for $i = 1, 2, \ldots, n$, then $(X, \varrho) \leqslant (U, \tau)$ as well.*

Proof. The subspace $\boldsymbol{L}$ introduced in the definition of (U, τ), is closed. Indeed, if $\{(u_{1,m}, \ldots, u_{n,m}) \colon m = 1, 2, \ldots\} \subset \boldsymbol{L}$ tends to some $(u_1, \ldots, u_n) \in U_1 \times \ldots \times U_n$ for $m \to \infty$, then every $\{u_{i,m}\}$ tends to u_1 in (U_0, τ_0) for $i = 1, 2, \ldots, n$ and having $u_{1,m} + \ldots + u_{n,m} = 0$, we obtain $u_1 + \ldots + u_n = 0$ and then $(u_1, \ldots, u_n) \in \boldsymbol{L}$. The product $(U_1 \times \ldots \times U_n, \tau_1 \times \ldots \times \tau_n)$ is an $(\mathscr{F})$-space and then, the quotient space (U, τ) must also be an $(\mathscr{F})$-space. To prove the second part of the Proposition it is sufficient to notice that the continuity of a pseudonorm in every (U_i, τ_i) amounts to its continuity in (U, τ). This way the Proposition is fully proved.

Take an $(\mathscr{I}\mathscr{F})$-sequence $\mathfrak{X}$ and a pseudonormed space $(S, \|\cdot\|)$. If for every n the space $(S, \|\cdot\|) \wedge (|\mathfrak{X}|_n, \tau_{\mathfrak{X},n})$ is well defined, then we write

$$(S, \|\cdot\|) \wedge \mathfrak{X} \overset{\mathrm{df}}{=} \{(S, \|\cdot\|) \wedge (|\mathfrak{X}|_n, \tau_{\mathfrak{X},n})\}.$$

Take a covering $\mathfrak{S}$ of $\mathfrak{X}$. We define the inductive sequence $\mathfrak{X}_{\mathfrak{S},n}$ setting

$$\mathfrak{X}_{\mathfrak{S},n} \overset{\mathrm{df}}{=} (|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n}) \wedge \mathfrak{X}.$$

Here the operation $\wedge$ is well defined in view of the inclusion $(|\mathfrak{X}|_k \cup |\mathfrak{S}|_i) \subset |\mathfrak{S}|$ that holds for every k and i.

A sequence $\{\mathfrak{X}_n\}$ of inductive sequences is called a *sequence of moderations* of $\mathfrak{X}$ by $\mathfrak{S}$ iff for some $\{m_n\}$ there is $\mathfrak{X}_{\mathfrak{S},m_n} \leqslant \mathfrak{X}_n$ and $\mathfrak{X}_{m_n} \leqslant \mathfrak{X}_{\mathfrak{S},n}$ for $n = 1, 2, \ldots$

Suppose now that we are provided with two coverings $\mathfrak{Z}$ and $\mathfrak{S}$ of $\mathfrak{X}$ and a subspace X_0 of $|\mathfrak{X}|$. Assume that $\mathfrak{Z} \leqslant \mathfrak{S}$.

The triplet $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the *Accessibility Property*, briefly the *(ACC)-Property*, iff the following condition holds:

(ACC) There exists a sequence $\{\mathfrak{X}_n\}$ of moderations of $\mathfrak{X}$ by $\mathfrak{S}$ such that to every n there correspond m for which the inclusion

$$|\mathfrak{S}|_n \frown |X_0 \overline{\frown} \mathfrak{X}_n|_p \subset |X_0 \overline{\frown} \mathfrak{Z}|_m$$

holds for every $p = 1, 2, \ldots$

Notice, that in Condition (ACC) there can always be substituted the special sequence of moderations, namely $\{\mathfrak{X}_{\mathfrak{S},n}\}$, and it will not alter the condition. Indeed, if $\mathfrak{X}_{\mathfrak{S},m_n} \leqslant \mathfrak{X}_n$ and $\mathfrak{X}_{m_n} \leqslant \mathfrak{X}_{\mathfrak{S},n}$, then for some $\{k_n\}$ and every p and n

$$|\mathfrak{S}|_n \frown |X_0 \overline{\frown} \mathfrak{X}_n|_p \subset |\mathfrak{S}|_{m_n} \frown |X_0 \overline{\frown} \mathfrak{X}_{\mathfrak{S},m_n}|_{k_p}$$

and

$$|\mathfrak{S}|_n \frown |X_0 \frown \mathfrak{X}_{\mathfrak{S},n}|_p \subset |\mathfrak{S}|_{m_n} \frown |X_0 \frown \mathfrak{X}_{m_n}|_{k_p}.$$

Hence, Condition (ACC) expressed with $\{\mathfrak{X}_{\mathfrak{S},n}\}$, applied to the first inclusion produces (ACC) expressed with $\{\mathfrak{X}_n\}$. Similarly, (ACC) with $\{\mathfrak{X}_n\}$ applied to the second inclusion produces (ACC) with $\{\mathfrak{X}_{\mathfrak{S},n}\}$. Therefore, we have proved the following result:

Proposition 4.2. *The validity of* (ACC) *for a given triplet* $(X_0, \mathfrak{Z}, \mathfrak{S})$ *does not depend on the choice of a sequence of moderations of* $\mathfrak{X}$ *by* $\mathfrak{S}$.

As to the meaning of the Accessibility Property, Condition (ACC) amounts to the fact that the elements of $|\mathfrak{S}|_n$ belonging at the same time to the closure of $X_0 \frown (|\mathfrak{X}|_p + |\mathfrak{S}|_n)$ in $(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n}) \wedge (|\mathfrak{X}|_p, \tau_{\mathfrak{X},p})$ are accessible within the closure of $X_0 \frown |\mathfrak{Z}|_n$ in $(|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})$, where m is assigned to n and stays the same no matter how great is p.

Let $\mathfrak{X}$ be an $(\mathscr{IF})$-sequence, Ξ — a basis of coverings of $\mathfrak{X}$ and X_0 — a subspace of $|\mathfrak{X}|$. Assume that to every $\mathfrak{Z} \in \Xi$ there correspond $\{p_n\}$ with $|\mathfrak{Z}|_n \frown |\mathfrak{X}| \subset |\mathfrak{X}|_{p_n}$ for $n = 1, 2, \ldots$

The subspace X_0 is said to be *well located* in $|\mathfrak{X}|$ with respect to Ξ iff the following condition holds:

To every $\mathfrak{Z} \in \Xi$ there correspond $\mathfrak{S} \in \Xi$, $\mathfrak{S} \geqslant \mathfrak{Z}$, such that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the Accessibility Property.

A basis Ξ of coverings of $\mathfrak{X}$ is said to admit *reflexive majorizations* iff to every $\mathfrak{Z} \in \Xi$ there correspond a reflexive Banach covering $\mathfrak{S}$ such that $\mathfrak{Z} \leqslant \mathfrak{S}$.

Theorem 4.1 ([5]). *Consider an* $(\mathscr{LF})$*-sequence* $\mathfrak{X}$ *and a basis* Ξ *of coverings of* $\mathfrak{X}$. *Let in the following* X_0 *be a subspace of* $|\mathfrak{X}|$.

In order that every sequentially continuous functional defined on X_0 *admit an extension to a sequentially continuous functional defined on the whole* $|\mathfrak{X}|$ *it is necessary that* X_0 *be good located in* $|\mathfrak{X}|$ *with respect to* Ξ.

If, in addition, Ξ *admits reflexive majorizations, then the good location of* X_0 *in* $|\mathfrak{X}|$ *with respect to* Ξ *is sufficient for every sequentially continuous functional defined on* X_0 *to admit an extension to a sequentially continuous functional defined on the whole* $|\mathfrak{X}|$.

Theorem 4.2. *Consider an* $(\mathscr{IF})$*-sequence* $\mathfrak{X}$ *and a subspace* X_0 *of* $|\mathfrak{X}|$. *If every sequentially continuous functional defined on* X_0 *admits an extension to a sequentially continuous functional defined on the whole* $|\mathfrak{X}|$, *then to every covering* $\mathfrak{Z}$ *of* $\mathfrak{X}$ *there corresponds a covering* $\mathfrak{S}$ *of* $\mathfrak{X}$, $\mathfrak{S} \geqslant \mathfrak{Z}$, *such that the following condition holds.*

To every n *there correspond* m *and* $\eta > 0$ *such that for every sequentially continuous* x_0' *defined on* X_0 *if* $\|x_0'\|^*_{\mathfrak{Z},m} < \eta$, *then there exists a sequentially continuous extension* x' *of* x_0' *to the whole* $|\mathfrak{X}|$ *such that* $\|x'\|^*_{\mathfrak{S},n} < 1$.

We shall postpone the proofs of Theorems 4.1 and 4.2 to the end of the paper when they easily follow from other results of the theory. However, it is only natural to show at once why Theorems 2.1 and 2.2 are particular cases of Theorems 4.1 and 4.2 respectively.

Proof of Theorem 2.1. It is clear that $\mathfrak{D} = \{(\mathscr{D}_n, \tau_n)\}$ is an $(\mathscr{LF})$-sequence. Consider the following four statements:

([5]) This is an improved version of Proposition 2 of [4].

a) *The family* $\Xi = \{\tilde{\mathfrak{D}}^{\mathfrak{h}} \colon \mathfrak{h} \in \mathscr{N}\}$, *where* $\tilde{\mathfrak{D}}^{\mathfrak{h}} \overset{\mathrm{df}}{=} \mathscr{D} \,\overline{\frown}\, \mathfrak{D}^{\mathfrak{h}}$, *form a basis of coverings in* $\mathfrak{D}$.

b) *To every* $\mathfrak{D}^{\mathfrak{h}}$ *there corresponds a reflexive inductive Banach sequence* $\mathfrak{Z}$ *such that* $\mathfrak{Z} \geqslant \tilde{\mathfrak{D}}^{\mathfrak{h}}$.

c) *For every* $\mathfrak{h} \in \mathscr{N}$ *the sequence* $\{\tilde{\mathfrak{D}}^{\mathfrak{h},m}\}$, *where* $\tilde{\mathfrak{D}}^{\mathfrak{h},m} \overset{\mathrm{df}}{=} \mathscr{D} \,\overline{\frown}\, \mathfrak{D}^{\mathfrak{h},m}$, *presents moderations of* $\mathfrak{D}$ *by* $\tilde{\mathfrak{D}}^{\mathfrak{h}}$.

d) *In the definition of the good location of a subspace* $\mathscr{D}_0$ *of* $\mathscr{D}$, *used in the Theorem 2.1, it can be substituted* $\tilde{\mathscr{D}}^{\mathfrak{h}}_k$ *for* $\mathscr{D}^{\mathfrak{h}}_k$ *and the closures* 1 *and* 2 *can be taken in* $(\tilde{\mathscr{D}}^{\mathfrak{h},k}_p, \tau^{\mathfrak{h},k}_p)$ *and* $(\tilde{\mathscr{D}}^{\mathfrak{k}}_m, \|\cdot\|^{\mathfrak{k}}_m)$ *instead of* $(\mathscr{D}^{\mathfrak{h},k}_p, \tau^{\mathfrak{h},k}_p)$ *and* $(\mathscr{D}^{\mathfrak{k}}_m, \|\cdot\|^{\mathfrak{k}}_m)$ *respectively which leads to an equivalent condition. Here* $\tilde{\mathscr{D}}^{\mathfrak{h},k}_p \overset{\mathrm{df}}{=} |\tilde{\mathfrak{D}}^{\mathfrak{h},k}|_p$ *and* $\tilde{\mathscr{D}}^{\mathfrak{k}}_m \overset{\mathrm{df}}{=} |\tilde{\mathfrak{D}}^{\mathfrak{k}}|_m$.

Theorem 2.1 will trivially follow from Theorem 4.1 if we only notice that by virtue of statements a, c and d, the definition of good location of $\mathscr{D}_0$ in $\mathscr{D}$ used in Theorem 2.1 is a particular case of the general definition of good location of $\mathscr{D}_0$ in $\mathscr{D}$ with respect to a basis of coverings, in this case the basis $\{\tilde{\mathfrak{D}}^{\mathfrak{h}} : \mathfrak{h} \in \mathscr{N}\}$, and, that in view of the statement b the requirements of Theorem 4.1 are fulfilled. Now, we turn to proving the statements a-d. Fix a special cover $\mathfrak{K} = \{K_n\}$ of Ω, where K_n are compact domains.

Ad a. Let $\{e_n\} \subset \mathscr{D}$ be a partition of unit that correspond to the cover $\mathfrak{K}$, i. e. let $\operatorname{supp} e_n \subset \operatorname{Int} K_n$ and $\sum_{n=1}^{\infty} e_n = 1$. Take any covering $\mathfrak{Z}$ of $\mathfrak{D}$. Without loss of generality we may assume that $|\mathfrak{Z}|_n = \mathscr{D}_n$ and that $\|\cdot\|_{\mathfrak{Z},n+1}$ coincide with $\|\cdot\|_{\mathfrak{Z},n}$ on $\mathscr{D}_n$. Hence, we can put $\|f\| \overset{\mathrm{df}}{=} \|f\|_{\mathfrak{Z},n}$ for $f \in \mathscr{D}_n$, $n = 1, 2, \ldots$, and the pseudonorm $\|\cdot\|$ is continuous in every $(\mathscr{D}_n, \tau_n)$. Therefore, there are sequences $\mathfrak{h} = \{h_n\} \in \mathscr{N}$ and $\{M_n\}$ such that $\|f\| \leqslant M_n \|f\|^{h_n}_{K_n}$ for $f \in \mathscr{D}$ with $\operatorname{supp} f$ contained in $\operatorname{Int} K_n$. Further, we have

$$f = \sum_{n=1}^{\infty} e_n f$$

and then, assuming that $\{M_n\}$ is monotone non-decreasing, we have

$$\|f\| \leqslant M_n \sum_{i=1}^{n} \|e_i f\|^{h_i}_{K_i}$$

for $f \in \mathscr{D}_n$, $n = 1, 2, \ldots$

Since $\|e_i f\|^{h_i}_{K_i} \leqslant N_i \|f\|^{h_i}_{K_i}$ for every $i = 1, 2, \ldots$, where N_i depends on e_i, we have

$$\|f\| \leqslant M_n N_n \sum_{i=1}^{n} \|f\|^{h_i}_{K_i} = M_n N_n \|f\|^{\mathfrak{h}}_n$$

provided $\{N_n\}$ is taken monotone non-decreasing. This proves that $\tilde{\mathfrak{D}}^{\mathfrak{h}} \geqslant 3$.

Ad b. Define

$$(f, g)_K = \int_K f(t)\overline{g(t)}\,dt, \qquad |||f|||_K = (f, f)^{1/2}$$

$$\text{and} \qquad |||f|||_K^n = \sum_{|p| \leqslant n} |||D^p f|||_K,$$

where K is a compact domain contained in Ω.

It is well known that for sufficiently great n the space $(\mathscr{C}^n(K), |||\cdot|||_K^n)$ can be completed within $\mathscr{C}(K)$. Let $(\mathscr{H}^n(K), |||\cdot|||_K^n)$ be such a completion. Furthermore, it is known that to every n there correspond m_n such that $(\mathscr{H}^{m_n}(K), |||\cdot|||_K^{m_n}) \geqslant (\mathscr{C}^n(K), \|\cdot\|_K^n)$. Setting

$$\mathscr{H}^{\mathfrak{h}} \overset{\mathrm{df}}{=} \{f \in \mathscr{C}(\Omega)\colon \operatorname{supp} f \text{ is compact}, f_{|K_n} \in \mathscr{H}^{h_n}(K_n) \text{ for all } n\},$$

$$\mathscr{H}_n^{\mathfrak{h}} \overset{\mathrm{df}}{=} \{f \in \mathscr{H}^{\mathfrak{h}}\colon \operatorname{supp} f \subset \bigcup_{i=1}^{n} K_i\},$$

$$|||f|||^{\mathfrak{h}} \overset{\mathrm{df}}{=} \sum_{i=1}^{\infty} |||f|||_{K_i}^{h_i}$$

for $f \in \mathscr{H}^{\mathfrak{h}}$,

$$|||f|||_n^{\mathfrak{h}} \overset{\mathrm{df}}{=} \sum_{i=1}^{n} |||f|||_{K_i}^{h_i} = |||f|||^{\mathfrak{h}}$$

for $f \in \mathscr{H}_n^{\mathfrak{h}}$, and finally

$$\mathfrak{H}^{\mathfrak{h}} = \{(\mathscr{H}_n^{\mathfrak{h}}, |||\cdot|||_n^{\mathfrak{h}})\},$$

we conclude that to every $\mathfrak{k} \in \mathscr{N}$ one can assign $\mathfrak{h} \in \mathscr{N}$ such that $\mathfrak{D}^{\mathfrak{k}} \leqslant \mathfrak{H}^{\mathfrak{h}}$. This proves b.

Ad c. Put $\tilde{\mathscr{D}}_p^{\mathfrak{h}} \overset{\mathrm{df}}{=} |\tilde{\mathfrak{D}}^{\mathfrak{h}}|_p$ and write briefly $\mathfrak{D}_2^{\mathfrak{h},m}$ for $\mathfrak{D}_{\mathfrak{H}^{\mathfrak{h}},m}$. Since $|\mathfrak{D}_2^{\mathfrak{h},m}|_p = \tilde{\mathscr{D}}_m^{\mathfrak{h}} + \mathscr{D}_p$, we have $|\tilde{\mathfrak{D}}^{\mathfrak{h},m}|_p \supset |\mathfrak{D}_2^{\mathfrak{h},m}|_p$. To prove that $\tilde{\mathfrak{D}}^{\mathfrak{h},m} \leqslant \mathfrak{D}_2^{\mathfrak{h},m}$ it is sufficient to show that convergence in $(|\mathfrak{D}_2^{\mathfrak{h},m}|_p, \tau_{\mathfrak{D}_2^{\mathfrak{h}},m,p})$ implies the convergence in $(\mathscr{D}_p^{\mathfrak{h},m}, \tau_p^{\mathfrak{h},m})$. Taking $\{f_n\} \subset |\mathfrak{D}_2^{\mathfrak{h},m}|_p$, $\tau_{\mathfrak{D}_2^{\mathfrak{h}},m,p}$-convergent to zero, we can find $\{g_n\} \subset \mathscr{D}_p$ and $\{h_n\} \subset \tilde{\mathscr{D}}_m^{\mathfrak{h}}$ both convergent to zero in $(\mathscr{D}_p, \tau_p)$ and $(\tilde{\mathscr{D}}_m^{\mathfrak{h}}, \|\cdot\|_m^{\mathfrak{h}})$ respectively such that $f_n = g_n + h_n$ for every n. Hence, both $\{g_n\}$ and $\{h_n\}$ are convergent to zero in $(\mathscr{D}_p^{\mathfrak{h},m}, \tau_p^{\mathfrak{h},m})$ and so is $\{f_n\}$. To show that to every m there correspond $k_m \geqslant m$ such that $\mathfrak{D}_2^{\mathfrak{h},k_m} \leqslant \tilde{\mathfrak{D}}^{\mathfrak{h},m}$ we proceed as follows. We take k_m such that

$$\{t \in \Omega\colon 1 = \sum_{i=1}^{k_m} e_i(t)\} \supset \bigcup_{j=1}^{m} K_j.$$

If $f \in |\tilde{\mathfrak{D}}^{\mathfrak{h},m}|_p$, then there exists $\{f_n\} \subset \mathscr{D}_p$, $\|f_n - f\|_m^{\mathfrak{h}}$ tending to zero for n tending to infinity, such that $f_{|G_m} \in \mathscr{E}(G_m)$, where $G_m = \Omega - \bigcup_{i=1}^{m} K_i$, and $\{f_{n|G_m}\}$ tends with respect to n uniformly with all derivatives to $f_{|G_m}$. Let $h = ef$, $h_n = er_n$ and $g = (1-e)f$, $g_n = (1-e)f_n$, where $e = \sum_{i=1}^{k_m} e_i$. Since $\operatorname{supp}(1-e)$ and $\bigcup_{i=1}^{m} K_i$ are disjoint, we have $g \in \mathscr{D}_p$. Further, $\{h_n\} \subset \mathscr{D}_{k_m}$, $\|h - h_n\|_{k_m}^{\mathfrak{h}} \underset{n}{\to} 0$ and then $h \in \tilde{\mathscr{D}}_{k_m}^{\mathfrak{h}}$. Hence, $f = g + h \in \mathscr{D}_p + \tilde{\mathscr{D}}_{k_m}^{\mathfrak{h}} = |\mathfrak{D}_2^{\mathfrak{h},k_m}|_p$ and we have $|\tilde{\mathfrak{D}}^{\mathfrak{h},m}|_p \subset |\mathfrak{D}_2^{\mathfrak{h},k_m}|_p$ for every p.

To show that the inclusion provides a continuous imbedding, take $\{f_n\} \subset |\tilde{\mathfrak{D}}^{\mathfrak{h},m}|_p$, $\{f_n\}$ tending to zero in $\tau_p^{\mathfrak{h},m}$-topology. By the same argument as above we show that $\{ef_n\} \subset \tilde{\mathscr{D}}_{k_m}^{\mathfrak{h}}$. Since $\{ef_n\}$ tends in $(\tilde{\mathscr{D}}_{k_m}^{\mathfrak{h}}, \|\cdot\|_{k_m}^{\mathfrak{h}})$ to zero and $\{(1-e)f_n\}$ tends to zero in $(\mathscr{D}_p, \tau_p)$ we conclude that $\tilde{\mathfrak{D}}^{\mathfrak{h},m} \geqslant \mathfrak{D}_2^{\mathfrak{h},k_m}$.

Ad d). To verify d, it is sufficient to prove that for every k the inclusion

$$\mathscr{D}_k^{\mathfrak{h}} \cap \operatorname{Closure}_1(\mathscr{D}_0 \cap \mathscr{D}_p) \subset \tilde{\mathscr{D}}_{k'}^{\mathfrak{h}} \cap \operatorname{Closure}_{1'}(\mathscr{D}_0 \cap \mathscr{D}_p)$$

holds for some $k' \geqslant k$, where the $\operatorname{Closure}_{1'}$ is taken in $(\tilde{\mathscr{D}}_p^{\mathfrak{h},k'}, \tau_p^{\mathfrak{h},k'})$. This, however, is rather obvious. If f belongs to $\mathscr{D}_k^{\mathfrak{h}} \cap \operatorname{Closure}_1(\mathscr{D}_0 \cap \mathscr{D}_p)$, then there exists $\{f_n\} \subset \mathscr{D}$ tending in $(\mathscr{D}_p^{\mathfrak{h},k}, \tau_p^{\mathfrak{h},k})$ to f. Put $e = \sum_{i=1}^{k'} e_i$. If we fix k' to have

$$\{t \in \Omega\colon\ e(t) = 1\} \supset \bigcup_{i=1}^{k} K_i,$$

then $\{ef_n\} \subset \mathscr{D}_{k'}$ and $\{ef_n\}$ tends to f in $(\mathscr{D}_{k'}^{\mathfrak{h}}, \|\cdot\|_{k'}^{\mathfrak{h}})$. Hence $f \in \tilde{\mathscr{D}}_{k'}^{\mathfrak{h}}$, and d follows. This way Theorem 2.1 is fully proved.

Similarly as in the case of distributions it is possible to develop, in connection with Theorem 4.1, the background that led us to Theorem 2.1′ as another formulation of Theorem 2.1.

Let $\mathfrak{X}$ be an $(\mathscr{IF})$-sequence. The topology $\tau_{\mathfrak{X}}$ of $|\mathfrak{X}|$ is the finest locally convex topology of $|\mathfrak{X}|$ with all the identical imbeddings of $(|\mathfrak{X}|_n, \tau_{\mathfrak{X},n})$ into $(|\mathfrak{X}|, \tau_{\mathfrak{X}})$ continuous, i. e., the finest locally convex topology of $|\mathfrak{X}|$ such that $(|\mathfrak{X}|_n, \tau_{\mathfrak{X},n}) \geqslant (|\mathfrak{X}|, \tau_{\mathfrak{X}})$ for every n. The space $(|\mathfrak{X}|, \tau_{\mathfrak{X}})$ is usually called the *inductive limit* of the sequence $\mathfrak{X}$.

It is obvious that the adjoint to $\mathfrak{X}$ coincides with the adjoint to $(|\mathfrak{X}|, \tau_{\mathfrak{X}})$. As it is always true in the case of strict $(\mathscr{IF})$-sequences, assume that the topology $\tau_{\mathfrak{X}}$ is such that every bounded sequence in $(|\mathfrak{X}|, \tau_{\mathfrak{X}})$ is contained and bounded in $(|\mathfrak{X}|_n, \tau_{\mathfrak{X},n})$ for at least one n. Under this assumption for any linear subspace X_0 of $|\mathfrak{X}|$ to say that a functional x'_0 defined on X_0 is $\tau_{\mathfrak{X}}$-bounded (i. e. maps $\tau_{\mathfrak{X}}$-bounded subsets of X_0 into bounded sets), is the same as to say that x'_0 is sequentially continuous. Furthermore, for any functional x'_0 defined on X_0 the existence of an extension of x'_0 to some x' from the adjoint to $(|\mathfrak{X}|, \tau_{\mathfrak{X}})$ is equivalent with the continuity of x'_0 on X_0 with respect to the relative topology of X_0 induced by the topology $\tau_{\mathfrak{X}}$ of $|\mathfrak{X}|$.

Then, assuming the existence of a basis of coverings of $\mathfrak{X}$ which admits reflexive majorizations, there can be introduced an equivalent formulation of Theorem 3 which runs as follows:

THEOREM 4.1′. *Consider an $(\mathcal{L}\mathcal{F})$-sequence $\mathfrak{X}$ provided with a basis of coverings Ξ that admits reflexive majorizations. Let in the following X_0 be a subspace of $|\mathfrak{X}|$.*

The subspace X_0 is good located in $|\mathfrak{X}|$ with respect to the basis of coverings Ξ if and only if every $\tau_{\mathfrak{X}}$-bounded functional defined on X_0 is continuous on X_0 with respect to the relative topology of X_0 induced by the topology $\tau_{\mathfrak{X}}$ of $|\mathfrak{X}|$.

Theorem 4.1′ permits to define well location of a subspace for an arbitrary linear locally convex topological space (X, τ). A subspace X_0 of X is said to be *good located* in X iff every τ-bounded linear functional defined on X_0 is τ-continuous.

The author knows no description of such a notion done by use of the topological properties of the space (X, τ).

5. The general theory. We start with some additional definitions. Consider an inductive sequence $\mathfrak{S}$ and a subspace U of $|\mathfrak{S}|$. Let in the following u' be a functional defined on U. Take a fixed n. If there exists $m \geqslant n$ such that u' is continuous in $(U \cap |\mathfrak{S}|_m, \|\cdot\|_{\mathfrak{S},m})$ and that the closure of $U \cap |\mathfrak{S}|_m$ in $(|\mathfrak{S}|_m, \|\cdot\|_{\mathfrak{S},m})$ contains $|\mathfrak{S}|_n$, then we put

$$u'_{\mathfrak{S},n} \overset{\mathrm{df}}{=} R_{|\mathfrak{S}|_n} E_{|U \overline{\cap} \mathfrak{S}|_m} R_{|\mathfrak{S}|_m} u',$$

where for any space V the symbol R_V denotes the operation of restriction of functionals to the intersection of their domains with V and E_V — the operation of extension of continuous functionals from dense subspaces of V over the whole V.

Since $\mathfrak{S}$ is inductive, it follows that the definition of $u'_{\mathfrak{S},n}$ does not depend on the choice of m. In the case when $\mathfrak{S} = U \overline{\cap} \mathfrak{S}$, we can always take $m = n$ and have

$$u'_{\mathfrak{S},n} = E_{|\mathfrak{S}|_n} R_{|\mathfrak{S}|_n} u'.$$

In the following, consider an $(\mathcal{I}\mathcal{F})$-sequence $\mathfrak{X}$, two coverings $\mathfrak{Z}$ and $\mathfrak{S}$ of $\mathfrak{X}$ and a subspace X_0 of $|\mathfrak{X}|$. Denote by X' the adjoint space of $\mathfrak{X}$ and by X'_0 the adjoint space of $X_0 \cap \mathfrak{X}$. Finally, let R denote the operation of restriction of functionals from their domains to the intersections of these domains with X_0, whenever such intersection is uniquely defined.

Below there are given four different properties of the triplet $(X_0, \mathfrak{Z}, \mathfrak{S})$ and some mutual relations among these properties and the Accessibility Property are thoroughly investigated.

It is said that the triplet $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the Property $(\cdot)$ iff the corresponding condition $(\cdot)$ among those listed below is satisfied, where one of the letters A_0, A, E_0, and E will appear in the parentheses. These letters are understood as shortenings for the following full terms: A_0 — weak approximation, A — approximation, E_0 — weak extension, E — extension (6).

(A_0) To every n there corresponds m such that $(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n}) \geqslant (|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})$ and that to every $z' \in |(\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})'|$ with $\|Rz'\|^*_{\mathfrak{Z},m} = 0$, every p and $\varepsilon > 0$ there correspond $x' \in X'$ such that $x'_{\mathfrak{S},n}$ exists, $\|x'_{\mathfrak{S},n} - z'_{\mathfrak{S},n}\|^*_{\mathfrak{S},n} < \varepsilon$ and $\|Rx'\|^*_{\mathfrak{Z},p} < \varepsilon$.

(A) To every n there corresponds m such that $(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n}) \geqslant (|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})$ and that to every $z' \in |(|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})'|$ with $\|Rz'\|^*_{\mathfrak{Z},m} = 0$, and every $\varepsilon > 0$ there corresponds $x' \in X'$ vanishing on X_0 such that $x'_{\mathfrak{S},n}$ exists and $\|x'_{\mathfrak{S},n} - z'_{\mathfrak{S},n}\|^*_{\mathfrak{S},n} < \varepsilon$.

(E_0) To every n there corresponds m such that for every $x'_0 \in X'_0$ with $\|x'_0\|^*_{\mathfrak{Z},m} < \infty$, every $\varepsilon > 0$ and p there corresponds $x' \in X'$ with $\|x'\|^*_{\mathfrak{S},n} < \infty$ such that $\|Rx' - x'_0\|^*_{\mathfrak{Z},p} < \varepsilon$.

(E) To every n there correspond m and $\eta > 0$ such that for every $x'_0 \in X'_0$ with $\|x'_0\|^*_{\mathfrak{Z},m} < \eta$ there corresponds $x' \in X'$ with $\|x'\|^*_{\mathfrak{S},n} < 1$ such that $Rx' = x'_0$.

Fix an $(\mathcal{I}\mathcal{F})$-sequence $\mathfrak{X}$, coverings $\mathfrak{S}$ and $\mathfrak{Z}$ of $\mathfrak{X}$, $\mathfrak{S} \geqslant \mathfrak{Z}$, and a subspace X_0 of $|\mathfrak{X}|$.

PROPOSITION 5.1. *If the triplet $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the Property $(\cdot)$, where in the parentheses there appears one of the letters ACC, A_0, E_0, A or E, then for every covering $\mathfrak{T} \geqslant \mathfrak{Z}$ with $|\mathfrak{X}| \cap \mathfrak{T} \geqslant \mathfrak{S}$ the triplet $(X_0, \mathfrak{Z}, \mathfrak{T})$ admits the property $(\cdot)$ with the same letter in the parentheses.*

Proof. The Proposition easily follows after completing a few simple calculations.

COROLLARY 5.1. *Let Λ be a family of equivalent coverings of $\mathfrak{X}$ and denote by $(\cdot)$ one of the conditions (A_0), (A), (E_0) and (E). The triplet $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits $(\cdot)$ with a fixed $\mathfrak{S}$ taken from Λ with $\mathfrak{S} \geqslant \mathfrak{Z}$ if and only if $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits $(\cdot)$ with every $\mathfrak{S} \in \Lambda$. Here we assume additionally that for at least one $\mathfrak{S} \in \Lambda$ there is $\mathfrak{S} \geqslant \mathfrak{Z}$.*

(6) An example communicated to the author by I. L. Glicksberg, indicating nontriviality of the question of producing extensions of functionals vanishing on subspaces fixed in advance, was of a great assistance in establishing properties (A) and (A_0) crucial for the whole theory of extensions.

Proof. This is an immediate consequence of Proposition 5.1.

The Corollary proves that all of the properties (ACC), (A_0), (E_0) and (E) can be expressed for triplets $(X_0, \mathfrak{Z}, \Lambda)$, where Λ denotes a class of equivalent coverings of $\mathfrak{X}$ with $\mathfrak{Z} \leqslant \mathfrak{S}$ for at least one (and then every) $\mathfrak{S} \in \Lambda$.

In view of this last statement we can always substitute for $\mathfrak{S}$ coverings with $\mathfrak{S} = |\mathfrak{X}| \mathbin{\overline{\frown}} \mathfrak{S}$ preserving all the generality.

PROPOSITION 5.2. *The condition* (E) *is equivalent to the one written below.*

($\hat{\mathrm{E}}$) *To every n there correspond m and η such that for every $x_0' \in X_0'$ with $\|x_0'\|_{\mathfrak{Z},m}^* < \eta$ and every p and $\varepsilon > 0$ there corresponds $x' \in X'$ such that $\|Rx' - x_0'\|_{\mathfrak{Z},p}^* < \varepsilon$ and $\|x'\|_{\mathfrak{Z},m}^* < 1$.*

Proof. Since the condition ($\hat{\mathrm{E}}$) of the Proposition trivially follows from (E), it remains to show the implication ($\hat{\mathrm{E}}$) $\to$ (E). In order to do that it is necessary to employ a result from [11]. At first we define two pre-($\mathscr{F}$)-sequences (cf. [11]). Let

$$|\mathfrak{V}|_n \overset{\mathrm{df}}{=} \{x_0' \in X_0' \colon \|x_0'\|_{\mathfrak{Z},n}^* < \infty\}.$$

The first pre-($\mathscr{F}$)-sequence to consider is $\mathfrak{V}$, with $\|\cdot\|_{\mathfrak{V},n} \overset{\mathrm{df}}{=} \|\cdot\|_{\mathfrak{Z},n}^*$ restricted to $|\mathfrak{V}|_n$, $n = 1, 2, 3, \ldots$ To define the other pre-($\mathscr{F}$)-sequence put

$$V \overset{\mathrm{df}}{=} \{x' \in X' \colon Rx' \in |\mathfrak{V}|\}.$$

The second pre-($\mathscr{F}$)-sequence to consider is $V \frown \mathfrak{S}^* = \{(V \frown |\mathfrak{S}^*|_n, \|\cdot\|_{\mathfrak{S},n}^*)\}$, where $|\mathfrak{S}^*|_n \overset{\mathrm{df}}{=} \{x' \in X' \colon \|x'\|_{\mathfrak{S},n}^* < \infty\}$. Let $R_1 \overset{\mathrm{df}}{=}$ the restriction of R to V.

It is easy to see that R_1 is a mapping of the pre-($\mathscr{F}$)-sequence $V \frown \mathfrak{S}^*$ into the pre-($\mathscr{F}$)-sequence $\mathfrak{V}$. Moreover, the condition ($\hat{\mathrm{E}}$) assures that R_1 is nearly-open mapping of $V \frown \mathfrak{S}^*$ into $\mathfrak{V}$. Then, to apply Proposition 12 of [11] and conclude that R_1 is open, which means exactly that the condition (E) is satisfied, it is sufficient to prove that R_1 is complete-closed in the sense of [11]. To prove that, take $\{x_n'\} \subset |V \frown \mathfrak{S}^*|$ which is a Cauchy sequence in $[V \frown \mathfrak{S}^*]$. Clearly, $\{x_n'\}$ tends pointwise to some x' defined on $|\mathfrak{X}|$ and for every p it can be found m_p such that $\|x' - x_{m_p}'\|_{\mathfrak{S},p}^* < \infty$. Then, $x' \in X'$ and, if the restrictions of $R_1 x_n'$ tend to some $z_0' \in |\mathfrak{V}|$, it must be $R_1 x' = z_0'$ and $x' \in V$. Hence R_1 is complete-closed and the Proposition follows.

Take an ($\mathscr{I}\mathscr{F}$)-sequence $\mathfrak{X}$. A subspace X_0 of $|\mathfrak{X}|$ admits the *separation principle* in $\mathfrak{X}$ iff to every p and every $x \notin \bigcup_n |X_0 \mathbin{\overline{\frown}} \mathfrak{X}|_n$ one can assign a functional x' from the adjoint to $\mathfrak{X}$ in such a way that $x'x = 1$ and $x'(X_0 \frown |\mathfrak{X}|_p) = \{0\}$.

Take an ($\mathscr{I}\mathscr{F}$)-sequence $\mathfrak{X}$, two coverings $\mathfrak{Z}$ and $\mathfrak{S}$ of $\mathfrak{X}$, $\mathfrak{Z} \leqslant \mathfrak{S}$, and a subspace X_0 of $|\mathfrak{X}|$. As before, R denotes the operation of restric-

tion to X_0 and X' — the adjoint space to $\mathfrak{X}$. Assume that for every n there is $|\mathfrak{Z}|_n \frown |\mathfrak{X}| \subset |\mathfrak{X}|_q$ for sufficiently great q.

PROPOSITION 5.3. I. *The Accessibility Property of $(X_0, \mathfrak{Z}, \mathfrak{S})$ follows from the Property* (A) *of the same $(X_0, \mathfrak{Z}, \mathfrak{S})$.*

II. *If in every $\mathfrak{X}_{\mathfrak{S},n}$, $n = 1, 2, \ldots$, X_0 admits the separation principle and $\mathfrak{S}$ is reflexive, then the Accessibility Property of $(X_0, \mathfrak{Z}, \mathfrak{S})$ implies the Property (A_0) for the same $(X_0, \mathfrak{Z}, \mathfrak{S})$.*

Proof. Ad I. Take an arbitrary n and adjust m to suit the requirements of (A). Consider any $z \notin |X_0 \frown \mathfrak{Z}|_m$. There exists a functional $z' \in |(|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})'|$ such that $z'z = 1$ and that z' vanishes on $|X_0 \frown \mathfrak{Z}|_m$. Applying (A) we find $x' \in X'$ such that x' vanishes on X_0 and $\|x_{\mathfrak{S},n}' - z_{\mathfrak{S},n}'\|_{\mathfrak{S},n}^* < 1/\|z\|_{\mathfrak{S},n}$, provided $z \in |\mathfrak{S}|_n$.

The functional x' restricted to $|\mathfrak{X}|_p + (|\mathfrak{X}| \frown |\mathfrak{S}|_n)$ can be subsequently extended to $\bar{x}' \in |(|\mathfrak{X}|_p + |\mathfrak{S}|_n, \tau_{\mathfrak{X},p} \wedge \tau_{\mathfrak{S},n})'|$ and then, with $z \in |\mathfrak{S}|_n \frown |X_0 \mathbin{\overline{\frown}} \mathfrak{X}_{\mathfrak{S},n}|_p$ we have $\bar{x}'z = 0$. Hence, $1 = |z'z| < (1/\|z\|_{\mathfrak{S},n})\|z\|_{\mathfrak{S},n} = 1$ which is contradictory. This concludes the proof of the first part of the Proposition.

Ad II. Take an arbitrary n and adjust m to have $(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n}) \geqslant (|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})$ and, simultaneously, to suit the requirements of (ACC). To prove II it is sufficient to show that the subspace

$$U' \overset{\mathrm{df}}{=} \{x_{\mathfrak{S},n}' \colon x' \in \text{ the adjoint to } \mathfrak{X}_{\mathfrak{S},n}, \|Rx'\|_{\mathfrak{Z},p}^* = 0\}$$

of $|(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n})'|$ is weak* dense in the subspace

$$V' \overset{\mathrm{df}}{=} \{z_{\mathfrak{S},n}' \colon z' \in |(|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})'|, \|Rz'\|_{\mathfrak{Z},m}^* = 0\}$$

of $|(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n})'|$ and use the reflexivity of $(|\mathfrak{S}|_n; \|\cdot\|_{\mathfrak{S},n})/L_n$, where $L_n \overset{\mathrm{df}}{=} \{s \in |\mathfrak{S}|_n \colon \|s\|_{\mathfrak{S},n} = 0\}$, to obtain the strong density that implies (A_0).

If U'^- denotes the weak* closure of U', then

$$U'^- = \{z' \in |(|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n})'| \colon \text{for } z \in |\mathfrak{S}|_n,\ U'z = \{0\} \text{ implies } z'z = 0\}.$$

To prove the weak* density it is sufficient to show that $V' \subset U'^-$, which amounts to showing that if $v'z \neq 0$ for some $v' \in V'$ and $z \in |\mathfrak{S}|_n \subset |\mathfrak{Z}|_m$, then there exists $u' \in U'$ such that $u'z \neq 0$.

Since $v'z \neq 0$ for $v' \in V'$ and $z \in |\mathfrak{Z}|_m$ means that $z \notin |X_0 \mathbin{\overline{\frown}} \mathfrak{Z}|_m$, applying the inclusion from (ACC) one can find that $z \notin |\mathfrak{S}|_n \frown |X_0 \mathbin{\overline{\frown}} \mathfrak{X}_{\mathfrak{S},n}|_q$ for every q. But for $z \in |\mathfrak{S}|_n$ it must be that $z \notin |X_0 \mathbin{\overline{\frown}} \mathfrak{X}_{\mathfrak{S},n}|_q$ for every q and applying the separation principle we find $x' \in X'$ with $\|x'\|_{\mathfrak{S},n} < \infty$ such that x' vanishes on $X_0 \frown |\mathfrak{X}_{\mathfrak{S},n}|_{q_0}$ for q_0 with $|\mathfrak{X}|_{q_0} \supset |\mathfrak{Z}|_p$ and the Proposition follows.

Consider an ($\mathscr{I}\mathscr{F}$)-sequence $\mathfrak{X}$ and coverings $\mathfrak{Z}, \mathfrak{S}$ of $\mathfrak{X}$, $\mathfrak{Z} \leqslant \mathfrak{S}$. Let in the following X_0 denote a subspace of $\mathfrak{X}$ and X', X_0' adjoint spaces of $\mathfrak{X}$ and $X_0 \frown \mathfrak{X}$ respectively.

PROPOSITION 5.4. *If* $(X_0, \mathfrak{Z}, \mathfrak{S})$ *admits the properties* (A_0) *and* (E_0), *then* $(X_0, \mathfrak{Z}, \mathfrak{S})$ *admits* (E) *as well.*

Proof. Let the assumptions of the Proposition be fulfilled. To verify the Proposition it is sufficient to show that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits $(\hat{E})$ and then apply Proposition 5.2.

Fix any n and adjust m to suit simultaneously the requirements of (A_0) and (E_0) and fulfil at the same time the relation $(|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m}) \leqslant (|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n})$. Take arbitrary $p \geqslant m$ and let N be such that $N\|x\|_{\mathfrak{Z},m} \geqslant \|x\|_{\mathfrak{Z},p}$ for $x \in |\mathfrak{Z}|_m$. Further, let M be such that $\|x\|_{\mathfrak{Z},m} \leqslant M\|x\|_{\mathfrak{S},n}$ for $x \in |\mathfrak{S}|_n$. Fix $\eta < \frac{1}{3}$. Take $x_0' \in X_0'$ with $\|x_0'\|^*_{\mathfrak{Z},m} < \eta$ and let ε be an arbitrary positive number. By virtue of (E_0) there exists $x_1 \in X'$ such that $\|x_1'\|^*_{\mathfrak{S},n} < \infty$ and $\|Rx_1' - x_0'\|^*_{\mathfrak{Z},p} < \min(\eta/N, \varepsilon/2)$. Let in the following $z_1' \in |(|\mathfrak{Z}|_m, \|\cdot\|_{\mathfrak{Z},m})'|$ denote an extension of x_1' restricted to $|\mathfrak{Z}|_m \cap X_0$ such that $\|Rx_1'\|^*_{\mathfrak{Z},m} = \|z_1'\|^*_{\mathfrak{Z},m}$. Put $-z' \overset{\text{df}}{=} z_1' - x'_{1\mathfrak{Z},m}$. Clearly, $\|Rz'\|^*_{\mathfrak{Z},m} = 0$ and then, applying (A_0), we can find $x_2' \in X'$ such that $\|Rx_2'\|^*_{\mathfrak{Z},p} < \min(\varepsilon/2, M\eta)$ and $\|x'_{2\mathfrak{S},n} - z'_{\mathfrak{S},n}\|^*_{\mathfrak{S},n} < \min(\varepsilon/2, M\eta)$. Put $x' \overset{\text{df}}{=} x_1' - x_2'$. We have

$$\|Rx' - x_0'\|^*_{\mathfrak{Z},p} \leqslant \|Rx_1' - x_0'\|^*_{\mathfrak{Z},p} + \|Rx_2'\|^*_{\mathfrak{Z},p} < \varepsilon.$$

Further,

$$\begin{aligned}\|x'\|^*_{\mathfrak{S},n} &\leqslant \|x'_{2\mathfrak{S},n} - z'_{\mathfrak{S},n}\|^*_{\mathfrak{S},n} + M\|z' - x'_{1\mathfrak{Z},m}\|^*_{\mathfrak{Z},m}\\ &\leqslant M\eta + M\|z_1'\|^*_{\mathfrak{Z},m} = M\eta + M\|Rx_1'\|^*_{\mathfrak{Z},m}\\ &\leqslant M\eta + M\|x_0'\|^*_{\mathfrak{Z},m} + M\|Rx_1' - x_0'\|^*_{\mathfrak{Z},m} \leqslant 2M\eta + MN\|Rx_1' - x_0'\|^*_{\mathfrak{Z},p}\\ &\leqslant 2M\eta + MN\eta/N = 3M\eta < 1.\end{aligned}$$

This proves $(\hat{E})$ for $(X_0, \mathfrak{Z}, \mathfrak{S})$ and then the Proposition follows.

Let $\mathfrak{X}$ be an $(\mathscr{LF})$-sequence and denote by X' the adjoint of $\mathfrak{X}$. A covering $\mathfrak{Z}$ of $\mathfrak{X}$ admits the property (D) iff the following condition is satisfied:

(D) To every n, every $z' \in |(|\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n})'|$ and $\varepsilon > 0$ there corresponds $x' \in X'$ such that $\|z' - x'_{\mathfrak{Z},n}\|^*_{\mathfrak{Z},n} < \varepsilon$.

Let in the following $\mathfrak{Z}$ and $\mathfrak{S}$ be two coverings of $\mathfrak{X}$, $\mathfrak{Z} \leqslant \mathfrak{S}$, and X_0 — a subspace of $\mathfrak{X}$. As usual, denote by R the operation of restriction from $|\mathfrak{X}|$ to X_0 and by X_0' — the adjoint of $X_0 \cap \mathfrak{X}$.

PROPOSITION 5.5. *Let* $\mathfrak{Z}$ *admits the property* (D). *Then, the triplet* $(X_0, \mathfrak{Z}, \mathfrak{S})$ *admits* (A) *provided it admits* (E).

Proof. Take $\{m_n\}$ and $\{M_n\}$, $M_n \geqslant 1$, such that $(|\mathfrak{Z}|_{m_n}, \|\cdot\|_{\mathfrak{Z},m_n}) \leqslant (|\mathfrak{S}|_n, \|\cdot\|_{\mathfrak{S},n})$ and $M_n\|x\|_{\mathfrak{S},n} \geqslant \|x\|_{\mathfrak{Z},m_n}$ for $x \in |\mathfrak{S}|_n$, $n = 1, 2, \ldots$ Let, additionally, m_n and $0 < \eta_n < 1$ correspond to n according to the requirements of (E). Fix n, take any $\varepsilon > 0$ and $z' \in |(|\mathfrak{Z}|_{m_n}, \|\cdot\|_{\mathfrak{Z},m_n})'|$ with $\|Rz'\|^*_{\mathfrak{Z},m_n} = 0$. Applying (D) we find $u' \in X'$ such that

$$\|z' - u'_{\mathfrak{Z},m_n}\|^*_{\mathfrak{Z},m_n} < \varepsilon\eta_n/2M_n.$$

Hence, $\|(2/\varepsilon)Ru'\|^*_{\mathfrak{Z},m_n} < \eta_n$ and applying (E) we find $v' \in X'$ with $Ru' = Rv'$ and $\|2v'/\varepsilon\|^*_{\mathfrak{S},n} < 1$. Setting $x' = u' - v'$ we have

$$\begin{aligned}&\|z' - x'_{\mathfrak{Z},m_n}\|^*_{\mathfrak{S},n}\\ &\leqslant \|z' - u'_{\mathfrak{Z},m_n}\|^*_{\mathfrak{S},n} + \|v'\|^*_{\mathfrak{S},n} \leqslant M_n\|z' - u'_{\mathfrak{Z},m_n}\|^*_{\mathfrak{Z},m_n} + \|v'\|^*_{\mathfrak{S},n} < \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

and the Proposition follows.

THEOREM 5.1. *If* $\mathfrak{Z}$ *is a covering of a strict* $(\mathscr{IF})$*-sequence* $\mathfrak{X}$ *such that for every* n *there exists* p *with* $|\mathfrak{Z}|_n \cap |\mathfrak{X}| \subset |\mathfrak{X}|_p$, *then for every subspace* X_0 *of* $|\mathfrak{X}|$ *the triplet* $(X_0, \mathfrak{Z}, \mathfrak{Z})$ *admits the property* (E_0).

Proof. Take any $X_0 \subset |\mathfrak{X}|$, any n and adjust p to have $|\mathfrak{X}|_p \supset |\mathfrak{Z}|_n \cap |\mathfrak{Z}|$. Further, take $(|\mathfrak{X}|_p, \|\cdot\|_p) \leqslant (|\mathfrak{X}|_p, \tau_{\mathfrak{X},p})$ such that $(|\mathfrak{Z}|_n \cap |\mathfrak{X}|, \|\cdot\|_p) \geqslant (|\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n})$.

Consider the space

$$(U, \|\cdot\|) \overset{\text{df}}{=} \big((|\mathfrak{X}| \cap |\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n}) \wedge (|\mathfrak{X}|_p \cap X_0, \|\cdot\|_p)\big) \wedge (|\mathfrak{X}|_p, \|\cdot\|_p).$$

We have

$$(|\mathfrak{X}| \cap |\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n}) \wedge (|\mathfrak{X}|_p \cap X_0, \|\cdot\|_p) \leqslant (X_0 \cap U, \|\cdot\|),$$

$$(U, \|\cdot\|) \leqslant (|\mathfrak{X}| \cap |\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n}).$$

The second of this relations follows directly from the definition of $\wedge$ and the relation $(|\mathfrak{X}| \cap |\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n}) \wedge (|\mathfrak{X}|_p \cap X_0, \|\cdot\|_p) \leqslant \big((|\mathfrak{X}| \cap |\mathfrak{Z}|_n) + (|\mathfrak{X}|_p \cap X_0), \|\cdot\|_p\big)$ implies the first.

For $x_0' \in X_0'$ with $\|x_0'\|^*_{\mathfrak{Z},n} < \infty$ we fix $(|\mathfrak{X}|_p, \|\cdot\|_p)$ to have $\|x_0'\|^*_p < \infty$. Then it must be $\|x_0'\|^* < \infty$ and x_0' can be extended to $u' \in |(U, \|\cdot\|)'|$. Hence, $\|u'\|^*_{\mathfrak{Z},n} < \infty$ and u' belongs to $|(|\mathfrak{X}|_p, \tau_{\mathfrak{X},p})'|$. Any extension x' of u' to a sequentially continuous functional over $|\mathfrak{X}|$ satisfies the requirements of (E_0). This way the Proposition has been proved.

Consider an $(\mathscr{IF})$-sequence $\mathfrak{X}$ and a subspace X_0 of $|\mathfrak{X}|$. Denote by X_0' the adjoint to $X_0 \cap \mathfrak{X}$. A functional $x_0' \in X_0'$ is said to be $\mathfrak{X}$*-majorizable* iff there exists a covering $\mathfrak{Z}$ of $\mathfrak{X}$ such that $\|x_0'\|^*_{\mathfrak{Z},n} < \infty$ for every n.

THEOREM 5.2. *If* $\mathfrak{X}$ *is strict, then every* $x_0' \in X_0'$ *is* $\mathfrak{X}$*-majorizable.*

Proof. Take $\|\cdot\|_1$ defined on $|\mathfrak{X}|_1$ in such a way that $\|x_0'\|^*_1 < \infty$. Put $|\mathfrak{Z}|_1 = |\mathfrak{X}|_1$ and $\|\cdot\|_{\mathfrak{Z}_1,1} = \|\cdot\|_1$. Suppose that it has been produced $(|\mathfrak{Z}|_i, \|\cdot\|_{\mathfrak{Z},i})$ with $\|x_0'\|^*_{\mathfrak{Z},i} < \infty$ for $i = 1, 2, \ldots, n$ and $(|\mathfrak{Z}|_i, \|\cdot\|_{\mathfrak{Z},i}) \geqslant (|\mathfrak{Z}|_{i+1}, \|\cdot\|_{\mathfrak{Z},i+1})$ for $i = 1, 2, \ldots, n-1$. We assume that $|\mathfrak{Z}|_i = |\mathfrak{X}|_i$ for $i = 1, 2, \ldots, n$.

Put $p = n+1$ and take $(|\mathfrak{X}|_p, \|\cdot\|_p)$ with $(|\mathfrak{X}|_p, \|\cdot\|_p) \leqslant (|\mathfrak{X}|_p, \tau_{\mathfrak{X},p})$, $(|\mathfrak{X}|_n, \|\cdot\|_p) \geqslant (|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{Z},n})$ and $\|x_0'\|^*_p < \infty$, where $\|\cdot\|^*_p$ is the polar pseudonorm induced by $(|\mathfrak{X}|_p, \|\cdot\|_p)$.

As in the proof of Theorem 5.1 we consider the space

$$(U, \|\cdot\|) \overset{\mathrm{df}}{=} \big((|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{Z},n}) \wedge (|\mathfrak{X}|_p, \cap X_0, \|\cdot\|_p)\big) \wedge (|\mathfrak{X}|_p, \|\cdot\|_p)$$

and we verify that

$$(|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{Z},n}) \wedge (|\mathfrak{X}|_p \cap X_0, \|\cdot\|_p) \leqslant (X_0 \cap U, \|\cdot\|), \qquad (U, \|\cdot\|) \leqslant (|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{Z},n}).$$

Since $\|x_0'\|_p^* < \infty$, we have $\|x_0'\|^* < \infty$, where $\|\cdot\|^*$ is the polar pseudonorm induced by $(U, \|\cdot\|)$. Setting $(|\mathfrak{Z}|_{n+1}, \|\cdot\|_{\mathfrak{Z},n+1}) \overset{\mathrm{df}}{=} (U, \|\cdot\|)$ we construct the next element of the covering $\mathfrak{Z}$ with $\|x_0'\|_{\mathfrak{Z},n}^* < \infty$ for every n. By virtue of the principle of induction the Theorem follows.

Consider an $(\mathcal{LF})$-sequence $\mathfrak{X}$ and a subspace X_0 of $|\mathfrak{X}|$. Let X' and X_0' be the adjoint spaces of $\mathfrak{X}$ and $X_0 \cap \mathfrak{X}$ respectively. As before R denotes the operation of restriction of functionals to the intersections of their domains with X_0.

Proposition 5.6. *If $RX' = X_0'$, then to every covering $\mathfrak{Z}$ of $\mathfrak{X}$ there correspond a covering $\mathfrak{S}$ of $\mathfrak{X}$ such that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the property* (E).

To prove the Proposition we shall need some additional definitions and a lemma. Take a covering $\mathfrak{Z}$ of $\mathfrak{X}$. As it was already defined, the polar $\mathfrak{Z}^*$ of $\mathfrak{Z}$ is a sequence of pseudonormed spaces defined as follows. We set $|\mathfrak{Z}^*|_n = \{x' \in X' \, \|x'\|_{\mathfrak{Z},n}^* < \infty\}$, where $\|\cdot\|_{\mathfrak{Z},n}^*$ is the polar pseudonorm induced by $(|\mathfrak{X}| \cap |\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n})$, and for $\|\cdot\|_{\mathfrak{Z}*,n}$ we put the mentioned polar pseudonorm restricted to $|\mathfrak{Z}^*|_n$, $n = 1, 2, \ldots$ It is easy to find that the polar $\mathfrak{Z}^*$ of any covering $\mathfrak{X}$ is an $(\mathcal{F})$-sequence.

According to the Example I of [12] we define a σ^2-family $\mathfrak{X}^*$ as follows. Let $\{\|\cdot\|_{k,n}: k = 1, 2, \ldots\}$, $n = 1, 2, \ldots$, be pointwise non-decreasing sequences of pseudonorms inducing topologies $\tau_{\mathfrak{X},n}$ in each $|\mathfrak{X}|_n$ respectively. Denote by $\|\cdot\|_{k,n}^*$ the polar pseudonorms induced by $(|\mathfrak{X}|_n, \|\cdot\|_{k,n})$ respectively and let $X_{k,n}^* \overset{\mathrm{df}}{=} \{x' \in X : \|x'\|_{k,n}^* < \infty\}$.

The double sequence $\{X_{k,n}^*, \|\cdot\|_{k,n}^*\}$, where $\|\cdot\|_{k,n}^*$ is restricted to $X_{k,n}^*$, decomposes a certain σ^2-family which, according to the notation accepted in Example I of [12], is written $\mathfrak{X}^*$. Clearly, $X' = |\mathfrak{X}^*|$. Denote by $\mathcal{N}$ the set of all increasing sequences of natural numbers [2]. For $\mathfrak{k} \in \mathcal{N}$, $\mathfrak{k} = \{k_n\}$, we put

$$(|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{k},n}) \overset{\mathrm{df}}{=} (|\mathfrak{X}|_1, \|\cdot\|_{k_1,1}) \wedge \ldots \wedge (|\mathfrak{X}|_n, \|\cdot\|_{k_n,n}).$$

It is clear that $\mathfrak{X}_\mathfrak{k} \overset{\mathrm{df}}{=} \{(|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{k},n})\}$ is a covering of $\mathfrak{X}$. It is easy to see, that for an arbitrary covering $\mathfrak{Z}$ of $\mathfrak{X}$, there always exists $\mathfrak{k} \in \mathcal{N}$ such that $\mathfrak{X}_\mathfrak{k} \geqslant \mathfrak{Z}$.

Lemma 5.1. *If $\|\cdot\|_{k,n}^*$ denotes polar pseudonorms induced by $(|\mathfrak{X}|_n, \|\cdot\|_{k,n})$ respectively and $\|\cdot\|_{\mathfrak{k},n}^*$ — the polar pseudonorm induced by $(|\mathfrak{X}|_n, \|\cdot\|_{\mathfrak{k},n})$, then for every u' with the domain containing $|\mathfrak{X}|_n$ we have*

$$\|u'\|_{\mathfrak{k},n}^* = \max\{\|u'\|_{k_i,i}^*: i = 1, 2, \ldots, n\}.$$

Proof. Take any u' defined on $U \supset |\mathfrak{X}|_n$. We have

$$|u'x| \leqslant \|u'\|_{\mathfrak{k},n}^* \|x\|_{\mathfrak{k},n} \leqslant \|u'\|_{\mathfrak{k},n}^* \|x\|_{k_i,i} \quad \text{for} \quad x \in |\mathfrak{X}|_i,\ i = 1, 2, \ldots, n.$$

Finally, $\|u'\|_{k_i,i}^* \leqslant \|u'\|_{\mathfrak{k},n}^*$ or, setting $|||u'||| \overset{\mathrm{df}}{=} \max\{\|u'\|_{k_i,i}^* : i = 1, 2, \ldots, n\}$, $|||u'||| \leqslant \|u'\|_{\mathfrak{k},n}^*$. On the other hand, we have

$$|u'x| \leqslant \|u'\|_{k_i,i} \|x\|_{k_i,i}^* \leqslant |||u'||| \|x\|_{k_i,i} \quad \text{for} \quad x \in |\mathfrak{X}|_i,\ i = 1, 2, \ldots, n.$$

Hence, $|u'x| \leqslant |||u'||| \|x\|_{\mathfrak{k},n}$ for $x \in |\mathfrak{X}|_n$, $\|u'\|_{\mathfrak{k},n}^* \leqslant |||u'|||$ and the Lemma follows.

Proof of Proposition 5.6. Take an arbitrary covering $\mathfrak{Z}$ of $\mathfrak{X}^*$. To define $\mathfrak{V}_1$ put $|\mathfrak{V}_1|_n = \{x_0' \in X_0': \|x_0\|_{\mathfrak{Z},n}^*\| < \infty\}$ and $\|\cdot\|_{\mathfrak{V}_1,n} = \|\cdot\|_{\mathfrak{Z},n}^*$ restricted to $|\mathfrak{V}_1|_n$, where $\|\cdot\|_{\mathfrak{Z},n}^*$ is the polar pseudonorm induced by $(|\mathfrak{Z}|_n, \|\cdot\|_{\mathfrak{Z},n})$. It is easy to find that $\mathfrak{V}_1$ is an $(\mathcal{F})$-sequence. Put $Y = \{x' \in X': Rx' \in |\mathfrak{V}_1|\}$. By virtue of Lemma 5.1 and Propositions 1 and 2 of [12] we find that the family $\{\mathfrak{X}_\mathfrak{k}^*: \mathfrak{k} \in \mathcal{N}\}$ of components of $\mathfrak{X}^*$ overwhelms in $\mathfrak{X}^*$. Here $\mathfrak{X}_\mathfrak{k}^*$ denotes the polar of the covering $\mathfrak{X}_\mathfrak{k}$ of $\mathfrak{X}'$. Since, as it can be easily checked, the mapping R of Y onto $|\mathfrak{V}_1|$ is closed relative to $\mathfrak{X}^*$, we can apply Theorem 1 of [12] and conclude that for some $\mathfrak{h} \in \mathcal{N}$ the mapping R is open from $[Y \cap \mathfrak{X}_\mathfrak{h}^*]$ to $[\mathfrak{V}_1]$. This, however, means that the triplet $(X_0, \mathfrak{Z}, \mathfrak{S})$, where $\mathfrak{S} \overset{\mathrm{df}}{=} \mathfrak{X}_\mathfrak{h}$, admits (E). It is always possible to choose $\mathfrak{X}_\mathfrak{h} \geqslant \mathfrak{Z}$ and then the Proposition is fully proved.

Theorem 5.3 [7]. *Consider an $(\mathcal{LF})$-sequence $\mathfrak{X}$ and a basis Ξ of coverings of $\mathfrak{X}$. Let in the following X_0 be a subspace of $|\mathfrak{X}|$, X' the adjoint to $\mathfrak{X}$, X_0' the adjoint to $X_0 \cap \mathfrak{X}$ and, finally, denote by R the operation of restriction of functionals from $|\mathfrak{X}|$ to X_0. Assume that $|\mathfrak{Z}|_h \cap |\mathfrak{X}| \subset |\mathfrak{X}|_{q_n}$ for $\mathfrak{Z} \in \Xi$.*

I. *If $RX' = X_0'$, then to every $\mathfrak{Z} \in \Xi$ there corresponds $\mathfrak{S} \in \Xi$, $\mathfrak{S} \geqslant \mathfrak{Z}$, such that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits* (E).

II. *Consider the following condition:*

(i) *To every $\mathfrak{Z} \in \Xi$ there correspond $\mathfrak{S} \in \Xi$, $\mathfrak{S} \geqslant \mathfrak{Z}$, such that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the* (ACC) *property and the property* (E_0).

If every covering from Ξ admits the property (D), *then in order that $RX' = X_0'$ it is necessary that the condition* (i) *shall be satisfied.*

If to every $\mathfrak{Z} \in \Xi$ there correspond a reflexive covering $\mathfrak{S}$ such that $\mathfrak{S} \geqslant \mathfrak{Z}$ and that X_0 admits the separation principle in every moderation $\mathfrak{X}_{\mathfrak{S},n}$, $n = 1, 2, \ldots$, then condition (i) *is sufficient for RX' to contain all $x_0' \in X_0'$.*

Proof. Take any $\mathfrak{Z} \in \Xi$. According to Proposition 5.6 there exists a covering $\mathfrak{T}$ of $\mathfrak{X}$,

$$(|\mathfrak{T}|_n, \|\cdot\|_{\mathfrak{T},n}) \overset{\mathrm{df}}{=} (|\mathfrak{X}|_1, \|\cdot\|_{k_1,1}) \wedge \ldots \wedge (|\mathfrak{X}|_n, \|\cdot\|_{k_n,n})$$

such that $(X_0, \mathfrak{Z}, \mathfrak{T})$ admits (E). According to the definition of the basis

[7] This is an improved version of Proposition 2 of [5].

of coverings, there exists $\mathfrak{S} \in \Xi$, $\mathfrak{S} \geqslant \mathfrak{Z}$, such that $|\mathfrak{X}| \frown \mathfrak{S} \geqslant \mathfrak{T}$ and from Proposition 5.1 we conclude that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits (E) as well. This finishes the proof of the first part of the Theorem.

Now, knowing that elements of Ξ admit the Property (D), we find from Proposition 5.5 that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the Property (A). Then, from Proposition 5.3 it follows that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the (ACC) property. This shows that the condition (i) is necessary to have $RX' = X_0'$.

The only remaining statement of the Theorem that still needs proof is the one concerning the existence of extensions of functionals from X_0' that are bounded on elements of coverings of $\mathfrak{X}$. Take $x_0' \in X_0'$ and fix $\mathfrak{Z}$ to have $\|x_0'\|_{\mathfrak{Z},n}^* < \infty$ for some n, where $\mathfrak{Z}$ is a covering of $\mathfrak{X}$. It follows from the definition of the basis of coverings that we can always have $\mathfrak{Z} \in \Xi$. Then, applying the condition (i) we find $\mathfrak{S} \in \Xi$, $\mathfrak{S} \geqslant \mathfrak{Z}$, such that $(X_0, \mathfrak{Z}, \mathfrak{S})$ admits the properties (ACC) and (E_0). It follows easily from the definition of (ACC) that it holds for $(X_0, \mathfrak{Z}, \mathfrak{T})$ for $\mathfrak{T} \geqslant \mathfrak{S}$ provided it holds for $(X_0, \mathfrak{Z}, \mathfrak{S})$. Taking $\mathfrak{T}$ reflexive and such that X_0 admits the separation principle in every $\mathfrak{X}_{\mathfrak{T},n}$, $n = 1, 2, \ldots$, we find from Proposition 5.3, part II, that $(X_0, \mathfrak{Z}, \mathfrak{T})$ admits (A_0). Since (E_0) for $(X_0, \mathfrak{Z}, \mathfrak{T})$ follows from (E_0) for $(X_0, \mathfrak{Z}, \mathfrak{S})$, $\mathfrak{S} \leqslant \mathfrak{T}$, we can apply Proposition 5.4 and find that $(X_0, \mathfrak{Z}, \mathfrak{T})$ admits (E). Therefore x_0', having some $\|x_0'\|_{\mathfrak{Z},m}^*$ finite, must admit an extension $x' \in X'$. This finishes the proof of Theorem 5.3.

Theorems 4.1 and 4.2 follow trivially from Theorem 5.3 and Theorem 5.1, the latter proved for strict $(\mathcal{IF})$-sequences.

Proof of Theorem 4.1. Since from Theorem 5.1 it follows that the part concerning the Property (E_0) in the condition (i) of Theorem 5.3, II is always satisfied for strict $(\mathcal{IF})$-sequences, the condition (i) in the case of strict $(\mathcal{IF})$-sequence amounts to stating that X_0 is well located in $\mathfrak{X}$ with respect to the basis of coverings Ξ.

Now, for strict $\mathfrak{X}$ every covering must admit the Property (D) and every subspace of $\mathfrak{X}$ admits the separation principle in every moderation $\mathfrak{X}_{\mathfrak{S},n}$ of $\mathfrak{X}$. Theorem 5.2 independently shows that for every $x_0' \in X_0'$ there exists a covering $\mathfrak{Z}$ of $\mathfrak{X}$ such that all $\|x_0'\|_{\mathfrak{Z},n}^*$ are finite. All these facts confronted with the part II of Theorem 5.3 prove that for strict $\mathfrak{X}$ the part II of Theorem 5.3 amounts to Theorem 4.1.

Proof of Theorem 4.2. Theorem 4.2 is the fully written version of the part I of Theorem 5.3.

References

[1] L. Ehrenpreis, *Solutions of some problems of division, IV*, Amer. J. Math. 82 (1960), p. 522-588.

[2] L. Hörmander, *On the range of convolution operators*, Ann. of Math. 76 (1962), p. 148-170.

[3] L. Schwartz, *Théorie des distributions, I-II*, Paris 1950-51.

[4] W. Słowikowski, *On the continuity of inverse operators in (LF)-spaces*, Bull. Amer. Math. Soc. 69 (1963), p. 832-834.

[5] — *The concept of inductive families of (F)-spaces in connection with solvability of linear equations*, Bull. Acad. Polon. Sci., Ser. sci. math., astr. et phys., 11 (1963), p. 517-520.

[6] — *Linear operators in spaces of distributional type*, ibidem, p. 521-524.

[7] — *On extensions of sequentially continuous functionals to distributions*, ibidem 13 (1965), p. 411-414.

[8] — *An example of a linear functional with no extension to a distribution*, ibidem, p. 415-419.

[9] — *Extensions of sequentially continuous functionals in (LF)-spaces*, ibidem, p. 455-458.

[10] — *Extensions of linear sequentially continuous functionals in inductive sequences of (F)-spaces*, ibidem, p. 575-578.

[11] — *On the theory of (F)-sequences*, Studia Mathematica 25 (1965), p. 281-296.

[12] — *On the theory of inductive families*, ibidem, p. 1-10.

[13] — *Linear equation in inductive families of (F)-spaces*, to appear.

[14] V. Pták, *Some open mappings theorems in LF-spaces and their application to existence theorems for convolution equations*, Math. Scand. 16 (1965), p. 75-93.

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