STUDIA MATHEMATICA, T. XXVI. (1966)

On some classes of modular spaces

by

W. ORLICZ (Poznań)

1. In this section we define some terms and introduce some notation we shall be using. By *φ-function* we mean a continuous, non-decreasing function $\varphi(u)$, defined for $u \geqslant 0$, vanishing only at $u = 0$, and tending to ∞ with $u \to \infty$. φ-functions will be denoted by $\varphi, \psi, \ldots$ and their inverse functions by $\varphi_{-1}, \psi_{-1}, \ldots$

A φ-function φ will be called *s-convex*, $0 < s \leqslant 1$, if

$$\varphi(\alpha u+\beta v) \leqslant \alpha^s \varphi(u)+\beta^s \varphi(v)$$

for $\alpha, \beta \geqslant 0$, $\alpha^s+\beta^s = 1$ and arbitrary $u, v \geqslant 0$. A 1-convex φ-function will be called briefly *convex*, as customary. It is readily seen that s-convex φ-functions are strictly increasing in $(0, \infty)$. A φ-function φ is said to satisfy the condition (0_1) if $\varphi(u)/u \to 0$ as $u \to 0$; it satisfies the condition (∞_1) if $\varphi(u)/u \to \infty$ as $u \to \infty$. For a φ-function φ which satisfies conditions (0_1) and (∞_1) a *complementary* function can be defined by the formula

$$\varphi^*(v) = \sup_{u\geqslant 0} \big(uv-\varphi(u)\big).$$

The complementary function is always a convex φ-function, and it satisfies conditions (0_1), (∞_1).

The function $\bar{\varphi}(u) = (\varphi^*)^*(u)$ is the greatest convex φ-function satisfying (0_1), (∞_1), and for which $\varphi(u) \geqslant \bar{\varphi}(u)$ for any $u \geqslant 0$.

The letter T will always stand for an abstract set on which real-valued functions $x, y, z, \ldots$ are defined. For a set e of elements of T, χ_e denotes its characteristic function $\chi_e(t) = 0$ for $t \in T$, if e is an empty set. We use very often the notation a, instead of $a\chi_T$, and we occasionally write $\mathit{1}$ instead of χ_T. A *simple function* is a function of the form $a_1 \chi_{e_1}(t)+a_2 \chi_{e_2}(t)+\ldots+a_n \chi_{e_n}(t)$. For any φ-function φ the symbol $\varphi(|x|)$ denotes the function $\varphi\big(|x(t)|\big)$ defined on T, $\sup x$ denotes $\sup x(t)$, where the supremum is taken over all $t \in T$; the symbols $x+y$, xy etc. have the usual meaning. By $x_n \to x$ as $n \to \infty$ or $x_n \rightrightarrows x$ as $n \to \infty$ we always denote that $x_n(t)$ converges pointwise or uniformly in T respectively, to $x(t)$ as $n \to \infty$. The symbol $x \leqslant y$ denotes that $x(t) \leqslant y(t)$ for any $t \in T$,

$x \vee y$ or $x \wedge y$ stand for the function $\sup(x(t), y(t))$ or $\inf(x(t), y(t))$, respectively.

1.1. Throughout this paper X will always denote a collection of real-valued functions defined and bounded on T and satisfying the following conditions:

1° The class of sets $e \subset T$, for which $\chi_e \in X$, is a Boolean algebra of sets.

2° X is a real linear space.

3° For any $\varepsilon > 0$ there exists a simple function $a_1\chi_{e_1} + a_2\chi_{e_2} + \ldots + a_n\chi_{e_n}$, such that $\chi_{e_i} \in X$, and

$$|x - (a_1\chi_{e_1} + a_2\chi_{e_2} + \ldots + a_n\chi_{e_n})| < \varepsilon.$$

4° If $x_n \in X$, $x_n \rightrightarrows x$ as $n \to \infty$, then $x \in X$.

By E we shall always denote the Boolean algebra of sets defined in 1°. By a *partition* of T we mean a finite class of non-empty sets $e_1, e_2, \ldots, e_n$ such that $\bigcup e_i = T$, $e_i \in E$, $e_i \cap e_j = 0$ for $i \neq j$, $i, j = 1, 2, \ldots, n$. In the sequel when speaking of simple functions we tacitly assume that $\chi_{e_i} \in X$ for $i = 1, 2, \ldots, n$.

1.2. Any simple function x can be represented in the canonical form

$$(+) \qquad x = a_1\chi_{e_1} + a_2\chi_{e_2} + \ldots + a_n\chi_{e_n},$$

where $(e_1, e_2, \ldots, e_n)$ is a partition of T. This representation is unique for a given partition of T; $|x| = |a_1|\chi_{e_1} + |a_2|\chi_{e_2} + \ldots + |a_n|\chi_{e_n}$, and more general $\varphi(|x|) = \varphi|(a_1)|\chi_{e_1} + \varphi(|a_2|)\chi_{e_2} + \ldots + \varphi(|a_n|)\chi_{e_n}$. For any simple functions x, y in X there exist canonical representation of x and y corresponding to a common partition of T. If (+) and

$$y = b_1\chi_{e_1} + b_2\chi_{e_2} + \ldots + b_n\chi_{e_n}$$

are such representations for x and y, respectively, then $x \vee y$ or $x \wedge y$ can be represented in the form $c_1\chi_{e_1} + c_2\chi_{e_2} + \ldots + c_n\chi_{e_n}$, where $c_i = \sup(a_i, b_i)$ or $c_i = \inf(a_i, b_i)$, respectively.

1.3. (a) *If $x, y \in X$, then $xy \in X$*; (b) *if $x \in X$, then for any φ-function φ, $\varphi(|x|) \in X$*; (c) *X is a vector lattice, assuming the natural relation of order in X; the join of elements x, y in X is $x \vee y$; and $x \wedge y$ is their meet.*

Ad (a). Because of the equality $xy = ((x+y)^2 - (x-y)^2)/4$ it is sufficient to prove that $x^2 \in X$. But this is evidently true for any simple function, so also for any x in X, by 3° and 4°.

Ad (b). Let γ be a continuous function for $u \geqslant 0$ and, for a given $\varepsilon > 0$, let be $|\gamma(u_1) - \gamma(u_2)| < \varepsilon$ if $|u_1 - u_2| < \delta$, $|u_1|, |u_2| \leqslant u_0$, where $\sup|x| \leqslant u_0/2$. If y is a simple function such that $|x-y| < \inf(\delta, u_0/2)$, then $\sup|y| < u_0$, $|\gamma(|x|) - \gamma(|y|)| < \varepsilon$, $\gamma(|y|)$ is a simple function, and in view of 3°, 4° the assertion (b) follows.

Ad (c). Elements $x \wedge y$, $x \vee y$ belong to X with x, y, for X is a linear space, and $|z| \in X$ if $z \in X$.

1.4. *For any $\varepsilon > 0$, $x \in X$, $x \geqslant 0$ there exists a simple function y such that $0 \leqslant y \leqslant x$, $|x-y| < \varepsilon$.*

In fact, if for a simple function z we have $|x-z| < \varepsilon/2$, then $y = (z - \varepsilon/2) \vee 0$ is a simple function for which $0 \leqslant y \leqslant x$, $|x-y| = |(x - z + \varepsilon/2) \wedge x| \leqslant x - z + \varepsilon/2 < \varepsilon$.

1.4.1. A function x is said to be *measurable with respect to E* if for any real number a both sets $\{t: x(t) > a\}$, $\{t: x(t) < a\}$ belong to E.

(a) *If x is bounded in T and measurable with respect to E, then $x \in X$.*

(b) *Any simple function in X is measurable with respect to E.*

(c) *If E is a σ-algebra, then each function in X is measurable with respect to E.*

Let us prove (c), for example. It is enough to show that $e = \{t: x(t) < a\} \in E$. For $n = 1, 2, \ldots$ there exists a simple function y_n for which $|x - y_n| < 1/n$. The sets $e_n = \{t: y_n(t) < a - 1/n\}$ are measurable with respect to E, and so is $e = \bigcup_{n=1}^{\infty} e_n$.

1.5. *A class of functions X, subject to the following conditions, satisfies conditions 1°-4° in 1.3:*

1° $1 \in X$; 2° *if $x \in X$, then $x^2 \in X$*; 3° *if $x_n \to x$, $x_n \in X$, $|x_n| \leqslant 1$ for $n = 1, 2, \ldots$, then $x \in X$*; 4° *X is a real linear space.*

Under these assumptions E is a σ-algebra and X a σ-complete linear lattice.

By 2° and 4° it follows that $xy \in X$ if $x, y \in X$. This and 4° implies that any finite union and intersection of sets e_i belonging to E is also in E. Because of 1° $T \subset E$, and by 3° E is σ-complete, i. e. $e_1 \subset e_2 \subset \ldots$, $e_n \in E$ implies $\lim_{n\to\infty} e_n \in E$. Given a function $\gamma(u)$ continuous in $(-\infty, \infty)$. For any $u_0 > 0$ we can find a polynomial $w_k(u)$ such that $|\gamma(u) - w_k(u)| < 1/k$ if $|u| < u_0$, for $k = 1, 2, \ldots$ we have $w_k(x) \in X$, hence $w_k(x) \rightrightarrows \gamma(x)$, whence $\gamma(x) \in X$, and in particular $|x| \in X$. This proves that X is a linear lattice, assuming the natural relation of order in X. Let us prove now the property 1.1, 3°. In view of 1.4.1(a) we have only to show that each $x \in X$ is measurable with respect to E. As readily seen it is enough to prove the measurability of the set $e = \{t: x(t) < 1\}$, where $x \geqslant 0$. But $(x \wedge 1)^n \in X$ for $n = 1, 2, \ldots, 1 - (x \wedge 1)^n \to \chi_e$ as $n \to \infty$ and so, by 1°, 3°, the function χ_e belongs to X.

2. Let X be defined as in section 1.1 and suppose that a real-valued functional $\overline{m}(\cdot)$ is defined on X, which fulfils the following conditions:

(1) $\overline{m}(1) = 1$.

(2) $\overline{m}(|x|) = \overline{m}(x)$.

(3) $\overline{m}(x) \leqslant \overline{m}(y)$ for $0 \leqslant x \leqslant y$.
(4) $\overline{m}(x+y) \leqslant \overline{m}(x)+\overline{m}(y)$.
(5) $\overline{m}(\lambda x) = |\lambda|\overline{m}(x)$ for any real λ.

Conditions (2)-(5) are nothing but usual axioms defining a monotone pseudonorm in function spaces. In view of purposes which we have in mind it seems more appropriate here to call $\overline{m}(\cdot)$ a *subadditive* (or an upper) *mean value* in the space X, or shortly a *mean value*. In section 8 some examples of mean values, which may be of some interest from the viewpoint of applications, are given. Let us remark that spaces X provided with the pseudonorm $\overline{m}(\cdot)$ are closely connected with so called Banach (normed) function spaces. During the last ten years many authors have contributed to the general theory of normed function spaces. Particularly Luxemburg and Zaanen give a systematic presentation of the theory in question in a series of papers, which they started to publish 3 years ago [3] (cf. also [2]). In their papers the basic space X is the space of functions measurable with respect to a σ-additive and σ-finite (totally finite) measure μ, while our space X fulfils more general conditions. Besides, we are only concerned in this paper with some special questions connected with the theory of spaces $L^{*\varphi}$ (spaces of φ-integrable functions) [1], [5]. In the theory of space $L^{*\varphi}$ (called also *Orlicz spaces*) one defines a modular by means of an integral. Instead of an integral the notion of mean value to define modulars is used systematically in this paper. We restrict ourselves to the class of bounded functions. But this restriction is not always necessary, and we intend to return, in the second part of the paper, to some questions concerning the generalized spaces of φ-integrable functions, without this restriction.

The following lemmata will be used often:

2.1. $|\overline{m}(x)-\overline{m}(y)| \leqslant \overline{m}(x-y)$.

2.1.1. *If* $x_n \rightrightarrows x$ *as* $n \to \infty$, *then* $\overline{m}(x_n) \to \overline{m}(x)$.

2.2. Let us introduce the notation $\overline{m}(e)$ for $\overline{m}(\chi_e)$, where $e \in E$. If e is an empty set, then $\overline{m}(e) = 0$. The set function $\overline{m}(\cdot)$ is a finitely subadditive and monotone measure on E, i. e.

$$\overline{m}\Big(\bigcup_1^n e_i\Big) \leqslant \overline{m}(e_1)+\ldots+\overline{m}(e_n),$$

where $e_i \in E$, $\overline{m}(e_1) \leqslant \overline{m}(e_2)$, if $e_1, e_2 \in E$, $e_1 \subset e_2$. However, $\overline{m}(\cdot)$ need not be countably subadditive in general. In the example 8, II, 1α), $\overline{m}(e)$ is the upper relative Lebesgue measure on the half-line $u \geqslant 0$; in this case there exist disjoint sets e_i for which $T = \bigcup_1^\infty e_i$, $\overline{m}(e_i) = 0$ for $i = 1, 2, \ldots$ but $\overline{m}(T) = 1$. We shall call $\overline{m}(e)$ the $\overline{m}$-measure of the set e. In general, the range of the set function $\overline{m}(\cdot)$ is contained in $\langle 0, 1\rangle$ but need not be identical with it. If the set of values $\overline{m}(e)$, $e \in E$, is dense in $\langle 0, 1\rangle$ the algebra of sets E is said to fulfil the *property* $(\mathscr{D})$ (with respect to the given $\overline{m}(\cdot)$). In the example 8,I(a) 2), (b) 2), II, 1α), 2β), the corresponding algebra E fulfil the property $(\mathscr{D})$, exactly speaking, the range of $\overline{m}(\cdot)$ is $\langle 0, 1\rangle$. The possibility that under such general assumptions on E and $\overline{m}(\cdot)$ as in this paper, the values $\overline{m}(e)$ lie dense in $\langle 0, 1\rangle$, but do not fill up $\langle 0, 1\rangle$ is not a priori excluded, although we cannot actually give an example for such a situation.

2.3. *If* $\lambda_0 > 1$, $\lambda > 0$, $a_\lambda = \{t: |x(t)| > \lambda\}$ $(a_\lambda = \{t: |x(t)| \geqslant \lambda\})$, *there exists an* $\overline{m}$*-measurable set* e_λ *such that* $a_\lambda \subset e_\lambda$

$$(*)\qquad \lambda\overline{m}(e_\lambda) \leqslant \lambda_0\overline{m}(x).$$

Let $0 < \eta < \lambda$; by 1.4 there exists a non-negative simple function y such that $\big||x|-y\big| < \eta$. In virtue of 1.4.1(b) the set $e_\lambda = \{t: y(t) \geqslant \lambda-\eta\}$ is measurable with respect to E and $a_\lambda \subset e_\lambda$, for $|x| < y+\eta$. Since $|\overline{m}(x)-\overline{m}(y)| \leqslant \eta$, the inequality $(\lambda-\eta)\overline{m}(e_\lambda) \leqslant \overline{m}(y) \leqslant \overline{m}(x)+\eta$ holds. If $\overline{m}(x) > 0$, we assume η sufficiently small to fulfil the inequalities $\lambda(\sqrt{\lambda_0})^{-1} < \lambda-\eta$, $\overline{m}(x)+\eta < \sqrt{\lambda_0}\,\overline{m}(x)$; we obtain $\lambda(\sqrt{\lambda_0})^{-1}\overline{m}(e_\lambda) \leqslant \sqrt{\lambda_0}\,\overline{m}(x)$, and $(*)$ follows. If $\overline{m}(x) = 0$, the inequality $(*)$ is also true, for in this case $\overline{m}(e_\lambda) = 0$. In fact, we can assume $y \leqslant |x|$, whence $\overline{m}(e_\lambda)(\lambda-\eta) \leqslant \overline{m}(y) \leqslant \overline{m}(x) = 0$, $\overline{m}(e_\lambda) = 0$.

2.3.1. *Let* $|x_n| \leqslant x$ *for* $n = 1, 2, \ldots$ *If, for a* φ*-function* ψ, $\overline{m}\big(\psi(x_n)\big) \to 0$ *as* $n \to \infty$, *then for any* φ*-function* $\overline{m}\big(\varphi(|x_n|)\big) \to 0$ *as* $n \to \infty$.

Let $a_\lambda^n = \{t: \psi\big(|x_n(t)| \geqslant \psi(\lambda)\big\}$. In view of 2.3 there exist sets $e_\lambda^n \in E$ such that $a_\lambda^n \subset e_\lambda^n$, $\psi(\lambda)\overline{m}(e_\lambda^n) \leqslant 2\overline{m}\big(\psi(|x_n|)\big)$. We choose $\lambda > 0$ in such a manner that the inequality $\psi(u) \leqslant \psi(\lambda)$ implies $\varphi(u) \leqslant \varepsilon$. If $t \in T-e_\lambda^n$, then $\psi\big(|x_n(t)|\big) \leqslant \psi(\lambda)$ and consequently $\varphi\big(|x_n(t)|\big) \leqslant \varepsilon$. We have the inequalities

$$\overline{m}\big(\varphi(|x_n|)\big) = \overline{m}\big(\varphi(|x_n|)\chi_{e_\lambda^n}+\varphi(|x_n|)(1-\chi_{e_\lambda^n})\big) \leqslant \sup\varphi(|x|)\,\overline{m}(e_\lambda^n)+\varepsilon$$
$$\leqslant 2\sup\varphi(|x|)\,\psi(\lambda)^{-1}\overline{m}\big(\psi(|x_n|)\big)+\varepsilon,$$

and the relation $\overline{m}\big(\varphi(|x_n|)\big) \to 0$ as $n \to \infty$ follows.

2.3.2. *If, for a* φ*-function* ψ, $\overline{m}\big(\psi(|x|)\big) = 0$, *then for any* φ*-function* $\overline{m}\big(\varphi(|x|)\big) = 0$.

This is a trivial consequence of 2.3.1.

2.4. *A necessary and sufficient condition for* $\overline{m}(x) = 0$ *is that for any* $\lambda > 0$ *there exists a set* e_λ *with* $\overline{m}$*-measure* 0, *such that* $a_\lambda \subset e_\lambda$, *where* $a_\lambda = \{t: |x(t)| > \lambda\}$.

The necessity follows from 2.3, immediately. To prove the sufficiency we use the inequality

$$\overline{m}(x) \leqslant \overline{m}(x\chi_{e_\lambda})+\overline{m}\big(x(1-\chi_{e_\lambda})\big) \leqslant \sup|x|\,\overline{m}(e_\lambda)+\lambda\overline{m}(\mathit{1}).$$

2.5. By 2.4 it is seen that if, for a function x, $\overline{m}(x)=0$, then the set $e=\{t\colon |x(t)|\neq 0\}$ is in the union of countably many sets of $\overline{m}$-measure 0. Let us remark that the converse assertion is not true in general.

The class of all $\overline{m}$-measurable sets with the $\overline{m}$-measure 0 will be denoted by E_0. Evidently E_0 is a ring, but not a σ-ring, even when E is a σ-algebra. Let us still observe, that if E is a σ-algebra, then in 2.5 we can assume $a_\lambda=e_\lambda$. Under this hypothesis the necessary and sufficient condition for $\overline{m}(x)=0$ is $a_\lambda\in E_0$ for any positive λ.

2.6. *For a φ-function φ and $x\in X$, $\overline{m}(\varphi(\lambda|x|))\to 0$ as $\lambda\to 0+$, holds.*

2.6.1. (a) *For any φ-function, $\overline{m}(\varphi(\lambda|x|))$ is a continuous non-decreasing function for $\lambda\geqslant 0$.*

The inequality $\overline{m}(\varphi(\lambda_1|x|))\leqslant\overline{m}(\varphi(\lambda_2|x|))$ for $0\leqslant\lambda_1\leqslant\lambda_2$ follows immediately from $\varphi(\lambda_1|x|)\leqslant\varphi(\lambda_2|x|)$. Given an $\varepsilon>0$ we choose a $\delta>0$ such that $|\lambda-\lambda_0|\sup|x|<\delta$ implies $|\varphi(\lambda|x|)-\varphi(\lambda_0|x|)|<\varepsilon$, hence

$$\left|\overline{m}(\varphi(\lambda|x|))-\overline{m}(\varphi(\lambda_0|x|))\right|\leqslant\overline{m}(\varphi(\lambda|x|)-\varphi(\lambda_0|x|))\leqslant\varepsilon,$$

and the continuity of $\overline{m}(\varphi(\lambda|x|))$ follows.

2.6.2. *If a φ-function φ is s-convex, then $\overline{m}(\varphi(\lambda|x|))$ is strictly increasing for $\lambda\geqslant 0$.*

The same statement is true for any φ-function for which $\inf\limits_{u>0}\varphi(\lambda u)/\varphi(u)>1$ for any $\lambda>1$.

2.6.3. *For any φ-function $\overline{m}(\varphi(\lambda x))\to\infty$ as $\lambda\to\infty$, if $\overline{m}(x)>0$.*

There exists a simple function $y\geqslant 0$ such that $y\leqslant|x|$, $|x-y|<\varepsilon$. We can assume $\overline{m}(y)>0$, for $|\overline{m}(x)-\overline{m}(y)|<\varepsilon$. The sets $e_\lambda=\{t\colon y(t)>\lambda\}$ are $\overline{m}$-measurable and, by 2.4, $\overline{m}(e_{\lambda_0})>0$ for a $\lambda_0>0$. Because of the inequality

$$\overline{m}(\varphi(\lambda|x|))\geqslant\overline{m}(\varphi(\lambda y))=\overline{m}(\varphi(\lambda\lambda_0\lambda_0^{-1}y))\geqslant\overline{m}(e_{\lambda_0})\varphi(\lambda\lambda_0)$$

we obtain $\overline{m}(\varphi(\lambda|x|))\to\infty$ as $\lambda\to\infty$.

2.7. The *essential supremum* of a function x, which will be written $\sup^*x$, is by definition the infimum of numbers λ for which $\overline{m}(x\vee\lambda-\lambda)=0$. This definition is equivalent to the following one:

The essential supremum of x is the infimum of λ's with the property that $a_\lambda=\{t\colon x(t)>\lambda\}$ can be covered by an $e_\lambda\in E_0$.

Indeed, if $a_\lambda\subset e_\lambda$, $e_\lambda\in E_0$, then, in virtue of $x\vee\lambda-\lambda=(x-\lambda)\chi_{a_\lambda}$, we have $\overline{m}(x\vee\lambda-\lambda)=0$. If $\overline{m}(x\vee\lambda-\lambda)=0$, then, by 2.4, there exists a set $e_\lambda\in E_0$ which covers the set $\{t\colon x\vee\lambda-\lambda>\varepsilon\}=\{t\colon x(t)>\lambda+\varepsilon\}$.

Evidently $\sup^*|x|=0$ implies $\overline{m}(x)=0$ and conversely; $\sup^*|x|\leqslant\sup|x|$.

2.7.1. *The functional $\sup^*|x|$ is a subadditive mean value on X. For any $x,y\in X$ the inequality*

$$(*)\qquad \overline{m}(xy)\leqslant\sup{}^*|x|\overline{m}(y)$$

holds.

We shall prove the condition 2 (4), for example. Since

$$(|x|\vee\lambda_1-\lambda_1)+(|y|\vee\lambda_2-\lambda_2)\geqslant|x+y|\vee(\lambda_1+\lambda_2)-(\lambda_1+\lambda_2),$$

we have $\overline{m}((x+y)\vee(\lambda_1+\lambda_2)-(\lambda_1+\lambda_2))=0$ when $\sup^*|x|<\lambda_1$, $\sup^*|y|<\lambda_2$, hence $\sup^*|x+y|\leqslant\lambda_1+\lambda_2$, $\sup^*|x+y|\leqslant\sup^*|x|+\sup^*|y|$. To prove $(*)$ it suffices to remark that

$$|x|\,|y|\leqslant(|x|\,|y|\vee\lambda-\lambda|y|)+\lambda|y|,$$

$$\overline{m}(xy)\leqslant\overline{m}(|x|\,|y|\vee\lambda-\lambda|y|)+\lambda\overline{m}(y)=\lambda\overline{m}(y)\quad\text{ for }\quad\lambda>\sup{}^*|x|,$$

and consequently $\overline{m}(xy)\leqslant\sup^*|x|\overline{m}(y)$.

2.7.2. *Let y be a simple function whose canonical form is*

$$(*)\qquad y=a_1\chi_{e_1}+a_2\chi_{e_2}+\ldots+a_n\chi_{e_n}+b_1\chi_{\bar e_1}+b_2\chi_{\bar e_2}+\ldots+b_s\chi_{\bar e_s},$$

where $(e_1,e_2,\ldots,e_n,\bar e_1,\bar e_2,\ldots,\bar e_s)$ is a partition, $\overline{m}(e_i)>0$ for $i=1,2,\ldots,n$, $\overline{m}(\bar e_j)=0$ for $j=1,2,\ldots,s$. Then $\sup^*y=\sup\limits_i a_i$.

Since

$$y\vee\lambda-\lambda=(a_1\vee\lambda-\lambda)\chi_{e_1}+\ldots+(a_n\vee\lambda-\lambda)\chi_{e_n}+$$
$$+(b_1\vee\lambda-\lambda)\chi_{\bar e_1}+\ldots+(b_s\vee\lambda-\lambda)\chi_{\bar e_s},$$

we get $\overline{m}((a_i\vee\lambda-\lambda)\chi_{e_i})=(a_i\vee\lambda-\lambda)\overline{m}(e_i)=0$, if $\overline{m}(y\vee\lambda-\lambda)=0$, and consequently $a_i\vee\lambda=\lambda$, $a_i\leqslant\lambda$, $\sup\limits_i a_i\leqslant\sup^*y$. Conversely, if $\lambda>\sup a_i$, then $a_i\vee\lambda-\lambda=0$ for $i=1,2,\ldots,n$, hence $\overline{m}(y\vee\lambda-\lambda)=0$, $\sup^*y\leqslant\sup\limits_i a_i$.

2.7.3. *Let $0<\lambda<\sup^*|x|$. There exists a set e with positive $\overline{m}$-measure such that $e\subset\{t\colon |x(t)|>\lambda\}$.*

Choose a simple function y such that $0\leqslant y\leqslant|x|$, $\big||x|-y\big|<\varepsilon$. Since, by 2.7.1,

$$\left|\sup{}^*|x|-\sup{}^*|y|\right|\leqslant\sup{}^*|x-y|\leqslant\varepsilon,$$

it can be assumed $\sup^*|y|>\lambda$. Representing $|y|$ in the canonical form 2.7.2 $(*)$ we have, for a certain k, $a_k>\lambda$, and since $a_i,b_i\geqslant 0$, for $t\in e=e_k$ the inequality $|x(t)|\geqslant a_k>\lambda$ holds, moreover $\overline{m}(e)>0$.

2.7.4. *For any φ-function* $\sup^*\varphi(|x|) = \varphi(\sup^*|x|)$.

Assume first x to be a non-negative simple function y, represented in the canonical form 2.7.2 (*). Then, $a_i, b_i \geqslant 0$, $\sup^* y = \sup_i a_i$, and

$$\varphi(y) = \varphi(a_1)\chi_{e_1}+\varphi(a_2)\chi_{e_2}+\ldots+\varphi(a_n)\chi_{e_n}+\varphi(b_1)\chi_{\bar e_1}+\varphi(b_2)\chi_{\bar e_2}+\ldots+ +\varphi(b_s)\chi_{\bar e_s}$$

consequently

$$\sup^*\varphi(y) = \sup_i\varphi(a_i) = \varphi(\sup_i a_i) = \varphi(\sup^* y).$$

We verify the formula $\sup^*\varphi(|x|) = \varphi(\sup^*|x|)$ for an arbitrary function $x\in X$, approximating $|x|$ uniformly by simple functions.

2.8. The mean value $\overline{m}(\cdot)$ is said *extreme mean value*, whenever $\overline{m}(x) = \sup^*|x|$ for any x in X.

Each of the following conditions is necessary and sufficient for a mean value to be an extreme mean value:

A. *For any $e\in E$ is either $\overline{m}(e) = 1$ or $\overline{m}(e) = 0$.*

B. $\overline{m}(x^2) = (\overline{m}(x))^2$ *for any* $x\in X$.

Ad A. The condition is sufficient. Let $0 < \sup^*|x|$ and $0 < \lambda < \sup^*|x|$. By 2.7.3 there exists a set e with positive $\overline{m}$-measure such that $|x|\chi_e > \lambda$ for $t\in e$. By A we have $\overline{m}(e) = 1$, whence $\lambda \leqslant \overline{m}(x)$, and $\sup^*|x| \leqslant \overline{m}(x)$. On the other hand, we have $\overline{m}(x) \leqslant \sup^*|x|$ for every x in X.

Ad B. The necessity follows by 2.7.4. The sufficiency is trivial since for any $e\in E$, $\overline{m}(e) = \overline{m}(\chi_e^2) = \overline{m}(e)^2$.

3. In this section as well as in the following sections the notation $x = y[\overline{m}]$ will be used for a pair of elements $x, y\in X$, whenever $\overline{m}(x-y) = 0$. If $x = y[\overline{m}]$, then the functions x, y are called *$\overline{m}$-equal*. Clearly, $\chi_{e_1} = \chi_{e_2}[\overline{m}]$, for $e_1, e_2\in E$, if and only if $\overline{m}\big((e_1-e_2)\cup(e_2-e_1)\big) = 0$; $\overline{m}(x) = \overline{m}(y)$ if $x = y[\overline{m}]$. It is easily seen that the relation $\cdot = \cdot[\overline{m}]$ is an equivalence relation. We introduce still the notation $X_0 = \{x\in X: \overline{m}(x) = 0\}$; of course X_0 is a linear subspace of X.

3.1. (a) *If $x_1 = x_2[\overline{m}]$, $y_1 = y_2[\overline{m}]$, then for any reals α, β,*

$$\alpha x_1+\beta y_1 = \alpha x_2+\beta y_2[\overline{m}];$$

(b) *if $x_1 = x_2[\overline{m}]$, then for any φ-function* $\varphi(|x_1|) = \varphi(|x_2|)[\overline{m}]$;

(c) *if $x_1 = x_2[\overline{m}]$, $y_1 = y_2[\overline{m}]$, then* $x_1y_1 = x_2y_2[\overline{m}]$.

Ad (b). From the inequality $\big||x_1|-|x_2|\big| \leqslant |x_1-x_2|$ it follows $|x_1| = |x_2|[\overline{m}]$. For an arbitrarily prescribed number $\varepsilon > 0$ let us choose $\delta > 0$ such that $\big||x_1(t)|-|x_2(t)|\big| < \delta$ implies $\big|\varphi(|x_1(t)|)-\varphi(|x_2(t)|)\big| < \varepsilon$. Since $x_1 = x_2[\overline{m}]$, there exists a set in E_0 which covers the set

$$a = \{t\colon \big||x_1(t)|-|x_2(t)|\big| \geqslant \delta\}.$$

Therefore

$$\overline{m}\big(\varphi(|x_1|)-\varphi(|x_2|)\big) \leqslant \overline{m}(\varepsilon) = \varepsilon \quad\text{and}\quad \varphi(|x_1|) = \varphi(|x_2|)[\overline{m}]$$

follows.

Ad (c). It suffices to apply the inequality

$$\overline{m}(x_1y_1-x_2y_2) \leqslant \sup|x_1|\,\overline{m}(y_1-y_2)+\sup|y_2|\,\overline{m}(x_1-x_2) = 0.$$

3.2. *If $x_1 = x_2[\overline{m}]$, $y_1 = y_2[\overline{m}]$, then $x_1\vee y_1 = x_2\vee y_2[\overline{m}]$, $x_1\wedge y_1 = x_2\wedge y_2[\overline{m}]$.*

3.2.1. *If $x_1 = x_2[\overline{m}]$, then* $\sup^* x_1 = \sup^* x_2$.

In virtue of 3.1 and 3.2 we have $x_1\vee\lambda-\lambda = x_2\vee\lambda-\lambda[\overline{m}]$, $\overline{m}(x_1\vee\lambda-\lambda) = \overline{m}(x_2\vee\lambda-\lambda)$ and we apply now the definition of the essential supremum.

3.3. In order to provide the quotient space X/X_0 with a vector lattice structure we will now introduce an order relation in an appropriate way. We will say that an element x *is less or equal to y in the sense $\overline{m}$*, and denote this by $x\leqslant y[\overline{m}]$, whenever $\overline{m}(x\vee y-y) = 0$, or, equivalently, $x\vee y = y[\overline{m}]$. It follows from this definition that $|x| \leqslant |y|[\overline{m}]$ implies $\overline{m}(|x|) \leqslant \overline{m}(|y|)$. If $x\leqslant y[\overline{m}]$, then there exists a function $\overline{m}$-equal to y (to x) and $\geqslant x$ ($\leqslant y$). In fact, since $x\vee y-y = x-x\wedge y$, the equality $x = x\wedge y[\overline{m}]$ holds, hence $z_1 = x\wedge y\leqslant y$, $x\leqslant x\vee y = z_2$. Conversely, if $z_1 = x[\overline{m}]$, $z_2 = y[\overline{m}]$, $z_1\leqslant z_2$, then $x\leqslant y[\overline{m}]$.

3.3.1. (a) *If $x_1\leqslant x_2[\overline{m}]$ and $x_2\leqslant x_1[\overline{m}]$, then $x_1 = x_2[\overline{m}]$, and conversely.*

(b) *If $x_1 = x_2[\overline{m}]$, $y_1 = y_2[\overline{m}]$, $x_1\leqslant y_1[\overline{m}]$, then $x_2\leqslant y_2[\overline{m}]$.*

(c) *If $x\leqslant y[\overline{m}]$, $\lambda > 0$, then $\lambda x\leqslant\lambda y[\overline{m}]$.*

(d) *If $x\leqslant y[\overline{m}]$, then $x+z\leqslant y+z[\overline{m}]$ for any $z\in X$.*

(e) *If $x\leqslant z[\overline{m}]$, $y\leqslant z[\overline{m}]$, then $x\vee y\leqslant z[\overline{m}]$; if $z\leqslant x[\overline{m}]$, $z\leqslant y[\overline{m}]$, then $z\leqslant x\wedge y[\overline{m}]$.*

(f) *If $|x_1|\leqslant|x_2|[\overline{m}]$, then $\varphi(|x_1|)\leqslant\varphi(|x_2|)[\overline{m}]$ for any φ-function φ.*

For example, we shall prove (b). We have $x_1\vee y_1 = y_1[\overline{m}]$, and, by 3.2, $x_1\vee y_1 = x_2\vee y_2[\overline{m}]$, therefore $y_2 = x_2\vee y_2[\overline{m}]$, $x_2\leqslant y_2[\overline{m}]$.

4. In the sequel the letter $\mathscr{X}$ always stands for the quotient space X/X_0. It follows from the lemmata given in section 3, that defining the addition of classes of $\overline{m}$-equal functions and their multiplication by real scalars, in a natural way, $\mathscr{X}$ becomes a real linear space. It follows also from 3.3.1 that the relation „$\cdot\leqslant\cdot[\overline{m}]$” makes $\mathscr{X}$ to a linear structure. Supremum with respect to the ordering of classes represented by the elements x or y respectively, is the class represented by $x\vee y$, and analogously $x\wedge y$ represents the infimum of these classes.

From now on we will freely use the letters $x, y, z, \ldots$ either as symbols of individual functions or as symbols of classes of $\overline{m}$-equal elements, to which they belong, i. e. as symbols of elements of $\mathscr{X}$. In a similar way the symbols $x \leqslant y[\overline{m}]$, $x = y[\overline{m}]$, $\sup^*|x|$, $\overline{m}(|x|)$, xy, $\varphi(|x|)$ etc. will be used, and that is motivated by corresponding invariant properties with respect to $\overline{m}$-equality, as given in 3. What concerns symbols $x \leqslant y$, $x_n \to x$, $x_n \Rightarrow x$ etc. we attach the same meaning to them as before, i. e. they will be applied only to functions as elements of X.

4.1. *For any φ-function φ, $\overline{m}(\varphi(|x|))$ is a modular in $\mathscr{X}$ in the sense of* [8], [12], *i. e. this functional possesses the following properties:*

A. $\overline{m}(\varphi(|x|)) = 0$ *if and only if* $x = 0[m]$.

B. $\overline{m}(\varphi(|x_1|)) \leqslant \overline{m}(\varphi(|x_2|))$ *if* $|x_1| \leqslant |x_2|[\overline{m}]$.

C. $\overline{m}(\varphi(|x_1| \vee |x_2|)) \leqslant \overline{m}(\varphi(|x_1|)) + \overline{m}(\varphi(|x_2|))$.

D. $\overline{m}(\varphi(\lambda|x|)) \to 0$ *as* $\lambda \to 0+$.

Property A follows by 2.3.2, to prove C let us remark that $\varphi(|x_1| \vee |x_2|) \leqslant \varphi(|x_1|) + \varphi(|x_2|)$ for any $x_1, x_2 \in X$. The property D is a consequence of 2.6.

Suppose now φ to be an s-convex function, then the inequality

$$\varphi(\alpha|x_1| + \beta|x_2|) \leqslant \alpha^s\varphi(|x_1|) + \beta^s\varphi(|x_2|) \quad \text{for} \quad \alpha, \beta \geqslant 0,\ \alpha^s + \beta^s = 1$$

holds, which implies

C_s. $\overline{m}(\varphi(\alpha|x_1| + \beta|x_2|)) \leqslant \alpha^s\overline{m}(\varphi(|x_1|)) + \beta^s\overline{m}(\varphi(|x_2|))$
for $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$.

In particular, if φ is convex, the modular $\overline{m}(\varphi(|x|))$ is a convex functional on $\mathscr{X}$.

4.2. It follows from the general theory of modular spaces [7], [8] that in $\mathscr{X}$ an F-norm can be defined, by the formula

$$\|x\|_\varphi = \inf\{\varepsilon > 0 : \overline{m}(\varphi(|x|/\varepsilon)) \leqslant \varepsilon\}.$$

For an s-convex φ-function two others norms — both s-homogeneous — can be defined, as follows [5], [10]:

$$\|x\|_\varphi^s = \inf\{\varepsilon > 0 : \overline{m}(\varphi(|x|/\varepsilon^{1/s})) \leqslant 1\},$$
$$\|x\|_\varphi^{0s} = \inf_{\lambda > 0}\{\lambda^{-s} + \lambda^{-s}\overline{m}(\varphi(\lambda x))\}.$$

If φ is convex, i. e. $s = 1$, the norms $\|\cdot\|_\varphi^1$, $\|\cdot\|_\varphi^{01}$ are homogeneous. For these homogeneous norms the symbols $\|\cdot\|_\varphi^c$ or $\|\cdot\|_\varphi^0$ respectively will be used, instead of $\|.\|_\varphi^1$ or $\|\cdot\|_\varphi^{01}$ respectively. Let us notice that all norms mentioned above are monotonic, and for an s-convex φ-function, they are equivalent (in $\mathscr{X}$) each to the other. An immediate consequence of the definition of $\|\cdot\|_\varphi$ is, that the relation $\|x_n\|_\varphi \to 0$ as $n \to \infty$, and the relation $m(\varphi(\lambda|x_n|)) \to 0$ as $n \to \infty$, for any $\lambda > 0$, are equivalent.

4.3. *The norm $\|\lambda x\|_\varphi$ is continuous with respect to λ, and if $\|x\|_\varphi > 0$, it tends to ∞ with λ.*

We have $\big|\|\lambda x\|_\varphi - \|\lambda_0 x\|_\varphi\big| \leqslant \|(\lambda - \lambda_0)x\|_\varphi$ as $\lambda \to \lambda_0$. The second part of the statement follows by 2.6.3.

4.4. *For any $x \neq 0[\overline{m}]$ in $\mathscr{X}$ there holds the equation $\overline{m}(\varphi|x|/\|x\|_\varphi) = \|x\|_\varphi$, and the equation $\overline{m}(\varphi(|x|/(\|x\|_\varphi^s)^{1/s})) = 1$ (under the assumption that φ is s-convex). The number $\varepsilon = \|x\|_\varphi$ or $\varepsilon = (\|x\|_\varphi^s)^{1/s}$ respectively, is the unique solution, of the first equation or the second one, respectively.*

The first part of the statement is a consequence of 2.6.1 and 2.6.3. If $\overline{m}(\varphi(|x|\varepsilon^{-1})) = \varepsilon$, $\overline{m}(\varphi(|x|\varepsilon_0^{-1})) = \varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$, then $\overline{m}(\varphi(|x|\varepsilon^{-1})) \geqslant \overline{m}(\varphi(|x|\varepsilon_0^{-1}))$ — a contradiction. As concerns the norm $\|x\|_\varphi^s$ it suffices to apply 2.6.2.

4.5. *Assume $|x_n| \leqslant |x|[\overline{m}]$ for $n = 1, 2, \ldots$ Then the relation $\overline{m}(\varphi(|x_n|)) \to 0$ as $n \to \infty$, and the relation $\|x_n\|_\varphi \to 0$ as $n \to \infty$ are equivalent. In particular, the relation $\overline{m}(e_n) \to 0$ as $n \to \infty$ and $\|\chi_{e_n}\|_\varphi \to 0$ as $n \to \infty$, one implies the other.*

By 2.3.1 $\overline{m}(\varphi(\lambda|x_n|)) \to 0$ as $n \to \infty$, for every $\lambda > 0$, and so $\|x_n\|_\varphi \to 0$ as $n \to \infty$, follows. The converse implication is trivial, for $\overline{m}(\varphi(|x_n|)) \leqslant \|x_n\|_\varphi$, when $\|x_n\|_\varphi \leqslant 1$.

4.6. (a) *If $\overline{m}(e) > 0$, then the equalities $m(e) = \varepsilon[\varphi(\varepsilon^{-1})]^{-1}\lambda$ and $\|\lambda\chi_e\|_\varphi = \varepsilon\lambda$, are equivalent.*

(b) *If $\overline{m}(e) > 0$, then $\|\chi_e\|_\varphi^s = [\varphi_{-1}(\overline{m}(e)^{-1})]^{-s}$; in particular, for a convex φ-function we have $\|\chi_e\|_\varphi^c = [\varphi_{-1}(\overline{m}(e)^{-1})]^{-1}$.*

In conclusion of this section we will give the formulae for norms under consideration in the classical case $\varphi(u) = u^\alpha$ or u^α/α, $\alpha > 0$. Straightforward computation shows that if $\varphi(u) = u^\alpha$, we get

(a) If $0 < \alpha < 1$, $\alpha = s$, $\|x\|_\varphi^s = \overline{m}(|x|^\alpha)$,

(b) if $1 \leqslant \alpha$, $\|x\|_\varphi^c = \overline{m}(|x|^\alpha)^{1/\alpha}$,

(c) if $0 < \alpha < 1$, $\alpha = s$, $\|x\|_\varphi^{0s} = \overline{m}(|x|^\alpha)$,

(d) if $1 \leqslant \alpha$, $\|x\|_\varphi^0 = \overline{m}(|x|^\alpha)^{1/\alpha}$;

when $\varphi(u) = 1/\alpha \cdot u^\alpha$, where $\alpha > 1$, $1/\alpha + 1/\alpha' = 1$, the following formulae hold:

(a) $$\|x\|_\varphi^c = \alpha^{-1/\alpha}\overline{m}(|x|^\alpha)^{1/\alpha},$$

(b) $$\|x\|_\varphi^0 = \alpha'^{1/\alpha'}\overline{m}(|x|^\alpha)^{1/\alpha}.$$

There also holds the formula $\|x\|_\varphi = \overline{m}(|x|^\alpha)^{\frac{1}{(1+\alpha)}}$ for $\varphi(u) = u^\alpha$ and any $\alpha > 0$.

5. Suppose E fulfils the property $(\mathscr{D})$. If for any $x_n \in \mathscr{X}$, the relation $\|x_n\|_\varphi \to 0$ as $n \to \infty$, implies the relation $\|x_n\|_\psi \to 0$ as $n \to \infty$, then

(*) $\quad \psi(u) \leqslant a\varphi(ku)$ for $u \geqslant u_0$, where a, k are positive constants.

We choose an $\varepsilon > 0$ such that the inequality $\|x\|_\varphi \leqslant \varepsilon$ implies $\|x\|_\psi \leqslant 1$. Let $\overline{m}(e) > 0$. By 4.3 there exists u for which $\|u\chi_e\|_\varphi = \varepsilon$; whence

$$\varphi(u/\varepsilon)\overline{m}(e) = \varepsilon. \tag{+}$$

But $\|u\chi_e\|_\psi = \overline{m}(e)\psi\big(u(\|u\chi_e\|_\psi)^{-1}\big) \leqslant 1$, therefore $\overline{m}(e)\psi(u) \leqslant 1$ and by (+) we get

$$\varepsilon\psi(u) \leqslant \varphi(u/\varepsilon). \tag{++}$$

Since the set of the values $\overline{m}(e)$ is dense in $(0, 1)$, the set of those u which satisfy (+) is dense in $\langle u_0, \infty\rangle$, where u_0 satisfies the condition $\varphi(u_0/\varepsilon) = \varepsilon$, consequently (++) is satisfied for any $u \geqslant u_0$.

5.1. *If the inequality 5 (*) is satisfied, then for any $x_n \in \mathscr{X}$ the relation $\|x_n\|_\varphi \to 0$ as $n \to \infty$, implies the relation $\|x_n\|_\psi \to 0$ as $n \to \infty$.*

It is enough to show that if $\overline{m}\big(\varphi(\lambda|x_n|)\big) \to 0$ as $n \to \infty$, for any $\lambda > 0$, then $\overline{m}\big(\psi(|x_n|)\big) \to 0$ as $n \to \infty$.

Let $\lambda > 1$. For any t in T for which $\lambda|x_n(t)| \geqslant u_0$ we have, in virtue of 5 (*), the inequality

$$\psi\big(|x_n(t)|\big) \leqslant \psi\big(\lambda|x_n(t)|\big) \leqslant a\varphi\big(k\lambda|x_n(t)|\big),$$

if $|x_n(t)| < u_0\lambda^{-1}$, then $\psi\big(|x_n(t)|\big) \leqslant \psi(u_0\lambda^{-1})$. Consequently

$$\psi(|x_n|) \leqslant a\varphi(k\lambda|x_n|) + \psi(u_0\lambda^{-1}),$$

$$\overline{m}(\psi|x_n|) \leqslant a\overline{m}\big(\varphi(k\lambda|x_n|) + \psi(u_0\lambda^{-1})\big),$$

and from this,

$$\limsup_{n\to\infty} \overline{m}\big(\psi(|x_n|)\big) \leqslant \psi(u_0\lambda^{-1}), \qquad \overline{m}\big(\psi(|x_n|)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

5.2. *Let φ_n be a strictly increasing φ-function for $n = 1, 2, \ldots$ The following conditions are equivalent:*

(a) $\limsup_{n\to\infty} \varphi_n(u) \leqslant 1$ *for* $0 \leqslant u < 1$, $\lim_{n\to\infty} \varphi_n(u) = \infty$ *for* $u > 1$;

(b) $\lim(\varphi_n)_{-1}(u) = 1$ *for* $u > 1$.

For example, we shall prove $(a) \Rightarrow (b)$. Let $0 < u' < 1 < u''$, $v > 1$. Since $\varphi_n(u') < v < \varphi_n(u'')$ for sufficiently large n, we get $u' < (\varphi_n)_{-1}(v) < u''$, whence

$$u' \leqslant \liminf_{n\to\infty}(\varphi_n)_{-1}(v) \leqslant \limsup(\varphi_n)_{-1}(v) \leqslant u''.$$

5.2.1. *If φ_n are s-convex φ-functions, then condition 5.2 (a) implies $\lim(\varphi_n)_{-1}(1) = 1$.*

Let be $0 < u < 1$, $u < \overline{u} < 1$, then $\varphi_n(u) = \varphi_n(\overline{u}u\overline{u}^{-1}) \leqslant (u\overline{u}^{-1})^s \leqslant \varphi_n(\overline{u})$, $\limsup_{n\to\infty} \varphi_n(u) < 1$. If $0 < u' < 1 < u''$, then $\varphi_n(u') < 1 < \varphi_n(u'')$ for sufficiently large n, hence $u' < (\varphi_n)_{-1}(1) < u''$, $\lim(\varphi_n)_{-1}(1) = 1$.

5.3. *Let φ_n be φ-functions such that:*

(*) $\varphi_n(u) \to 0$ *as* $n \to \infty$, *for* $0 \leqslant u < 1$, $\varphi_n(u) \to \infty$ *as* $n \to \infty$, *for* $u > 1$.

For each $\lambda > 0$ and $e \in E$, with positive $\overline{m}$-measure, there exists the limit

$$\|\lambda\chi_e\|_{\varphi_n} \to \lambda \quad \text{as} \quad n \to \infty. \tag{**}$$

Conversely, if for some e with positive $\overline{m}$-measure the limit (**) exists, then (*) holds.

Define ε_n in such a manner that $\lambda\varepsilon_n = \|\lambda\chi_e\|_{\varphi_n}$ or equivalently $\overline{m}(e) = \varepsilon_n[\varphi_n^{-1}(\varepsilon_n^{-1})]^{-1}\lambda$. Let $\varepsilon_0 > 1$. Then for $n \geqslant n_0$ the inequality $\varepsilon_0[\varphi_n(\varepsilon_0^{-1})]^{-1}\lambda > \overline{m}(e)$ holds, and since $\varepsilon[\varphi_n(\varepsilon^{-1})]^{-1}$ is strictly increasing with ε, it must be $\varepsilon_n \leqslant \varepsilon_0$, whence

$$\limsup_{n\to\infty} \varepsilon_n \leqslant 1.$$

Similarly, it can be shown that

$$\liminf_{n\to\infty} \varepsilon_n \geqslant 1,$$

so that

$$\lim_{n\to\infty} \varepsilon_n = 1.$$

In order to prove the second part of the theorem let us assume $\varepsilon_n\lambda = \|\lambda\chi_e\|_{\varphi_n} \to \lambda$ as $n \to \infty$, where $\overline{m}(e) > 0$, for any $\lambda > 0$. Let $u' < 1 < u''$. Since $\varepsilon_n \to 1$ as $n \to \infty$, we get $u' < \varepsilon_n < u''$ for $n > n_0$, hence

$$u'[\varphi_n(u'^{-1})]^{-1}\lambda < \overline{m}(e) = \varepsilon_n[\varphi_n(\varepsilon_n^{-1})]^{-1}\lambda < u''[\varphi_n(u''^{-1})]^{-1}\lambda.$$

But, because of $\overline{m}(e) > 0$, the last inequalities can be satisfied for an arbitrary positive λ, only if the condition (*) is satisfied.

5.4. (a) *If for s-convex φ-functions the condition 5.2 (a) is satisfied, then for any e with positive $\overline{m}$-measure, there exists the limit*

$$\|\chi_e\|_{\varphi_n}^s \to 1 \quad \text{as} \quad n \to \infty. \tag{*}$$

(b) *If for a mean value $\overline{m}(\cdot)$ the corresponding class E fulfils the property ($\mathscr{D}$) and (*) is satisfied for any e in E with positive measure, then the condition 5.2 (a) is satisfied.*

Ad (a). If $0 < \overline{m}(e) \leqslant 1$, then by 5.2(b) and 5.2.1

$$(\varphi_n)_{-1}\big(\overline{m}(e)^{-1}\big) \to 1 \quad \text{as} \quad n \to \infty.$$

Ad (b). Since the values $u = \overline{m}(e)^{-1}$ are dense in $\langle 1, \infty)$, the relation $[(\varphi_n)_{-1}(u)]^{-s} \to 1$ for $n \to \infty$ holds in a set which is dense in $\langle 1, \infty)$, and by the monotony of $(\varphi_n)_{-1}$, the limit $(\varphi_n)_{-1}(u) \to 1$ as $n \to \infty$ exists for any $u > 1$. It is enough to apply 5.2.

5.5. *If φ is a convex φ-function satisfying conditions* (o_1), (∞_1), *then for any $v>0$ there exists $\lambda_0>0$ such that*

$$v\varphi_{-1}^{*}\left(\frac{1}{v}\right)=\frac{1}{\lambda_0}+\frac{v}{\lambda_0}\varphi(\lambda_0)=\inf_{\lambda>0}\left(\frac{1}{\lambda}+\frac{v}{\lambda}\varphi(\lambda)\right).$$

Indeed, for $v_0=\varphi_{-1}(v^{-1})$ we can find $\lambda_0>0$ for which the equality $v_0\lambda_0=\varphi(\lambda_0)+\varphi^*(v_0)$ holds.

5.5.1. (a) *For strictly increasing φ-functions φ_n, the conditions $\varphi_n(u)\to u$ as $n\to\infty$, for $u>1$, and $(\varphi_n)_{-1}(u)\to u$ as $n\to\infty$, for $u>1$, are equivalent.*

(b) *For an s-convex φ_n, from $\varphi_n(u)\to u$ for $u>1$ it follows $(\varphi_n)_{-1}(1)\to 1$.*

Ad (b). φ_n are strictly increasing, as follows from the s-convexity. By (a) we have

$$\limsup_{n\to\infty}(\varphi_n)_{-1}(1)\leqslant u \quad \text{for} \quad u>1,$$

hence

$$\limsup_{n\to\infty}(\varphi_n)_{-1}(1)\leqslant 1.$$

Let $0<\alpha<1$; since $\varphi_n(\alpha^{1/s}u)\leqslant\alpha\varphi_n(u)$ for $u\geqslant 0$, we get $\alpha^{1/s}(\varphi_n)_{-1}(u)\leqslant(\varphi_n)_{-1}(\alpha u)$ for $u\geqslant 0$. For $u>1$, $\alpha=1/u$ it follows

$$u^{-1/s}(\varphi_n)_{-1}(u)\leqslant(\varphi_n)_{-1}(1),\quad u^{-1/s}u\leqslant\liminf_{n\to\infty}(\varphi_n)_{-1}(1),\quad 1\leqslant\liminf_{n\to\infty}(\varphi_n)_{-1}(1)$$

and consequently $(\varphi_n)_{-1}(1)\to 1$ as $n\to\infty$.

5.5.2. *Let φ_n be a convex φ-function satisfying conditions* (o_1), (∞_1) *for $n=1,2,\ldots$*

(a) *If $\lim_{n\to\infty}\varphi_n(u)=\infty$ for $u>1$, then $\limsup_{n\to\infty}\varphi_n^*(v)\leqslant v$ for $v\geqslant 0$, and conversely.*

(b) *If $\lim_{n\to\infty}\varphi_n(u)=0$ for $0\leqslant u<1$, then $\liminf_{n\to\infty}\varphi_n^*(v)\geqslant v$ for $v\geqslant 0$, and conversely.*

(c) *If $\liminf_{n\to\infty}\varphi_n^*(v)\geqslant v$ for $v>1$, then $\limsup_{n\to\infty}\varphi_n(u)\leqslant u$ for $0\leqslant u<1$.*

(Some analogous lemmata can be found in [13]).

Ad (a). Let $\lim_{n\to\infty}\varphi_n(u)=\infty$ for $u>1$. Suppose $v>0$ is given. We choose u_n such that

$$(+)\qquad vu_n=\varphi_n(u_n)+\varphi_n^*(v).$$

Let $u_0>1$. It must be $\limsup_{n\to\infty}u_n\leqslant u_0$, for if not so it would be

$$v\geqslant\frac{\varphi_{n_i}(u_{n_i})}{u_{n_i}}\geqslant\frac{\varphi_{n_i}(u_0)}{u_0}$$

for an increasing sequence of indices n_i, but this is contradictory to $\varphi_{n_i}(u_0)\to\infty$ as $i\to\infty$. It follows that

$$\limsup_{n\to\infty}u_n\leqslant 1,$$

and by (+) we obtain

$$v\geqslant v\limsup_{n\to\infty}u_n\geqslant\limsup_{n\to\infty}\varphi_n^*(v).$$

If

$$\limsup_{n\to\infty}\varphi_n^*(v)\leqslant v \quad \text{for} \quad v\geqslant 0,$$

then for a given $0<\lambda<1$, $u>1$, such that $\lambda u>1$, we have $\varphi_n^*(v)\leqslant\lambda uv$ for $n\geqslant n_0$, and since $\varphi_n(u)\geqslant uv-\varphi_n^*(v)$ we get $\varphi_n(u)\geqslant uv(1-\lambda)$ for any $v>0$, whence

$$\lim_{n\to\infty}\varphi_n(u)=\infty.$$

Ad (b) and (c). Applying the inequality $uv\leqslant\varphi_n(u)+\varphi_n^*(v)$ for $0<u<1$, we get

$$v\leqslant\liminf_{n\to\infty}\varphi_n^*(v) \quad \text{for} \quad v\geqslant 0$$

if $\varphi_n(u)\to 0$ for $0<u<1$. If for a v_n the equation $uv_n=\varphi_n(u)+\varphi_n^*(v_n)$ holds, where $0<u<1$, we have $u\geqslant\varphi_n^*(v_n)\cdot v_n^{-1}$. It follows from the last inequality and by the convexity of φ_n^* that

$$\limsup_{n\to\infty}v_n\leqslant 1,$$

and so

$$u\geqslant u\limsup_{n\to\infty}v_n\geqslant\limsup_{n\to\infty}\varphi_n(u)$$

holds. If

$$\liminf_{n\to\infty}\varphi_n^*(v)\geqslant v \quad \text{for any } v\geqslant 0,$$

then, analogously as above, we can prove $v_n\to 0$ as $n\to\infty$, which implies

$$\lim_{n\to\infty}\varphi_n(u)=0 \quad \text{for} \quad 0\leqslant u<1.$$

5.5.3. It follows from 5.5.2 that conditions

(α) $\lim_{n\to\infty}\varphi_n(u)=0$ for $0\leqslant u<1$,

(β) $\lim_{n\to\infty}\varphi_n(u)=\infty$ for $u>1$,

imply

(γ) $\lim_{n\to\infty}\varphi_n^*(v)=v$ for $v>1$ (for $v\geqslant 0$).

If (γ) is satisfied, then

$$\limsup_{n\to\infty} \varphi_n^*(v) \leqslant v \quad \text{for any } v \geqslant 0$$

and consequently (β) is satisfied, and besides

$$\limsup_{n\to\infty} \varphi_n(u) \leqslant u \quad \text{for} \quad 0 \leqslant u < 1.$$

5.6. (a) *Let φ_n be a convex φ-function, satisfying conditions (o_1), (∞_1), for $n = 1, 2, \ldots$ If the condition*

$$(*) \qquad \varphi_n^*(v) \to v \quad \text{for} \quad v > 1$$

is satisfied, then for any $e \in E$ with positive $\overline{m}$-measure there exists the limit

$$(**) \qquad \|\chi_e\|_{\varphi_n}^0 \to 1 \quad \text{as} \quad n \to \infty.$$

(b) *If for a mean value $\overline{m}(\cdot)$ the corresponding class E fulfils the property $(\mathfrak{D})$ and $(**)$ is satisfied for any e in E with positive $\overline{m}$-measure, then condition $(*)$ holds.*

Ad (a). By 5.5 and the definition of $\|\cdot\|_{\varphi_n}^0$ we get

$$\|\chi_e\|_{\varphi_n}^0 = \inf_{\lambda>0}\left(\frac{1}{\lambda} + \frac{\overline{m}(e)}{\lambda}\varphi_n(\lambda)\right) = \overline{m}(e)(\varphi_n^*)_{-1}\left(\frac{1}{\overline{m}(e)}\right).$$

In virtue of $(*)$ and 5.5.1 it follows $\overline{m}(e)(\varphi_n^*)_{-1}\big(\overline{m}(e)^{-1}\big) \to 1$ as $n \to \infty$.

Ad (b). The set of values $v = \overline{m}(e)^{-1}$ is dense in $\langle 1, \infty)$ and for any such value $v^{-1}(\varphi_n^*)_{-1}(v) \to 1$ as $n \to \infty$. But by the monotony of $(\varphi_n^*)_{-1}$ this relation holds for any $v > 1$, and so, by 5.5.1, $\varphi_n^*(u) \to u$ as $n \to \infty$, for any $u > 1$.

5.7. *If φ_n, φ are φ-functions, then the following conditions are equivalent:*

(a) $\lim_{n\to\infty} \|x\|_{\varphi_n} = \|x\|_\varphi$ *for* $x \in \mathscr{X}$.

(b) $\lim_{n\to\infty} \varphi_n(u) = \varphi(u)$ *for* $u \geqslant 0$.

(b) $\Rightarrow$ (a). Let $\varepsilon_n = \overline{m}\big(\varphi_n(|x|\varepsilon_n^{-1})\big)$, $\varepsilon = \overline{m}\big(\varphi(|x|\varepsilon^{-1})\big)$. The continuity and monotony of φ_n, φ ensure $\varphi_n(u) \rightrightarrows \varphi(u)$ in any interval $\langle 0, u_0\rangle$. Suppose $\varepsilon_{n_i} \leqslant \varepsilon$ where $n_i \to \infty$. Then $\varepsilon_{n_i} \geqslant \overline{m}\big(\varphi_{n_i}(|x|\varepsilon^{-1})\big)$, and owing to the uniform convergence of $\varphi_n\big(|x(t)|\varepsilon^{-1}\big)$ to $\varphi\big(|x(t)|\varepsilon^{-1}\big)$ we get $\overline{m}\big(\varphi_n(|x|\varepsilon^{-1})\big) \to \overline{m}\big(\varphi(|x|\varepsilon^{-1})\big)$, whence

$$\liminf_{i\to\infty} \varepsilon_{n_i} \geqslant \varepsilon,$$

and consequently

$$\lim_{i\to\infty} \varepsilon_{n_i} = \varepsilon.$$

Assume now $\varepsilon_{n_i} \geqslant \varepsilon$ as $n_i \to \infty$. Then

$$\overline{m}\big(\varphi_{n_i}(|x|\varepsilon_{n_i}^{-1})\big) - \overline{m}\big(\varphi(|x|\varepsilon_{n_i}^{-1})\big) \to 0 \quad \text{as} \quad n \to \infty$$

that is to say,

$$\varepsilon_{n_i} - \overline{m}\big(\varphi(|x|\varepsilon_{n_i}^{-1})\big) \to 0 \quad \text{as} \quad n_i \to \infty.$$

The sequence ε_{n_i} is bounded, for $\overline{m}\big(\varphi(|x|\varepsilon_{n_i}^{-1})\big) \leqslant \varepsilon$. For any accumulation point of the sequence ε_{n_i} the equation $\varepsilon_0 = \overline{m}\big(\varphi(|x|\varepsilon_0^{-1})\big)$ holds, and since, by 4.4, the equation $\varepsilon = \overline{m}\big(\varphi(|x|\varepsilon^{-1})\big)$ is satisfied only for $\varepsilon = \|x\|_\varphi$, we get $\varepsilon_0 = \|x\|_\varphi$, $\varepsilon_{n_i} \to \varepsilon = \|x\|_\varphi$.

(a) $\Rightarrow$ (b). Suppose $\varepsilon > 0$ and $e \in E$, $\overline{m}(e) > 0$ given. Choose $\lambda > 0$ such that $\varepsilon\lambda = \|\lambda\chi_e\|_\varphi$, that is to say, $\overline{m}(e) = \varepsilon[\varphi(\varepsilon^{-1})]^{-1}\lambda$. Choose ε_n such that $\varepsilon_n\lambda = \|\lambda\chi_e\|_{\varphi_n}$, or equivalently $\overline{m}(e) = \varepsilon_n[\varphi_n(\varepsilon_n^{-1})]^{-1}\lambda$. (a) implies $\varepsilon_n \to \varepsilon$, $\varphi_n(\varepsilon_n^{-1}) \to \varphi(\varepsilon^{-1})$ as $n \to \infty$. In other words, for any $u > 0$ there exists a sequence u_n such that $u_n \to u$, $\varphi_n(u_n) \to \varphi(u)$. If u belongs to (u', u''), where $0 < u' < u''$, are arbitrarily given, we get $u' < u_n < u''$, $\varphi_n(u') \leqslant \varphi_n(u_n) \leqslant \varphi_n(u'')$ for sufficiently large n, and it follows

$$(+) \qquad \varphi(u) \leqslant \liminf_{n\to\infty} \varphi_n(u''), \qquad \limsup_{n\to\infty} \varphi_n(u') \leqslant \varphi(u).$$

Letting $u \to u'' - 0$ or $u \to u' + 0$, we obtain

$$\liminf_{n\to\infty} \varphi_n(u'') \geqslant \varphi(u''), \qquad \limsup_{n\to\infty} \varphi_n(u') \leqslant \varphi(u')$$

and consequently

$$\lim_{n\to\infty} \varphi_n(u) = \varphi(u) \quad \text{for} \quad u \geqslant 0.$$

5.8. (a) *If φ-functions φ_n satisfy condition 5.3 $(*)$, then the relation*

$$(*) \qquad \|x\|_{\varphi_n} \to \sup{}^*|x| \quad \text{as} \quad n \to \infty,\ x \in \mathscr{X},$$

holds.

(b) *If $(*)$ is satisfied, then φ_n fulfils condition 5.3 $(*)$.*

We can assume $\sup^*|x| > 0$, for $\sup^*|x| = 0$ implies $\|x\|_{\varphi_n} = 0$ for $n = 1, 2, \ldots$ Suppose $0 < \lambda < \sup^*|x|$. By 2.7.3 we have

$$\|\lambda\chi_e\|_{\varphi_n} \leqslant \|x\|_{\varphi_n} \leqslant \|\sup{}^*|x|\chi_T\|_{\varphi_n},$$

where $e \in E$, $\overline{m}(e) > 0$. Hence, by 5.3, the relation $(*)$ holds.

(b) immediately follows by 5.3 and in virtue of the fact that $\sup^*\lambda\chi_e = \lambda$, when $\lambda > 0$, $\overline{m}(e) > 0$.

5.9. *Let φ_n be an s-convex φ-function for $n = 1, 2, \ldots$*

(a) *Under the assumption of 5.2 (a) the relation*

$$(*) \qquad \|x\|_{\varphi_n}^s \to (\sup{}^*|x|)^s \quad \text{as} \quad n \to \infty,\ x \in \mathscr{X},$$

holds.

(b) *If* $(*)$ *is satisfied, and* E *fulfils the property* $(\mathcal{D})$, *then* 5.2 (a) *holds.*

To prove (a) we apply 5.4 (a) and the inequality

$$\lambda^s\|\chi_e\|_{\varphi_n}^s \leqslant \|x\|_{\varphi_n}^s \leqslant \|\chi_T\|_{\varphi_n}^s(\sup{}^*|x|)^s;$$

here λ, e have the same meaning as in the proof of 5.8 (a).

(b) follows immediately from 5.4 (b).

5.10. *Let* φ_n *be a convex* φ-*function, satisfying* (o_1), (∞_1), *for* $n = 1, 2, \ldots$

(a) *Under the assumption of* 5.6 $(*)$ *the relation*

$$(*)\qquad \|x\|_{\varphi_n}^0 \to \sup{}^*|x| \quad \textit{as} \quad n\to\infty,\ x\in\mathcal{X},$$

holds.

(b) *If* E *possesses the property* $(\mathcal{D})$ *and* $(*)$ *holds, then* φ_n *satisfy condition* 5.6 $(*)$.

The proof of (a) follows by 5.6 and by the application of the inequality

$$\lambda\|\chi_e\|_{\varphi_n}^0 \leqslant \|x\|_{\varphi_n}^0 \leqslant \|\chi_T\|_{\varphi_n}^0\sup{}^*|x|;$$

here λ, e have the same meaning as in the proof of 5.8 (a); (b) follows immediately from 5.6 (b).

6.1. In this section we are concerned, in the first place, with the following question:

On what conditions on φ-functions φ, ψ does the following inequality

$$\|x\|_\psi^c \leqslant \|x\|_\varphi^c \quad \text{for any } x\in\mathcal{X},$$

hold. If for a pair φ, ψ the last written inequality is satisfied, then it will be said they possess the *property* (C, ψ, φ). Assuming φ, ψ to be convex φ-functions, and E (for a given $\overline{m}(\cdot)$) to fulfil the property $(\mathcal{D})$, we obtain for (C, ψ, φ), by 4.6 (b), and by setting $x = \chi_e$, the following necessary condition:

$$\psi(u) \leqslant \varphi(u) \quad \text{for} \quad u \geqslant \psi_{-1}(1).$$

On the other hand, the inequality

$$\psi(u) \leqslant \varphi(u) \quad \text{for} \quad u \geqslant 0$$

is obviously sufficient to have the property (C, ψ, φ) for any mean value. But the last mentioned inequality certainly does not give a necessary condition for the property (C, ψ, φ), for if $\psi(u) = u^\beta$, $\varphi(u) = u^\alpha$, $1 \leqslant \beta < \alpha$, then (C, ψ, φ) is fulfilled.

Let us introduce the following notation: $\omega(u) = \varphi(\psi_{-1}(u))$.

6.2. *Let us assume that* ω *satisfies the following condition:*

For any $\varepsilon > 0$ *there exists a* λ, $0 < \lambda \leqslant 1$, *such that*

$$(*)\qquad u \leqslant \lambda\omega(u) + (1-\lambda) \quad \textit{for} \quad u \geqslant \varepsilon.$$

Under this assumption the property (C, ψ, φ) *is fulfilled for any upper mean value.*

It is enough to prove that $\|x\|_\varphi^c = 1$ implies $\|x\|_\psi^c \leqslant 1$.

Denote by λ_n the number λ for which the inequality $(*)$ is satisfied when $\varepsilon = \psi(1/n)$, and set $x_n = |x| \vee 1/n - 1/n$. By $(*)$ we have

$$\psi(|x_n|) \leqslant \lambda_n\varphi(|x_n|) + (1-\lambda_n),$$

hence

$$\overline{m}\big(\psi(|x_n|)\big) \leqslant \lambda_n\overline{m}\big(\varphi(|x_n|)\big) + (1-\lambda_n),$$

and

$$(+)\qquad \overline{m}\big(\psi(|x_n|)\big) \leqslant 1,$$

for $\overline{m}\big(\varphi(|x_n|)\big) \leqslant \overline{m}\big(\varphi(|x|)\big) = \|x\|_\varphi^c = 1$. But $x_n \to |x|$, $\overline{m}\big(\psi(|x_n|)\big) \to \overline{m}\big(\psi(|x|)\big)$ as $n\to\infty$, so, by (+), we get $\overline{m}\big(\psi(|x|)\big) \leqslant 1$ and consequently $\|x\|_\psi^c \leqslant 1$.

6.2.1. Under the same assumption on ω as in 6.2 there holds the inequality

$$\|x\|_\psi^s \leqslant \|x\|_\varphi^s \quad \text{for} \quad x\in\mathcal{X}.$$

6.3. *Each of the following conditions is sufficient for convex* φ-*functions* φ, ψ *to fulfil the property* (C, ψ, φ).

A. $\varphi(1) = \psi(1) = 1$; *there exists a* λ, $0 < \lambda \leqslant 1$, *such that* $\omega'(u) \geqslant 1/\lambda$ *for almost every* $u \geqslant 1$, $\omega'(u) \leqslant 1/\lambda$ *for almost every* $0 < u \leqslant 1$.

B. ω *is a convex function in* $(0, \infty)$, $\omega(1) \geqslant 1$, *and* ω *satisfies condition* (o_1).

C. ω *satisfies condition* (o_1), *if* $\overline{\omega}$ *denotes the greatest convex function such that* $\omega(u) \geqslant \overline{\omega}(u)$ *for* $u \geqslant 0$, *then* $\overline{\omega}(1) \geqslant 1$.

A sufficient condition for s-convex φ-*functions* φ, ψ *to satisfy the inequality*

$$\|x\|_\psi^s \leqslant \|x\|_\varphi^s \quad \textit{for any } x\in\mathcal{X},$$

is A, *and under the assumption of local absolute continuity of* φ, B *or* C, *as well.*

Ad A. Since $\varphi(1) = \psi(1) = 1$, we get $\omega(1) = 1$. Evidently the following conditions are equivalent:

(α) For a λ, $0 < \lambda \leqslant 1$, the inequality $u \leqslant \lambda\omega(u) + (1-\lambda)$ for $u \geqslant 0$, holds;

(β) there hold the inequalities

$$(+)\qquad \frac{1-\omega(u)}{1-u} \leqslant 1/\lambda \quad \text{for} \quad 0 \leqslant u < 1;$$

$$(++)\qquad \frac{\omega(u)-1}{u-1} \geqslant 1/\lambda \quad \text{for} \quad u > 1,$$

where $0 < \lambda \leqslant 1$.

Since $\varphi'(u)$, $\psi'(u)$ exist almost everywhere and $\psi'(u) > 0$ if $u > 0$, the derivative $\omega'(u)$ exist almost everywhere as well. Moreover φ, ψ satisfy the Lipschitz condition in any finite interval so does ω in any interval $\langle a, u_0\rangle$, where $0 < a$. From A and $\omega(1) = 1$ the inequalities (β) follow and this implies 6.2 (∗), where λ is independent of ε.

Ad B. From the integral representation

$$\omega(u) = \int_0^u \omega'_+(t)\,dt,$$

where ω'_+ is non-decreasing in $\langle 0, \infty\rangle$ and by $\omega(1) \geqslant 1$ we get $\omega(u)-1 \geqslant \omega'_+(1)(u-1)$ for $u > 1$, $1-\omega(u) \leqslant \omega'_+(1)(1-u)$ for $0 < u < 1$. It is enough to prove that $\lambda^{-1} = \omega'_+(1) > 1$. Clearly

$$1 \leqslant \omega(1) = \int_0^1 \omega'_+(t)\,dt \leqslant \omega'_+(1).$$

Assuming $\omega'_+(1) = 1$ we must have

$$\int_0^1 \big(\omega'_+(1) - \omega'_+(t)\big)\,dt = 0,$$

$\omega'_+(t) = 1$ for $0 < t < 1$, $\omega(u) = u$ for $0 \leqslant u \leqslant 1$, which is contradictory to the fact that ω fulfils the condition (o_1). Hence (β) is satisfied.

Assuming additionally that ω fulfils the condition (∞_1) another proof can be given. It is instructive, because of using the notion of the complementary function, and it runs the following lines.

There exists $u_0 > 0$ such that $u_0 = \omega(1) + \omega^*(u_0)$. By the Young's inequality $uu_0 \leqslant \omega(u) + \omega^*(u_0)$, $u \leqslant u_0^{-1}\omega(u) + u_0^{-1}\omega^*(u_0)$. But $\omega(1) \geqslant 1$, hence $1 \geqslant u_0^{-1} + u_0^{-1}\omega^*(u_0)$ and it suffices to set $\lambda = u_0^{-1}$. Evidently $0 < \lambda < 1$, for $u_0 > 1$.

Ad C. In virtue of $\omega(u) \geqslant \overline{\omega}(u)$, $\overline{\omega}$ satisfies the condition (o_1).

We have $u \leqslant \lambda\overline{\omega}(u) + (1-\lambda) \leqslant \lambda\omega(u) + (1-\lambda)$, where $\lambda^{-1} = \overline{\omega}'_+(1)$. Since $\overline{\omega}(1) \geqslant 1$, by A, $0 < \lambda < 1$ follows.

It follows from 6.3 C that if $\omega(u)$ satisfies (o_1) and $\omega(u) \geqslant u^\gamma$, where $\gamma > 1$, then the property (C, ψ, φ) is satisfied. If φ, ψ are s-convex, then the analogous inequality $\|x\|_\psi^s \leqslant \|x\|_\varphi^s$ for $x \in \mathscr{X}$ is satisfied too. Let us set $\psi(u) = u^\beta$, $\varphi(u) = u^\alpha$, where $1 \leqslant \beta < \alpha$. The well-known theorem of the monotonic increase of the mean value $\overline{m}(|x|^\alpha)^{1/\alpha}$ in the interval $\langle 1, \infty)$ of α is a direct consequence of the above remark. To obtain the monotony of $\overline{m}(|x|^\alpha)^{1/\alpha}$ for the interval $\alpha \in (0, 1)$ let us remark that if $0 < \beta \leqslant \alpha \leqslant 1$, then φ is β-convex, $\|x\|_\varphi^\beta = \overline{m}(|x|^\alpha)^{\beta/\alpha}$. For the function $\omega(u) = u^{\beta/\alpha}$ the condition 6.3 can be applied and by 6.2.1 we get $\overline{m}(|x|^\beta) \leqslant \overline{m}(|x|^\alpha)^{\beta/\alpha}$, or equivalently $\overline{m}(|x|^\beta)^{1/\beta} \leqslant \overline{m}(|x|^\alpha)^{1/\alpha}$.

6.4. The following property remains closely related to the property (C, ψ, φ): A pair of convex φ-functions φ, ψ is said to possess the *property* (H, φ, ψ) if the inequality

$$(+) \qquad \overline{m}(xy) \leqslant \|x\|_\varphi^c \|y\|_\psi^c \quad \text{for} \quad x, y \in \mathscr{X}$$

is satisfied for any upper mean value. The inequality (+) presents one of the possible types of Hölder's inequality. The other types of Hölder's inequality are known in the theory of Orlicz spaces [1] and they readily can be generalized to an arbitrary upper mean value; namely we have the following theorem:

6.4.1. *For an arbitrary mean value $\overline{m}(\cdot)$ there holds the following inequality*:

$$(+) \qquad \overline{m}(xy) \leqslant \|x\|_\varphi^c \|y\|_{\varphi*}^0 \quad \textit{for any} \quad x, y \in \mathscr{X}.$$

Indeed, for any $\lambda > 0$ we get, by the Young's inequality,

$$|xy| \leqslant \lambda^{-1}\big(\varphi(|x|)\big) + \lambda^{-1}\varphi^*(\lambda|y|),$$
$$\overline{m}(xy) \leqslant \lambda^{-1}\overline{m}\big(\varphi(|x|)\big) + \lambda^{-1}\overline{m}\big(\varphi^*(\lambda|y|)\big).$$

If $\|x\|_\varphi^c = \overline{m}\big(\varphi(|x|)\big) = 1$, then, owing to the definition of $\|y\|_{\varphi*}^0$, we obtain $\overline{m}(xy) \leqslant \|y\|_{\varphi*}^0$, whence $\overline{m}(xy) \leqslant \|x\|_\varphi^c\|y\|_{\varphi*}^0$.

In spite of inequality 6.4.1 (+), which is generally valid, the property (H, φ, φ^*) is only true under a special assumption on φ, and it is not directly deducible from 6.4.1 (+). The reason for this is that only the inequality $\|y\|_{\varphi*}^c \leqslant \|y\|_{\varphi*}^0$ occurs. But, setting $\varphi(u) = u^\alpha$, $\alpha > 1$, we get (H, φ, φ^*) (the classical Hölder's inequality), for in this case

$$\|x\|_\varphi^c = \overline{m}(|x|^\alpha)^{1/\alpha}, \quad \|y\|_{\varphi*}^0 = \|y\|_{\varphi*}^c = \overline{m}(|y|^{\alpha'})^{1/\alpha'}.$$

6.4.2. *If a convex φ-function φ satisfies the conditions* (o_1), (∞_1), $\varphi(1) = 1$, $\psi(u) = \varphi^*(v_0 u)[\varphi^*(v_0)]^{-1}$, *where* $v_0 = p'_+(1)$, *then the functions* φ, ψ *possess the property* (H, φ, ψ).

As it is known the equation $v_0 = \varphi(1) + \varphi^*(v_0)$ is satisfied for $v_0 = p'_+(1)$ and since $uv_0v \leqslant \varphi(u) + \varphi^*(v_0 v)$, we obtain

$$uv \leqslant v_0^{-1}\varphi(u) + v_0^{-1}\varphi^*(v_0 v).$$

Let

$$\|x\|_\varphi^c = \overline{m}\big(\varphi(|x|)\big) = 1, \quad \|y\|_\psi^c = \overline{m}\big(\psi(|y|)\big) = 1;$$

substituting in the last inequality $u = |x(t)|$, $v = |y(t)|$ we get

$$\overline{m}(xy) \leqslant v_0^{-1}\overline{m}\big(\varphi(|x|)\big) + v_0^{-1}\varphi^*(v_0)\overline{m}\big(\psi(|y|)\big) \leqslant 1,$$

which implies inequality 6.4 (+).

7. A set $U \in \mathscr{X}$ is called *s-convex*, $0 < s \leqslant 1$, if for any α, β which satisfy the conditions $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, $x, y \in U$ implies $\alpha x + \beta y \in U$. A linear topological Hausdorff space is called *locally s-convex*, if there is a base of s-convex neighbourhoods of 0 in it.

7.1. *Let us assume E fulfils the following property:*

For a given natural n and a positive η, for which $n\eta \leqslant 1$, there exist in E n disjoint sets $e_1, e_2, \ldots, e_n$ such that

$$(*) \qquad \overline{m}(e_i) \leqslant \eta, \qquad n\eta = \overline{m}\Big(\bigcup_1^n e_i\Big).$$

If the topology generated by the norm $\|\cdot\|_\varphi$ in $\mathscr{X}$ is locally s-convex, then there is a φ-function $\chi(u) = \psi(u^s)$, where ψ is a convex φ-function, for which

$$(**) \qquad \chi(k_1 u) \leqslant \varphi(u) \leqslant \chi(k_2 u), \qquad k_1, k_2 > 0, \qquad \text{for} \qquad u \geqslant u_0.$$

Choose in $\mathscr{X}$ an s-convex neighbourhood U of zero and a $\delta > 0$ in such a manner that $\|x\|_\varphi \leqslant \delta$ implies $x \in U$, and $x \in U$ implies $\|x\|_\varphi \leqslant 1$. Given an α, $0 < \alpha \leqslant 1$, let us denote by n a non-negative integer for which

$$(+) \qquad \tfrac{1}{2} < n\alpha^s \leqslant 1.$$

Let us choose a u satisfying the conditions

$$(++) \qquad \varphi(u/\delta) > \delta, \qquad \alpha^s \varphi(u/\delta) \geqslant \delta,$$

and set $\eta = \delta[\varphi(u/\delta)]^{-1}$. Since $\eta \leqslant \alpha^s$, $n\eta \leqslant 1$, there exist n disjoint sets in E for which the condition $(*)$ holds. From the inequality

$$\|u\chi_{e_i}\|_\varphi [\varphi(u\|u\chi_{e_i}\|_\varphi^{-1})]^{-1} = \overline{m}(e_i) \leqslant \eta = \delta[\varphi(u/\delta)]^{-1}$$

it follows $\|u\chi_{e_i}\|_\varphi \leqslant \delta$, for $u\varphi(u^{-1})^{-1}$ is strictly increasing. The elements $u\chi_{e_i}$ belong to U and by the s-convexity of U, $x = \alpha u\chi_{e_1} + \ldots + \alpha u\chi_{e_n} \in U$, for we have $n\alpha^s \leqslant 1$. This implies

$$\overline{m}\big(\varphi(|x|)\big) = \overline{m}\Big(\varphi\big(\alpha u(\chi_{e_1} + \ldots + \chi_{e_n})\big)\Big) = \varphi(\alpha u)\,\overline{m}\Big(\bigcup_1^n e_i\Big) \leqslant 1.$$

In view of $(*)$ it follows

$$n\eta\varphi(\alpha u) \leqslant 1,$$

whence, and by $(+)$, we get

$$n\delta[\varphi(u/\delta)]^{-1}\varphi(\alpha u) \leqslant 2n\alpha^s.$$

We have proved the inequality

$$\binom{++}{+} \qquad \varphi(\alpha u) \leqslant 2\delta^{-1}\alpha^s \varphi(u/\delta)$$

for all α, u for which $0 \leqslant \alpha \leqslant 1$ and $(++)$ hold. From $\binom{++}{+}$ it follows that $\varphi(u/\delta) \geqslant cu^s$ for $u \geqslant u_0$, where c is a positive constant, and u_0

sufficiently large. If this is not so, then one can find numbers u_n, such that $u_n \to \infty$, $\varphi(u_n/\delta)/u_n^s \to 0$ as $n \to \infty$. Let us define α_n by the requirement $\alpha_n^s \varphi(u_n/\delta) = \delta$.

Since for sufficiently large n, $\alpha_n < 1$, $\varphi(u_n/\delta) > \delta$, so substituting in $\binom{++}{+}$ $\alpha = \alpha_n$, $u = u_n$, we obtain the inequality

$$\varphi\big(\delta^{1/s} u_n \varphi(u_n/\delta)^{-1/s}\big) \leqslant 2\delta^{-1}\delta = 2,$$

which is contradictory to $\varphi(u) \to \infty$ as $u \to \infty$. Let u_0 be such that $\varphi(u_0/\delta) > \delta$, and $u_2 \geqslant u_1 \geqslant \overline{u}_0 = \sup\big(u_0, (\delta/c)^{1/s}\big)$. If $\alpha = u_1/u_2$, then

$$\alpha^s \varphi(u_2/\delta) = u_1^s \varphi(u_2/\delta) u_2^{-s} \geqslant \delta, \qquad \varphi(u_2/\delta) > \delta,$$

and by $\binom{++}{+}$

$$\frac{\varphi(u_1)}{u_1^s} \leqslant 2\delta^{-1}\frac{\varphi(u_2/\delta)}{u_2^s} \qquad \text{for} \qquad u_2 \geqslant u_1 \geqslant \overline{u}_0.$$

By a theorem in [4], 2.6.2, 2.7, the last inequality implies the existence of a φ-function χ with the required properties.

7.2. *If for a φ-function φ there exists a φ-function χ satisfying the inequalities 7.1 $(**)$, and of the form $\chi(u) = \psi(u^s)$, where ψ is a convex φ-function, then the topology in $\mathscr{X}$, which is generated by the norm $\|\cdot\|_\varphi$, is locally s-convex.*

By 5.1 the convergence with respect to the norm $\|\cdot\|_\varphi$ implies the convergence with respect to the norm $\|\cdot\|_\chi$, and conversely. But the topology which is generated by $\|\cdot\|_\chi$ is locally s-convex, for we can choose a base of neighbourhoods of zero composed of the following s-convex neighbourhoods

$$U(\varepsilon) = \{x \in \mathscr{X} : \|x\|_\chi^s < \varepsilon\}.$$

Let us conclude this section with the following remarks. Theorems 7.1, 7.2 generalize some results of [6], [9].

In section 8 one can find some example of set-algebras which satisfy the condition $(*)$ in 7.1. The condition 7.1 $(*)$ implies, of course, the property $(\mathscr{D})$ for a given E.

8. In this section some typical examples of spaces X and subadditive mean values are given.

I. We write $I = (a, b)$, where a and b are finite.

(a) Let X be the space of all real-valued and bounded functions in I. Then the class E is the collection of subsets in I. We may define

1) $\overline{m}(x) = \sup\limits_I |x(t)|$;

2) $\overline{m}(x) = (b-a)^{-1}\int_a^{\bar{b}} |x(t)|\,dt$,

where $\overline{\int_a^b} \ldots$ means the Riemann upper integral. The mean value 1) is an extreme value, in this case $\overline{m}(e) = 0$ only for the empty set, $\overline{m}(e) = 1$ if e is non-empty.

(b) Let X be the space of real-valued bounded and measurable functions in (a, b). We may define

1) $\overline{m}(x) = \sup_I{}^*|x(t)|$, where $\sup_I{}^*$ denotes the essential supremum with respect to the ideal of Lebesgue-measurable sets with the measure 0;

2) $\overline{m}(x) = (b-a)^{-1}\int_a^b |x(t)|\,dt$, where $\int_a^b \ldots$ means the Lebesgue integral.

The mean value 1) is evidently extreme.

II. Let $k(t, \tau)$ be a non-negative integrable function for any $0 \leqslant t < t_0$, where $t_0 \leqslant \infty$, and either for τ belonging to $I_0 = \{0 < \tau < \tau^*\}$ or to $I_\infty = \{\tau : \tau > \tau^*\}$. Let

$$\int_0^{t_0} k(t, \tau)\,dt = 1 \quad \text{where} \quad (\alpha)\ \tau \in I_0,\ (\beta)\ \tau \in I_\infty.$$

For X we choose the space of real-valued, bounded and measurable functions in $(0, t_0)$. We define

1) $\overline{m}(x) = \limsup_{\tau \to \tau_g} \int_0^{t_0} k(t, \tau)|x(t)|\,dt$, where $\tau_g = 0$, if $k(t, \tau)$ is defined in $(0, t_0) \times I_0$, $\tau_g = \infty$ if τ is taken in I_∞.

2) $\overline{m}(x) = \sup_{\tau \in I_g} \int_0^{t_0} k(t, \tau)|x(t)|\,dt$, where I_g is either I_0 or I_∞.

The particularly important cases can be obtained setting $k(t, \tau) = \tau^{-1}$ for $0 < t \leqslant \tau$, $k(t, \tau) = 0$ for $t > \tau$. Assuming $t_0 = \infty$, $\tau^* = 0$ we obtain the following mean values:

1α) $\overline{m}(x) = \limsup_{\tau \to \infty} \tau^{-1}\int_0^\tau |x(t)|\,dt$,

2β) $\overline{m}(x) = \sup_{\tau > 0} \tau^{-1}\int_0^\tau |x(t)|\,dt$.

Assuming $t_0 = 1$, $\tau^* = 1$ we get

1α′) $\overline{m}(x) = \limsup_{\tau \to 0} \tau^{-1}\int_0^\tau |x(t)|\,dt$,

2β′) $\overline{m}(x) = \sup_{0 < \tau < 1} \tau^{-1}\int_0^\tau |x(t)|\,dt$.

III. Let X be the space of bounded sequences $\{t_i\}$ of reals; then E is the class of all subsets of the collection of natural numbers. Let a_{ni}

be non-negative and let they satisfy the condition $\sum_i a_{ni} = 1$ for any n. We define the following mean values

1) $\overline{m}(x) = \limsup_{n \to \infty} \sum_{i=1}^n a_{ni}|t_i|$,

2) $\overline{m}(x) = \sup_n \sum_{i=1}^n a_{ni}|t_i|$.

Setting $a_{ni} = 1/n$ if $i = 1, 2, \ldots, n$, $= 0$ if $i > n$, for $n = 1, 2, \ldots$, we obtain the following mean values, which are of some importance, when investigating methods of the strongly summable sequences:

1α) $\overline{m}(x) = \limsup_{n \to \infty} \dfrac{1}{n}\sum_{i=1}^n |t_i|$,

2α) $\overline{m}(x) = \sup_n \dfrac{1}{n}\sum_{i=1}^n |t_i|$.

8.1. The classes E which correspond to the spaces X in I-III, and the mean values I(a), 2), I(b), 2), II 1α), II 2β), III 1α), III 2α), possess the property ($\mathcal{D}$). Let us consider, for example, the mean value II 2β). If $e = \langle \tau_1, \tau_2 \rangle$, then $\overline{m}(e) = 1 - \tau_1/\tau_2$, which means that the values $\overline{m}(e)$ are dense in $(0, 1)$. In the case of III 2α), we obtain $\overline{m}(e) = 1 - p/q$, when $e = \{n : p \leqslant n < q\}$ and consequently the property ($\mathcal{D}$) is fulfilled. Let us yet consider the mean value II 1α). Let $0 < \tau_1 < \tau_2$ and choose the sets $e_n = \langle r_n\tau_1, r_n\tau_2 \rangle$, where r_n are positive integers such that $r_{n+1}\tau_1 > r_n\tau_2$, $r_n\tau_2/r_{n+1}\tau_1 < 1/n$ for $n = 1, 2, \ldots$ If we define $e_n = \bigcup_1^\infty e_n$ we obtain the following inequalities

$$\frac{1}{\tau}\int_0^\tau \chi_e(t)\,dt \leqslant 1 - \frac{\tau_1}{\tau_2} + \frac{1}{n} \quad \text{for} \quad r_{n+1}\tau_1 < \tau \leqslant r_{n+1}\tau_2,$$

$$1 - \frac{\tau_1}{\tau_2} \leqslant \frac{1}{\tau}\int_0^\tau \chi_e(t)\,dt \quad \text{if} \quad \tau = r_{n+1}\tau_2,$$

which implies

$$\limsup_{\tau \to \infty} \tau^{-1}\int_0^\tau \chi_e(t)\,dt = 1 - \frac{\tau_1}{\tau_2},$$

and the property ($\mathcal{D}$) is fulfilled. By similar arguments we can verify that property ($\mathcal{D}$) is fulfilled for E, if the mean value is III 1α).

8.2. The set algebra E fulfils the property 7.1 (*), which is more general as the property ($\mathcal{D}$), if the corresponding mean value is II 1α), II 2β), III 1α), III 2α) respectively. Let us consider the mean values II 1α), II 2β).

Define two increasing sequences of natural numbers l_r, k_r in such a manner that:

1) $l_r \geqslant 2$,

2) $l_{r+1} > l_r+k_r,\ k_{r+1} > l_r+k_r$ for $r = 1, 2, \ldots$

3) $\varepsilon_r = \dfrac{k_r}{l_r+k_r-1} \to 1$ as $r \to \infty$,

4) $\dfrac{k_1+k_2+\ldots+k_{r-1}}{l_r} < 1-\varepsilon_r$ for $r = 2, 3, \ldots$

For instance, we can choose l_r arbitrarily, but such that $l_1 \geqslant 2$, $l_{r+1} \geqslant l_r(r+1)$, $(1l_1+2l_2+\ldots+(r-1)l_{r-1})l_r^{-1} < \frac{1}{2}(1+r)^{-1}$ for $r = 2, 3, \ldots$ and set $k_r = rl_r$.

Suppose $0 < \eta$ and that, for a natural n, $n\eta \leqslant 1$. Define $i_{rk} = \langle l_r+ +k-1, l_r+k)$ for $k = 1, 2, \ldots, k_r$. If $n\eta = 1$, we decompose any i_{rk} in n consecutive subintervals i_{rk}^j of the length η, if $n\eta < 1$ in $n+1$ subintervals, where the first n consecutive are of the length η, and the $(n+1)$-th is of length $1-n\eta$. Evidently distinct subintervals i_{rk}^j are disjoint, we define

$$e_j = \bigcup_{r=1}^{\infty} \bigcup_{k=1}^{k_r} i_{rk}^j \quad \text{for} \quad j = 1, 2, \ldots, n, \qquad e = \bigcup_{j=1}^{n} e_j.$$

If $\tau \in i_{rk}$ for $r = 1, 2, \ldots,\ k = 1, 2, \ldots, k_r$, then

$$\frac{1}{\tau}\int_{l_r}^{\tau} \chi_{e_j}(t)\,dt \leqslant \frac{k}{l_r+k-1}\,\eta \leqslant \frac{k_r}{l_r+k_r-1}\,\eta = \varepsilon_r\eta,$$

and so this inequality is satisfied for $l_r \leqslant \tau < l_r+k_r$. We have also for τ within l_r and l_r+k

$$\frac{1}{\tau}\int_0^{l_r} \chi_{e_j}(t)\,dt = \frac{1}{\tau}\int_{l_1}^{l_1+k_1} \ldots + \frac{1}{\tau}\int_{l_2}^{l_2+k_2} \ldots + \ldots + \frac{1}{\tau}\int_{l_{r-1}}^{l_{r-1}+k_{r-1}} \ldots$$

$$\leqslant \frac{k_1+k_2+\ldots+k_{r-1}}{l_r}\,\eta < (1-\varepsilon_r)\eta,$$

and consequently

$$\sup_{l_r \leqslant \tau < l_r+k_r} \frac{1}{\tau}\int_0^{\tau} \chi_{e_j}(t)\,dt \leqslant (1-\varepsilon_r)\eta + \varepsilon_r\eta \leqslant \eta$$

for $r = 1, 2, \ldots$

$$(+) \qquad \sup_{\tau>0} \frac{1}{\tau}\int_0^{\tau} \chi_{e_j}(t)\,dt \leqslant \eta.$$

On the other hand, we have

$$(++) \qquad \frac{k_r}{l_r+k_r}\,\eta = \frac{1}{l_r+k_r}\int_{l_r}^{l_r+k_r} \chi_{e_j}(t)\,dt \leqslant \sup_{\tau>0} \frac{1}{\tau}\int_0^{\tau} \chi_{e_j}(t)\,dt,$$

and, by 3), $k_r(l_r+k_r)^{-1}\eta \to \eta$. It follows from (+), (++), that

$$\limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} \chi_{e_j}(t)\,dt = \sup_{\tau>0} \frac{1}{\tau}\int_0^{\tau} \chi_{e_j}(t)\,dt = \eta \quad \text{for} \quad j = 1, 2, \ldots, n.$$

On the same way we can check that

$$\limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} \chi_e(t)\,dt = \sup_{\tau>0} \frac{1}{\tau}\int_0^{\tau} \chi_e(t)\,dt = n\eta,$$

and it follows property 7.1 (*).

Similar arguments may be applied to prove that the mean values III 1α), III 2α) satisfy also property 7.1 (*).

References

[1] M. A. Krasnosel'skii, Ya. B. Rutickii, *Convex functions and Orlicz spaces*, Groningen 1961.

[2] W. A. J. Luxemburg, *Banach function spaces*, Thesis Delft Inst. of Techn., Assen (Netherlands) 1955.

[3] — and A. C. Zaaneen, *Notes on Banach functions spaces*, Proc. Acad. Sci., Amsterdam; *Note I*, A66, p. 135-147 (1963); *Note II*, A66, p. 148-153 (1963); *Note III* A66, p. 293-250 (1963); *Note IV*, A66, p. 251-263 (1963); *Note V*, A66, p. 496-504 (1963); *Note VI*, A66, p. 655-668 (1963); *Note VII*, A66, p. 669-681 (1963): *Note VIII*, A67, p. 104-119 (1964); *Note IX*, A67, p. 360-376 (1964); *Note X*, A67, p. 493-506 (1964); *Note XI*, A67, p. 507-518 (1964); *Note XII*, A67, p. 519-529 (1964); *Note XIII*, A67, p. 530-543 (1964).

[4] W. Matuszewska, *Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces*, Studia Math. 21 (1962), p. 317-344.

[5] — *On generalized Orlicz spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 8 (1960), p. 349-353.

[6] — and W. Orlicz, *A note on the theory of s-normed spaces of φ-integrable functions*, Studia Math. 21 (1961), p. 107-115.

[7] J. Musielak and W. Orlicz, *On modular spaces*, ibidem 18 (1959), p. 49-65.

[8] — *Some remarks on modular spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 7 (1959), p. 669-676.

[9] — *On modular spaces of strongly summable sequences*, Studia Math. 22 (1962), p. 127-146.

[10] W. Orlicz, *A note on modular spaces I*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 9 (1961), p. 157-162.

[11] — *On some spaces of strongly summable sequences*, Studia Math. 22 (1963), p. 331-336.

[12] — *A note on modular spaces VII*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 12 (1964), p. 305-309.

[13] — *On the convergence of norms in spaces of φ-integrable functions*, ibidem 13 (1965), p. 205-210.

Reçu par la Rédaction le 5. 3. 1965

STUDIA MATHEMATICA, T. XXVI. (1966)

Extensions of sequentially continuous linear functionals in inductive sequences of $(\mathscr{F})$-spaces

par

W. SŁOWIKOWSKI (Warszawa)

1. Introduction [1]. These investigations was inspired a long time ago by a problem communicated to the author by L. Ehrenpreis. The problem concerned extensibility of sequentially continuous linear functionals defined on subspaces of Schwartz's spaces $\mathscr{D}(\Omega)$ of infinitely differentiable functions with compact carriers contained in a fixed domain Ω (cf. [3]). It can be easily verified that distributions from the domain of a partial differential operator on $\mathscr{D}'(\Omega)$ can always be considered as extensions of sequentially continuous functionals defined on the range of the adjoint differential operator acting on $\mathscr{D}(\Omega)$. Hence, it becomes apparent that a necessary and sufficient condition for existence of such extensions must be closely connected with any set of conditions that are necessary and sufficient for the operator to map onto $\mathscr{D}'(\Omega)$. For convolution operators, including as a particular case differential operators with constant coefficients, such a set of conditions was given by Hörmander in [2].

Going one step further in generality, call $(\mathscr{I}\mathscr{F})$-sequence any sequence $\mathfrak{X}$ of $(\mathscr{F})$-spaces such that every linear space from $\mathfrak{X}$ is a subspace of the subsequent linear space from the sequence and that the identical injection of every $(\mathscr{F})$-space from $\mathfrak{X}$ into the following one is continuous (cf. [12]).

Situation that necessitates using such a notion arises, for instance, when we discuss factor spaces of the Schwartz's $(\mathscr{D}, \tau_{\mathscr{D}})$ space. Such factor spaces need not be $(\mathscr{L}\mathscr{F})$-spaces any more though they always naturally decompose into $(\mathscr{I}\mathscr{F})$-sequences.

Let X denote the union of linear spaces from an $(\mathscr{I}\mathscr{F})$-sequence $\mathfrak{X}$. A linear functional defined on a linear subspace of X is called *sequentially continuous* if it is continuous in every $(\mathscr{F})$-space from $\mathfrak{X}$. We formulate the general problem of extension as follows.

Given an $(\mathscr{I}\mathscr{F})$-sequence $\mathfrak{X}$ find a natural condition for a linear subspace X_0 of X defined above which is necessary and sufficient

[1] A substantial part of the results presented here was obtained when the author was at the Institute for Advanced Study in Princeton on the NSF Grant G-14600.