On some classes of modular spaces

by

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1. In this section we define some terms and introduce some notation we shall be using. By *ϕ*-function we mean a continuous, non-decreasing function *ϕ*(\(u\)), defined for \(u \geq 0\), vanishing only at \(u = 0\), and tending to \(\infty\) with \(u \to \infty\). *ϕ*-functions will be denoted by \(ϕ_1, ϕ_2, \ldots\) and their inverse functions by \(ϕ_{-1}, ϕ_{-2}, \ldots\)

A *ϕ*-function \(ϕ\) will be called \(s\)-convex, \(0 < s \leq 1\), if

\[
ϕ(au + βv) ≤ a^sϕ(u) + β^sϕ(v)
\]

for \(a, β \geq 0\), \(a^s + β^s = 1\) and arbitrary \(u, v \geq 0\). A 1-convex *ϕ*-function will be called briefly convex, as customary. It is readily seen that \(s\)-convex *ϕ*-functions are strictly increasing in \((0, \infty)\). A *ϕ*-function \(ϕ\) is said to satisfy the condition \((ο_1)\) if \(ϕ(u)/u \to 0\) as \(u \to 0\); it satisfies the condition \((ο_∞)\) if \(ϕ(u)/u \to ∞\) as \(u \to ∞\). For a *ϕ*-function \(ϕ\) which satisfies conditions \((ο_1)\) and \((ο_∞)\) a complementary function can be defined by the formula

\[
ϕ^*(v) = \sup_{u > 0} \{uv - ϕ(u)\}.
\]

The complementary function is always a convex *ϕ*-function, and it satisfies conditions \((ο_1)\), \((ο_∞)\).

The function \(ϕ^*(v) = (ϕ^*)^*(u)\) is the greatest convex *ϕ*-function satisfying \((ο_1)\), \((ο_∞)\), and for which \(ϕ(u) ≥ ϕ^*(u)\) for any \(u \geq 0\).

The letter \(T\) will always stand for an abstract set on which real-valued functions \(x, y, z, \ldots\) are defined. For a set \(e\) of elements of \(T\), \(χ_e\) denotes its characteristic function \(χ_e(t) = 0\) for \(t \not\in e\), \(t \in T\), if \(e\) is an empty set. We use very often the notation \(α_e\) instead of \(αχ_e\), and we occasionally write \(I\) instead of \(χ_T\). A simple function is a function of the form

\[
α_1χ_{e_1}(t) + α_2χ_{e_2}(t) + \cdots + α_nχ_{e_n}(t).
\]

For any *ϕ*-function \(ϕ\) the symbol \(ϕ(\{x\})\) denotes the function \(ϕ(∥x∥)\) defined on \(T\), \(supx\) denotes \(supx(t)\), where the supremum is taken over all \(t \in T\); the symbols \(x + y, xy\) etc., have the usual meaning. By \(x_n \to x\) as \(n \to ∞\) or \(x_n \to x\) as \(n \to ∞\) we always denote that \(x_n(t)\) converges pointwise or uniformly in \(T\) respectively, to \(x(t)\) as \(n \to ∞\). The symbol \(x \leq y\) denotes that \(x(t) \leq y(t)\) for any \(t \in T\),
$x \lor y$ or $x \land y$ stand for the function $\sup \{x(t), y(t)\}$ or $\inf \{x(t), y(t)\}$, respectively.

1.4. Throughout this paper $X$ will always denote a collection of real-valued functions defined and bounded on $T$ and satisfying the following conditions:

1. The class of sets $\mathcal{A} \subset T$, for which $\mathcal{X}_{\mathcal{A}} \subseteq X$, is a Boolean algebra of sets.

2. $X$ is a real linear space.

3. For any $\varepsilon > 0$ there exists a simple function $\sum_{i=1}^{n} a_i \delta_{x_i}$ such that $\|x - \sum_{i=1}^{n} a_i \delta_{x_i}\| < \varepsilon$.

4. If $x \in X$, $a_n \to \infty$ as $n \to \infty$, then $a_n x \in \mathcal{X}$.

By $E$ we shall always denote the Boolean algebra of sets defined in 1. By a partition of $T$ we mean a finite class of non-empty sets $A_1, A_2, \ldots, A_n$ such that $\bigcup_{i=1}^{n} A_i = T$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, n$. In the sequel when speaking of simple functions we tacitly assume $\mathcal{X}_{\mathcal{A}} \subseteq X$ for $i = 1, 2, \ldots, n$.

1.5. Any simple function $x$ can be represented in the canonical form

$$x = \sum_{i=1}^{n} a_i \delta_{x_i} + \sum_{i=1}^{n} a_i \delta_{x_i} + \ldots + a_n \delta_{x_n},$$

where $(\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n})$ is a partition of $T$. This representation is unique for a given partition of $T$; for $|x| = \sum_{i=1}^{n} a_i \delta_{x_i} + \sum_{i=1}^{n} a_i \delta_{x_i} + \ldots + a_n \delta_{x_n}$, and more general $\varphi(x) = \sum_{i=1}^{n} a_i \delta_{x_i} + \sum_{i=1}^{n} a_i \delta_{x_i} + \ldots + a_n \delta_{x_n}$.

For any simple functions $x, y \in X$ there exist canonical representation of $x$ and $y$ corresponding to a common partition of $T$. If $x$ and $y$ are such representations for $x$ and $y$, respectively, then $x \lor y$ or $x \land y$ can be represented in the form $\sum_{i=1}^{n} \max(a_i, b_i) \delta_{x_i}$, where $a_i = \sup(\{a_i, b_i\})$ or $b_i = \inf(\{a_i, b_i\})$.

1.5. (a) If $x, y \in X$, then $x \lor y \in X$; (b) if $x \in X$, then for any $\varphi$-function $\varphi$, $\varphi(x) \in X$; (c) $X$ is a vector lattice, assuming the natural relation of order in $X$, the join of elements $x$ and $y$ in $X$ is $x \lor y$ and $x \land y$ is their meet.

Ad (a). Because of the equality $x \lor y = \frac{(x+y) - |x-y|}{2}$ it is sufficient to prove that $x \lor y$. But this is evidently true for any simple function, so also for any $x \in X$, by $3^{th}$ and $4^{th}$.

Ad (b). Let $y$ be a continuous function for $u > 0$ and, for a given $\varepsilon > 0$, let be $|y(u)| < \varepsilon$ if $|y(u)| < |u| < \varepsilon$, where $u \in \mathcal{X}_{\mathcal{A}}$. If $y$ is a simple function such that $|x-y| < \varepsilon$, then $\sup |y| < \varepsilon$, $\max(|x|, |y|) < \varepsilon$, $\gamma(|y|)$ is a simple function, and in view of $3^{th}$, $4^{th}$ the assertion (b) follows.

2. Let $X$ be defined as in section 1.1 and suppose that a real-valued functional $\rho(x)$ is defined on $X$, which fulfills the following conditions:

1. $\rho(x) = 1$.
2. $\rho(|x|) = \rho(x)$.
identical with it. If the set of values \( \mathbb{M}(e), e \in E \), is dense in \( (0,1) \) the algebra of sets \( E \) is said to fulfill the property \( (2) \) (with respect to the given \( \mathbb{M}(\cdot) \)). In the example \( S_4(\|u\|_2) \), \( (b,2) \), \( Î, 1a, 2b \), the corresponding algebra \( E \) fulfill the property \( (2) \), exactly speaking, the range of \( \mathbb{M}(\cdot) \) is \( (0,1) \). The possibility that under some such general assumptions on \( E \) and \( \mathbb{M}(\cdot) \) as in this paper, the values \( \mathbb{M}(e) \) lie dense in \( (0,1) \), but do not fill up \( (0,1) \) is not a priori excluded, although we cannot actually give an example for such a situation.

2.3. If \( \lambda > 0, \lambda > 0, a_1 = \{t:|x(t)| > \lambda \} \) \( \{a_2 = \{t:|x(t)| > \lambda \} \), there exists an \( \mathbb{M} \)-measurable set \( e_i \), such that \( a_1 \subset e_i \).

\[ \lambda \mathbb{M}(a_1) < \lambda \mathbb{M}(x) \]
2.5. By 2.4 it is seen that if, for a function \( x \), \( \mathbb{M}(x) = 0 \), then the set \( E = \{ t : |x(t)| = 0 \} \) is in the union of countably many sets of \( \mathbb{M} \)-measure 0. Let us remark that the converse assertion is not true in general.

The class of all \( \mathbb{M} \)-measurable sets with the \( \mathbb{M} \)-measure 0 will be denoted by \( E_0 \). Evidently \( E_0 \) is a ring, but not a \( \sigma \)-ring, even when \( E \) is a \( \sigma \)-algebra. Let us still observe, that if \( E \) is a \( \sigma \)-algebra, then in 2.5 we can assume \( a_k = e_k \). Under this hypothesis the necessary and sufficient condition for \( \mathbb{M}(x) = 0 \) is \( a_k \in E_k \) for any positive \( x \).

2.6. For a \( \phi \)-function \( x \) and \( x \in X \), \( \mathbb{M}(\phi(x)) \to 0 \) as \( \lambda \to 0 \), holds.

2.6.1. (a) For any \( \phi \)-function, \( \mathbb{M}(\phi(x)) \) is a continuous non-decreasing function for \( \lambda \geq 0 \).

The inequality \( \mathbb{M}(\phi(x)) \leq \mathbb{M}(\phi(x)) \) for \( 0 \leq \lambda \leq \lambda \) follows immediately from \( \phi(x) \leq \phi(x) \). Given an \( \varepsilon > 0 \) we choose a \( \delta > 0 \) such that \( |\lambda - \lambda| \varepsilon \), hence

\[
\mathbb{M}(\phi(x)) - \mathbb{M}(\phi(x)) \leq \mathbb{M}(\phi(x)) - \mathbb{M}(\phi(x)) \leq \varepsilon,
\]

and the continuity of \( \mathbb{M}(\phi(x)) \) follows.

2.6.2. If a \( \phi \)-function \( x \) is convex, then \( \mathbb{M}(\phi(x)) \) is strictly increasing for \( \lambda \geq 0 \).

The same statement is true for any \( \phi \)-function for which \( \inf_{x \geq 0} \phi(x) \varepsilon > 0 \) for any \( \lambda > 1 \).

2.6.3. For any \( \phi \)-function \( \mathbb{M}(\phi(x)) \to \infty \) as \( \lambda \to \infty \), if \( \mathbb{M}(x) > 0 \).

There exists a simple function \( y \) such that \( \mathbb{M}(y) > 0 \), for \( |x-y| < \epsilon \). We can assume \( \mathbb{M}(y) > 0 \), for \( \mathbb{M}(x) - \mathbb{M}(y) < \epsilon \). The sets \( e_k = \{ t : x(t) > \lambda \} \) are measurable, and, by 2.4, \( \mathbb{M}(e_k) > 0 \) for a \( \lambda > 0 \). Because of the inequality

\[
\mathbb{M}(\phi(x)) \geq \mathbb{M}(\phi(y)) \geq \mathbb{M}(\phi(y)) \geq \mathbb{M}(\phi(x))
\]

we obtain \( \mathbb{M}(\phi(x)) \to \infty \) as \( \lambda \to \infty \).

2.7. The essential supremum of a function \( x \), which will be written \( \sup^* x \), is by definition the infimum of numbers \( \lambda \) for which \( \mathbb{M}(x \wedge \lambda) = 0 \). This definition is equivalent to the following one:

The essential supremum of \( x \) is the infimum of \( \lambda x \) with the property that \( a_k = \{ t : x(t) > \lambda \} \) can be covered by an \( e_k \in E_k \).

Indeed, if \( a_k = e_k \), then, in virtue of \( x \wedge \lambda = (x - \lambda) \chi_{X_\lambda} \), we have \( \mathbb{M}(x \wedge \lambda) = 0 \). Then, by 2.4, there exists a set \( e_k \in E_k \) which covers the set \( \{ t : x(t) > \lambda \} \) if \( a_k = e_k \).

Evidently \( \sup^* x = 0 \) implies \( \mathbb{M}(x) = 0 \) and conversely \( \sup^* x = \sup |x| \)

2.7.1. The functional \( \sup^* x \) is a subadditive mean value on \( X \). For any \( x, y \in X \) the inequality

\[
\mathbb{M}(xy) \leq \sup^* x|y|\mathbb{M}(y)
\]

holds.

We shall prove the condition 2 (4), for example. Since

\[
|\langle x \rangle \wedge \lambda - \lambda \rangle - \langle y \rangle \wedge \lambda - \lambda \rangle | \geq |\langle x \rangle \wedge \lambda - \lambda \rangle - \langle x \rangle \wedge \lambda - \lambda \rangle |
\]

we have \( \mathbb{M}(|x + y| \wedge \lambda - \lambda \rangle - \langle x \rangle \wedge \lambda - \lambda \rangle | = 0 \) when \( \sup^* x |y| \leq \lambda \), \( \sup^* y |x| \leq \lambda \), hence \( \sup^* |x + y| \leq \lambda \), \( \sup^* |x + y| \leq \sup^* |x| + \sup^* |y| \). To prove (4) it suffices to remark that

\[
|\langle x \rangle |y| \leq |\langle x \rangle | + \lambda | |y| + \lambda | |y|,
\]

\[
\mathbb{M}(xy) \leq \mathbb{M}(|y| + \lambda | |y| + \lambda | |y|, \mathbb{M}(y) = \lambda \mathbb{M}(y)
\]

for \( \lambda > \sup^* |x| \), and consequently \( \mathbb{M}(xy) \leq \sup^* |x| \sup^* |y| \).

2.7.2. Let \( y \) be a simple function whose canonical form is

\[
(*) \quad y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + b_1 x_1 + b_2 x_2 + \ldots + b_k x_k,
\]

where \( (e_1, e_2, \ldots, e_n, e_1, e_2, \ldots, e_k) \) is a partition, \( \mathbb{M}(e_i) > 0 \) for \( i = 1, 2, \ldots, n \), \( \mathbb{M}(e_i) > 0 \) for \( j = 1, 2, \ldots, k \). Then \( \sup^* y = \sup^* x \).

Since

\[
y \wedge \lambda = (a_1 \wedge \lambda - \lambda ) x_1 + \ldots + (a_n \wedge \lambda - \lambda ) x_n +
\]

\[
+ b_1 \wedge \lambda - \lambda ) x_1 + \ldots + (b_k \wedge \lambda - \lambda ) x_k,
\]

we get \( \mathbb{M}((a_1 \wedge \lambda - \lambda ) x_1) = \mathbb{M}(e_i) = 0 \), if \( \mathbb{M}(y \wedge \lambda - \lambda ) = 0 \), and consequently \( a_1 \wedge \lambda - \lambda ) x_1 \leq \lambda \), \( \sup^* a_1 \leq \sup^* y \). Conversely, if \( \lambda > \sup^* a_1 \), then \( a_1 \wedge \lambda - \lambda ) x_1 = 0 \) for \( i = 1, 2, \ldots, n \), hence \( \mathbb{M}(y \wedge \lambda - \lambda ) = 0 \), \( \sup^* y = \sup^* a_1 \).

2.7.3. Let \( 0 < \lambda < \sup^* x \). Choose a set \( e \subseteq X \) with positive \( \mathbb{M} \)-measure such that \( e \subseteq \{ t : \mathbb{M}(x(t)) > \lambda \} \).

Choose a simple function \( y \) such that \( 0 \leq y \leq \mathbb{M}_0 \), \( |x - y| < \epsilon \). Since, by 2.7.1,

\[
\mathbb{M}(x) - \mathbb{M}(y) \leq \sup^* |x - y| \leq \epsilon,
\]

it can be assumed \( \sup^* y > \lambda \). Representing \( y \) in the canonical form \( 2.7.3 \) we have, for a certain \( k \), \( a_k > \lambda \), and since \( a_k b_k > 0 \), for \( k \in e \subseteq \epsilon \), the inequality \( \mathbb{M}(x) \geq \lambda \) holds, moreover \( \mathbb{M}(e) > 0 \).
2.7.4. For any \( \varphi \)-function \( \sup \varphi (|x|) = \varphi (\sup |x|) \).
Assume first \( x \) to be a non-negative simple function \( y \), represented in the canonical form 2.7.2 (*). Then, 
\[ a_1, b_1 \geq 0, \sup b_1 = \sup a_1, \]
\[ \varphi (y) = \varphi (a_1) x_1 + \varphi (a_2) x_2 + \ldots + \varphi (a_n) x_n + \varphi (b_1) x_1 + \varphi (b_2) x_2 + \ldots + \varphi (b_k) x_k, \]

consequently
\[ \sup \varphi (y) = \sup \varphi (a_1) = \varphi (\sup a_1) = \varphi (\sup \varphi (y)). \]

We verify the formula \( \sup \varphi (|x|) = \varphi (\sup |x|) \) for an arbitrary function \( x \in X \), approximating \( |x| \) uniformly by simple functions.

2.8. The mean value \( \overline{x}(\cdot) \) is said extreme mean value, whenever \( \overline{x}(\cdot) = \sup \varphi (|x|) \) for any \( x \) in \( X \).

Each of the following conditions is necessary and sufficient for a mean value to be an extreme mean value:

A. For any \( x \in X \), \( \overline{x}(\cdot) = 1 \) or \( \overline{x}(\cdot) = 0 \).

B. \( \overline{x}(\cdot) = \overline{x}(\cdot)^2 \) for any \( x \in X \).

Ad A. The condition is sufficient. Let \( 0 < \sup \varphi (|x|) \) and \( 0 < \lambda < \sup \varphi (|x|) \) for all \( x \in X \). By 2.7.3 there exists a set \( \varepsilon > 0 \) with positive \( \overline{x} \)-measure such that \( |x| > \lambda \) for \( x \in \varepsilon \). By \( A \) we have \( \overline{x}(\cdot) = 1 \), whence \( \lambda = \overline{x}(\cdot) \), and \( \sup \varphi (|x|) = \overline{x}(\cdot) \). On the other hand, we have \( \overline{x}(\cdot) = \sup \varphi (|x|) \) for every \( x \) in \( X \).

Ad B. The necessity follows by 2.7.4. The sufficiency is trivial since for any \( x \in X \), \( \overline{x}(\cdot) = \overline{x}(\cdot)^2 = \overline{x}(\cdot) \).

3. In this section as well as in the following sections the notation \( x = y \) will be used for a pair of elements \( x, y \in X \), whenever \( \overline{x}(x - y) = 0 \). If \( x = y \) then the functions \( x, y \) are called \( \overline{x} \)-equal. Clearly, \( x_0 = y_0 \), if and only if \( \overline{x}(x_0 - x_0) = 0 \); \( \overline{x}(a) = \overline{x}(b) \) at \( x = y \) holds. It is easily seen that the relation \( = \overline{x} \) is an equivalence relation. We introduce still the notation \( X_0 = \{x \in X: \overline{x}(\cdot) = 0\} \); of course \( X_0 \) is a linear subspace of \( X \).

3.1. (a) If \( x_1 = x_1 [M], y_1 = y_1 [M], \) then for any \( \alpha, \beta \), 
\[ \alpha x_1 + \beta y_1 = \alpha x_1 + \beta y_1 [M]; \]

(b) if \( x_1 = x_1 [M], y_1 = y_1 [M], \) then \( x_1 y_1 = x_1 y_1 [M] \).

Ad B. From the inequality \( |x_1| = |x_1| \leq \overline{x}(a - a) \) it follows \( |x_1| = |x_1| [M] \). For an arbitrary prescribed number \( \varepsilon \), let us choose \( \delta > 0 \) such that \( |x_1(t) - x_1(t)| < \delta \) implies \( |x_1(t)| < \overline{x}(a - a) \). 
Since \( x_1 = x_1 [M] \), there exists a set in \( E_0 \) which covers the set 
\[ a = \{t : |x_1(t)| < \overline{x}(a - a) \} \].

Therefore
\[ \overline{x}(|x_1|) = \overline{x}(|x_1|) [M] \]

or simply
\[ \overline{x}(|x_1|) = \overline{x}(|x_1|) [M] \]

follows.

Ad c. If \( x \in X \) and \( y \in X \), then \( x, y \) are \( \overline{x} \)-equal, if and only if \( \overline{x}(x - y) = 0 \). In virtue of 3.1 and 3.2 we have \( \overline{x} \)-equality \( x = y [M] \).

3.2. If \( x_1 = x_1 [M], y_1 = y_1 [M], \) then \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \).

3.2.1. In virtue of 3.1 and 3.2 we have \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \).

In virtue of 3.1 and 3.2 we have \( \overline{x} \)-equality \( x = y \) holds, whence \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \).

Conversely, if \( x_1 = x_1 [M], y_1 = y_1 [M], x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \), then \( x_1 \overline{y} y_1 [M] \).

3.3. (a) If \( x_1 = x_1 [M] \) and \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \), then \( x_1 = x_1 \overline{y} y_1 [M] \) and conversely.

(b) If \( x_1 = x_1 [M], y_1 = y_1 [M], \) then \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \).

(c) If \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \) and \( \overline{x}(|x_1|) = \overline{x}(|y_1|) [M] \);

(d) If \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \) and \( \overline{x}(|x_1|) = \overline{x}(|y_1|) [M] \);

(e) If \( x_1 = x_1 [M] \) and \( y_1 = y_1 [M] \) then \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \).

(f) If \( x_1 = x_1 [M] \) and \( \overline{x}(|x_1|) = \overline{x}(|y_1|) [M] \).

For example, we shall prove (b). We have \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \), and, by 3.2.1, \( \overline{x} \)-equality \( x_1 \overline{y} y_1 = x_1 \overline{y} y_1 [M] \).

4. In the sequel the letter \( x \) always stands for the quotient space \( X_0 / X \). It follows from the lemmas given in section 3, that defining the addition of classes of \( \overline{x} \)-equal functions and their multiplication by real scalars, in a natural way, \( x \) becomes a real linear space. It follows also from 3.1 that the relation \( \leq \overline{x} \) makes \( x \) to a linear structure.

Supremum with respect to the ordering of classes represented by the elements \( x \) or \( y \) respectively, is the class represented by \( x \vee y \), and analogously \( x \wedge y \) represents the infimum of these classes.

From now on we will freely use the letters \( x, y, z, \ldots \) either as symbols of individual functions or as symbols of classes of \( \overline{x} \)-equal elements,
to which they belong, i.e., as symbols of elements of \( \mathcal{X} \). In a similar way
the symbols \( x \leq y \in \mathcal{X} \), \( x = y \in \mathcal{X} \), \( \text{sup}^\mathcal{X} \{x\} \), \( \text{inf} \mathcal{X}(\{x\}) \), \( x_\lambda \), \( \varphi(x) \) etc. will be used, and that is motivated by corresponding invariant properties with respect to \( \mathcal{X} \)-equality, as given in 3. What concerns symbols \( x \leq y \),
\( x = y \), etc. we attach the same meaning to them as before, i.e.,
they will be applied only to functions as elements of \( \mathcal{X} \).

4.1. For any \( \varphi \)-function \( \varphi \in \mathcal{M}(\varphi(x)) \) is a modular in \( \mathcal{X} \) in the sense
of [8], [12], i.e., this function possesses the following properties:
A. \( \mathcal{M}(\varphi(x)) = 0 \) if and only if \( x = 0 \).
B. \( \mathcal{M}(\varphi(x_1) + x_2) = \mathcal{M}(\varphi(x_1)) + \mathcal{M}(\varphi(x_2)) \).
C. \( \mathcal{M}(\varphi(x_1) \cdot x_2) = \mathcal{M}(\varphi(x_1)) \cdot \mathcal{M}(\varphi(x_2)) \).
D. \( \mathcal{M}(\varphi(\lambda x)) \rightarrow 0 \) as \( \lambda \rightarrow 0 \).

Property A follows by 2.2.2, to prove C let us remark that
\( \varphi(x_1) \cdot x_2 = \varphi(x_1) + \varphi(x_2) \) for any \( x_1, x_2 \in \mathcal{X} \). The property D is a
consequence of 2.6.

Suppose now \( \varphi \) to be an \( \mathcal{X} \)-convex function, then the inequality
\( \varphi(x_1) + \varphi(x_2) \leq \alpha \varphi(x_1) + \beta \varphi(x_2) \)
for \( \alpha, \beta \geq 0, \alpha + \beta = 1 \), holds, which implies
\( \mathcal{M}(\varphi(\lambda x)) \leq \lambda \mathcal{M}(\varphi(x)) \).

In particular, if \( \varphi \) is convex, then \( \mathcal{M}(\varphi(x_1)) \) is a convex function
on \( \mathcal{X} \).

4.2. It follows from the general theory of modular spaces [7], [8]
that in \( \mathcal{X} \) an \( \mathcal{X} \)-norm can be defined by the formula
\( ||x||_\mathcal{X} = \inf \{s > 0 : \mathcal{M}(\varphi(x/s)) \leq 1\} \).

For an \( \mathcal{X} \)-convex \( \varphi \)-function two other norms — both \( \mathcal{X} \)-homogeneous — can be defined, as follows [5], [10]:
\( ||x||'_\mathcal{X} = \inf \{s > 0 : \mathcal{M}(\varphi(x/s)) \leq 1\} \),
\( ||x||''_\mathcal{X} = \inf \{s > 0 : \mathcal{M}(\varphi(x/s)) \leq 1\} \).

If \( \varphi \) is convex, i.e., \( \varphi = 1 \), the norms \( ||x||'_\mathcal{X} \) and \( ||x||''_\mathcal{X} \) are homogeneous.
For these homogeneous norms the symbols \( \|x\|_\mathcal{X} \) or \( ||x||'_\mathcal{X} \) respectively will be used, instead of \( ||x||'_\mathcal{X} \) or \( ||x||''_\mathcal{X} \) respectively. Let us notice that all norms
mentioned above are monotonic, and for an \( \mathcal{X} \)-convex \( \varphi \)-function, they are equivalent (in \( \mathcal{X} \)) each to the other. An immediate consequence of the definition of \( ||x||_\mathcal{X} \), is, that the relation \( ||x||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \), and the relation \( ||x||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \), for any \( \lambda > 0 \), are equivalent.

4.3. The norm \( ||x||_\mathcal{X} \) is continuous with respect to \( \lambda \), and if \( ||x||_\mathcal{X} \rightarrow 0 \),
it tends to \( \infty \) with \( \lambda \).

We have \( ||x||_\mathcal{X} - ||x||_\mathcal{X} \leq ||\lambda - \lambda||_\mathcal{X} \) as \( \lambda \rightarrow \infty \). The second part
of the statement follows by 2.6.3.

4.4. For any \( x \neq 0 \in \mathcal{X} \) there holds the equation \( \mathcal{M}(\varphi(x)) = ||x||_\mathcal{X} \),
and the equation \( \mathcal{M}(\varphi(x)/||x||_\mathcal{X}) = 1 \) (under the assumption
that \( \varphi \) is \( \mathcal{X} \)-convex). The number \( \epsilon = ||x||_\mathcal{X} \), or \( \epsilon = ||x||_\mathcal{X}^{1/\lambda} \), respectively, is
the unique solution, of the first equation or the second one, respectively.

The first part of the statement is a consequence of 2.6.1 and 2.6.3.
If \( \mathcal{M}(\varphi(x)/||x||_\mathcal{X}) = \epsilon \), then \( \mathcal{M}(\varphi(x)/||x||_\mathcal{X}) = \epsilon \), and \( 0 < \epsilon < \epsilon_0 \),
then \( \mathcal{M}(\varphi(x)/||x||_\mathcal{X}) > \mathcal{M}(\varphi(x)/||x||_\mathcal{X}) = \epsilon \), a contradiction.
As concerns the norm \( ||x||_\mathcal{X} \) it suffices to apply 2.6.2.

4.5. Assume \( ||x||_\mathcal{X} \leq ||x||_\mathcal{X} \) for \( n = 1, 2, \ldots \). Then the relation \( \mathcal{M}(\varphi(x)) \rightarrow 0 \) as \( n \rightarrow \infty \),
and the relation \( ||x||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \) are equivalent. In particular,
the relation \( \mathcal{M}(x_n) \rightarrow 0 \) as \( n \rightarrow \infty \) and \( ||x_n||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \), one implies the other.

By 2.3.1 \( \mathcal{M}(\varphi(x)) \rightarrow 0 \) as \( n \rightarrow \infty \), for every \( \lambda > 0 \), and so \( ||x_n||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \), follows. The converse implication is trivial, for \( \mathcal{M}(\varphi(x)) \leq ||x||_\mathcal{X} \), when \( ||x_n||_\mathcal{X} \leq 1 \).

4.6. (a) If \( \mathcal{M}(e) > 0 \), then the equalities \( e = \varphi(\lambda e) = \lambda \varphi(e) \)
and \( ||x||_\mathcal{X} = e \), are equivalent.
(b) If \( \mathcal{M}(e) > 0 \), then \( ||x||_\mathcal{X} = \varphi(\lambda e) = \lambda \varphi(e) \).
In conclusion of this section we will give the formulæ for norms under consideration in the classical case \( \varphi(u) = u^a \) or \( u^a/\alpha, \alpha > 0 \). Straightforward computation shows that if \( \varphi(u) = u^a \), we get
\( (a) \) \( \alpha < 1 < a < 1, \alpha = a > 1, \alpha = 1 \).
\( (b) \) \( 0 < 1 < a < 1, \alpha = a > 1, \alpha = 1 \).
\( (c) \) \( 0 < a < 1, \alpha = a, \alpha = a > 1, \alpha = 1 \).
\( (d) \) \( 1 < a, \alpha = a, \alpha = a > 1, \alpha = 1 \).
when \( \varphi(u) = 1 + u^a \), where \( a > 1, 1/a + a + 1 = 1 \), the following formulæ hold:
\( (a) \) \( ||x||_\mathcal{X} = \varphi(\lambda x) = \varphi(\lambda x) \).
\( (b) \) \( ||x||_\mathcal{X} = \varphi(\lambda x) \).
There also holds the formulæ \( ||x||_\mathcal{X} = \varphi(\lambda x)^{1/\alpha} \).

5. Suppose \( B \) fulfills the property (9). If for any \( x, \lambda \in \mathcal{X} \),
the relation \( ||x||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \), implies the relation \( ||x||_\mathcal{X} \rightarrow 0 \) as \( n \rightarrow \infty \), then
\( (*) \) \( \varphi(u) \leq \varphi(u) \) for \( u > u_*, u_*, k \) are positive constants.
We choose an \( \varepsilon > 0 \) such that the inequality \( \|u\|_p \leq \varepsilon \) implies \( \|u\|_p \leq 1 \). Let \( M(e) > 0 \). By 4.3 there exists \( u \) for which \( \|\lambda u\|_p = e \); hence

\[ \varphi(u) = e. \]

But \( \|\lambda u\|_p = M(e) \varphi(u(\|\lambda u\|_p^{-1})) \leq 1 \). Therefore \( M(e) \varphi(u) \leq 1 \) and

by \((+)\) we get \( \varphi(u) \leq \varphi(u(e)) \).

Since the set of the values \( M(e) \) is dense in \((0, 1)\), the set of those \( u \) which satisfy \((+)\) is dense in \( (u_0, \infty) \), where \( u_0 \) satisfies the condition \( \varphi(u_0) = e \); consequently for any \( u \geq u_0 \).

5.1. If the inequality \((*)\) is satisfied, then for any \( s \in E \) the relation

\[ \|s_u\|_p \rightarrow 0 \text{ as } n \rightarrow \infty, \]

implies the relation \( \|s_u\|_p \rightarrow 0 \text{ as } n \rightarrow \infty. \)

It is enough to show that if \( \|u(\lambda u)\|_p \rightarrow 0 \text{ as } n \rightarrow \infty \), for any

\( \lambda > 0 \); then \( \|u(\lambda u)\|_p \rightarrow 0 \text{ as } n \rightarrow \infty \).

Let \( \lambda > 1 \). For any \( t \in T \) for which \( \|u(t)\|_p \geq u_0 \) we have, in virtue of \((*)\), the inequality

\[ \varphi(u(t)) < \varphi(u(\lambda u(t))) \leq \alpha p(\lambda u(t)), \]

if \( \|u(t)\|_p < u_0 \lambda^{-1} \), then \( \|u(t)\|_p \leq \varphi(u_0 \lambda^{-1}). \) Consequently

\[ \|u(t)\|_p < \alpha p(\lambda u(t)) + \varphi(u_0 \lambda^{-1}), \]

and from this,

\[ \limsup_{n \to \infty} \varphi(u(t)) \leq \varphi(u_0 \lambda^{-1}), \quad \varphi(u(\lambda u(t))) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

5.2. Let \( f_u \) be a strictly increasing \( \varphi \)-function for \( n = 1, 2, \ldots \) The following conditions are equivalent:

(a) \[ \limsup_{n \to \infty} f_u(\lambda u) \leq 1 \text{ for } 0 \leq u < 1, \quad \liminf_{n \to \infty} f_u(u) = \infty \text{ for } u > 1; \]

(b) \[ \lim_{n \to \infty} (f_u - 1) = 1 \text{ for } u > 1. \]

For example, we shall prove \((a) \implies (b)\). Let \( 0 < u' < u < u'', v > 1 \).

Since \( f_u(u') < v < f_u(u'') \) for sufficiently large \( n \), we get \( u' < (f_u)_{-1}(v) < u'' \), whence

\[ u' \leq \liminf_{n \to \infty} (f_u - 1) \leq \limsup_{n \to \infty} (f_u - 1) \leq u''. \]

5.2.1. If \( f_u \) are \( s \)-convex \( \varphi \)-functions, then condition 5.2.1 implies

\[ \lim_{n \to \infty} f_u(1) = 1. \]

Let \( 0 < u < 1, \) \( u < u < 1 \), then \( f_u(u) = f_u\left(\frac{u}{u-1}\right) \leq (u^{-1})^p \leq f_u(1) \), \( f_u(u) \leq 1. \) If \( 0 < u < 1 < u'' \), then \( f_u(u) < 1 < f_u(u'' \))

for sufficiently large \( n \), hence \( u' < (f_u - 1) < u'' \liminf_{n \to \infty} (f_u - 1) = 1. \)

5.3. Let \( f_u \) be \( \varphi \)-functions such that:

(a) \[ f_u(u) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad 0 \leq u < 1, \quad f_u(u) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{for } u > 1. \]

For each \( \lambda > 0 \) and \( s \in E \) with positive \( \varphi \)-measure, there exists the limit

\[ \|\lambda u\|_p \rightarrow \lambda \text{ as } n \rightarrow \infty. \]

Conversely, if for some \( e \) with positive \( \varphi \)-measure the limit \((***)\) exists, then \((*)\) holds.

Define \( e_0 \) in such a manner that \( \lambda e_0 = \|\lambda u\|_p \) or equivalently \( \varphi(e) = e_0 [\varphi(e)]^{-1} - \lambda \). Let \( e_0 > 1 \). Then for any \( n \geq e_0 \) the inequality \( e_n \varphi(e_n^{-1}) \rightarrow e_0 \) holds, and since \( e_n \varphi(e_n^{-1}) \) is strictly increasing with \( e_0 \), it must be \( e_n \leq e_0 \), whence

\[ \limsup_{n \to \infty} e_n \leq 1. \]

Similarly, it can be shown that

\[ \liminf_{n \to \infty} e_n \geq 1, \]

so that \( \lim_{n \to \infty} e_n = 1. \)

In order to prove the second part of the theorem let us assume \( e_n \lambda = \|\lambda u\|_p \rightarrow \lambda \text{ as } n \rightarrow \infty, \)

where \( \varphi(e) > 0 \), for any \( \lambda > 0 \). Let \( u' < u < u'' \).

Since \( e_n \rightarrow 1 \) as \( n \rightarrow \infty \), we get \( u' < e_n < u < u'' \) for \( n > e_0 \), hence

\[ u''(\varphi(e_n^{-1}))^{-1} < \varphi(e) = e_0 [\varphi(e_n^{-1})^{-1} - \lambda] < u''(\varphi(e_n^{-1}))^{-1}. \]

But, because of \( \varphi(e) > 0 \), the last inequalities can be satisfied for an arbitrary positive \( \lambda \), only if the condition \((*)\) is satisfied.

5.4. (a) If for \( s \)-convex \( \varphi \)-functions the condition 5.2.1 is satisfied, then for any \( e \) with positive \( \varphi \)-measure, there exists the limit

\[ \|\lambda u\|_p \rightarrow \lambda \text{ as } n \rightarrow \infty. \]

(b) If for a mean value \( \varphi \)-function the corresponding class \( E \) fulfills the property \((\Psi)\) and \((*)\) is satisfied for any \( e \in E \) with positive measure, then the condition 5.2.1 is satisfied.

Ad (a). If \( 0 < \varphi(e) < 1 \), then by 5.2(b) and 5.2.1

Ad (b). Since the values \( u = \varphi(e) \) are dense in \( (1, \infty) \), the relation

\[ (f_u - 1) \rightarrow 1 \text{ as } n \rightarrow \infty, \]

by the monotony of \( f_u \), the limit \( (f_u - 1) \rightarrow 1 \) as \( n \rightarrow \infty \) exists for any \( u > 1 \). It is enough to apply 5.2.
5.5. If \( q \) is a convex \( \varphi \)-function satisfying conditions \((o_1), (\infty)_1\), then for any \( v > 0 \) there exists \( \lambda_0 > 0 \) such that

\[
\varphi^* \left( \frac{1}{v} \right) = \frac{1}{\lambda_0} + \frac{v}{\lambda_0} \varphi(\lambda_0) = \inf_{\lambda > \lambda_0} \left( \frac{1}{\lambda} + \frac{v}{\lambda} \varphi(\lambda) \right).
\]

Indeed, for \( \lambda_0 = \varphi^* (v^{-1}) \) we can find \( \lambda_0 > 0 \) for which the equality \( v \lambda_0 = \varphi(\lambda_0) + \varphi^*(v) \) holds.

5.5.1. (a) For strictly increasing \( \varphi \)-functions \( \varphi_n \), the conditions \( \varphi_n(u) \to u \) as \( n \to \infty \), for \( u > 1 \), and \( \varphi_n(u) \to u \) as \( n \to \infty \), for \( u > 1 \), are equivalent.

(b) For an \( s \)-convex \( \varphi_n \), from \( \varphi_n(u) \to u \) for \( u > 1 \) it follows \( \varphi_n \) \( \to 1 \).

Ad (b). \( \varphi_n \) are strictly increasing, as follows from the \( s \)-convexity. 

By (a) we have

\[
\limsup \varphi_{n-1}(1) \leq u \quad \text{for} \quad u > 1,
\]

hence

\[
\limsup \varphi_{n-1}(1) \leq 1.
\]

Let \( 0 < \alpha < 1 \); since \( \varphi_n(\alpha^{1/n}) \leq \alpha \varphi_n(u) \) for \( u > 0 \), we get \( \alpha^{1/n} \varphi_n(u) \leq \varphi_n(\alpha u) \) for \( u > 0 \). For \( u > 1 \), \( \alpha = 1/u \) it follows

\[
u^{-1} \varphi_n(1) \leq (\varphi_n)_u \leq \inf_{n=0} \inf (\varphi_n)_u \leq \liminf_{n=0} (\varphi_n)_u \text{ and consequently } (\varphi_n)_u \to 1 \text{ as } n \to \infty.
\]

5.5.2. Let \( \varphi_n \) be a convex \( \varphi \)-function satisfying conditions \((o_1), (\infty)_1\) for \( n = 1, 2, \ldots \)

(a) If \( \lim \varphi_n(u) = \infty \) for \( u > 1 \), then \( \limsup \varphi^*_n(v) \leq v \) for \( v > 0 \), and conversely.

(b) If \( \lim \varphi_n(u) = 0 \) for \( 0 < u < 1 \), then \( \liminf \varphi^*_n(v) \geq v \) for \( v > 0 \), and conversely.

(c) If \( \liminf \varphi^*_n(v) \geq v \) for \( v > 1 \), then \( \limsup \varphi_n(u) \leq u \) for \( 0 < u < 1 \).

(An analogous lemmata can be found in [13].)

Ad (a). Let \( \varphi_n(u) = \infty \) for \( u > 1 \). Suppose \( v > 0 \) is given. We choose \( u_0 \) such that

\[
\text{(*) } \varphi_n(u_0) = \varphi_n(u_0) + \varphi^*_n(v).
\]

Let \( u_0 > 1 \). It must be \( \limsup \varphi_n(u_0) \leq u_0 \), for if not so it would be

\[
v \geq \frac{\varphi_n(u_0)}{u_0} = \frac{\varphi_n(u_0)}{u_0} > \varphi_n(u_0).
\]

for an increasing sequence of indices \( n \), but this is contradictory to \( \varphi_n(u_0) \to \infty \) as \( n \to \infty \). It follows that

\[
\limsup u_n \leq 1,
\]

and by (a) we obtain

\[
v \geq \frac{\varphi_n(u_0)}{u_0} \geq \varphi^*_n(v).
\]

If

\[
\limsup \varphi^*_n(v) \leq v \quad \text{for} \quad v > 0,
\]

then for a given \( 0 < \alpha < 1, \alpha > 1 \), such that \( \alpha \lambda > 1 \), we have \( \varphi^*_n(v) \leq \beta v \) for \( n > n_1 \), and since \( \varphi_n(u) \geq \varphi_n(u_0) + \varphi^*_n(v) \) we get \( \varphi_n(u) \geq \varphi_n(u_0) + \varphi^*_n(v) \) for any \( v > 0 \), whence

\[
\lim \varphi_n(u) = \infty.
\]

Ad (b) and (c). Applying the inequality \( \varphi_n(u) \leq \varphi^*_n(v) \) for \( 0 < u < 1 \), we get

\[
v \leq \liminf \varphi^*_n(v) \quad \text{for} \quad v > 0
\]

if \( \varphi_n(u) \to 0 \) for \( 0 < u < 1 \). If for a \( \epsilon_n \) the equation \( \varphi_n(u) = \varphi^*_n(v) + \epsilon_n \) holds, where \( 0 < u < 1 \), we have \( u \leq \varphi^*_n(v) + \epsilon_n \). It follows from the last inequality by the convexity of \( \varphi^*_n(v) \) that

\[
\limsup \epsilon_n \leq 1,
\]

and so

\[
\limsup \varphi_n \leq \limsup \varphi_n = u.
\]

5.5.3. It follows from 5.5.2 that conditions

(a) \( \lim \varphi_n(u) = 0 \) for \( 0 < u < 1 \),

(b) \( \lim \varphi_n(u) = \infty \) for \( u > 1 \),

imply

\[
(\gamma) \lim \varphi^*_n(v) = v \quad \text{for} \quad v > 1 \quad (\text{for} \quad v > 0).
\]
If (γ) is satisfied, then
\[
\limsup_{n \to \infty} \varphi_n^*(v) \leq v \quad \text{for any } v \geq 0
\]
and consequently (β) is satisfied, and besides
\[
\limsup_{n \to \infty} \varphi_n(u) \leq u \quad \text{for } 0 \leq u < 1.
\]

5.6. (a) Let \( \varphi_n \) be a convex \( \varphi \)-function, satisfying conditions (α₁), (α₂), for \( n = 1, 2, \ldots \). If the condition
\[
\varphi_n^*(v) \to v \quad \text{for } v > 1
\]
is satisfied, then for any \( \varepsilon \in (0, 1) \) E with positive \( \bar{\mu} \)-measure there exists the limit
\[
\lim_{n \to \infty} \|\varphi_n\|_{\mu} = 1 \quad \text{as } n \to \infty.
\]

(b) If for a mean value \( \bar{\mu}(\cdot) \) the corresponding class \( E \) fulfills the property (Ω) and (**) is satisfied for any \( \varepsilon \in E \) with positive \( \bar{\mu} \)-measure, then condition (**) holds.

Ad (a). By 5.5 and the definition of \( \|\varphi\|_{\mu} \) we get
\[
\|\varphi_n\|_{\mu} = \lim_{\lambda \to \infty} \left( \frac{1}{\lambda} + \frac{\bar{\mu}(\varepsilon)}{\lambda} \right) = \bar{\mu}(\varepsilon) \lim_{\lambda \to \infty} \left( \frac{1}{\lambda} \right)
\]
In virtue of (α) and 5.5.1 it follows \( \bar{\mu}(\varepsilon) \frac{\varphi_n(\lambda)}{\lambda} \to 1 \) as \( n \to \infty \).

Ad (b). The set of values \( v = \bar{\mu}(\varepsilon)^{-1} \) is dense in \( (0, 1) \) and for any such value \( \varphi_{n-1}(\varepsilon) - \varepsilon \to 1 \) as \( n \to \infty \). But by the monotony of \( \varphi_n \), this relation holds for any \( v > 1 \), and so, by 5.5.1, \( \varphi_n(u) \to u \) as \( n \to \infty \), for any \( u > 1 \).

5.7. If \( \varphi_n, \varphi \) are \( \varphi \)-functions, then the following conditions are equivalent:

(a) \( \lim_{n \to \infty} \|\varphi_n\|_{\mu} = \|\varphi\|_{\mu} \) for \( \varepsilon \in X \).

(b) \( \lim_{n \to \infty} \varphi_n(u) = \varphi(u) \) for \( u \geq 0 \).

(b) \Rightarrow (a). Let \( \varepsilon = \bar{\mu}(\varepsilon) \varphi_n\left(\varepsilon^{-1}\right) \varepsilon = \bar{\mu}(\varepsilon) \varphi\left(\varepsilon^{-1}\right) \). The continuity and monotony of \( \varphi_n, \varphi \) ensure \( \varphi_n(u) \to \varphi(u) \) in any interval \( (0, u_0) \). Suppose \( \varepsilon_{n-1} \leq \varepsilon \) where \( n \to \infty \). Then \( \varepsilon_{n} \geq \limsup_{n \to \infty} \varphi_n\left(\varepsilon^{-1}\right) \), and owing to the uniform convergence of \( \varphi_n\left(\varepsilon^{-1}\right) \to \varphi\left(\varepsilon^{-1}\right) \) we get
\[
\bar{\mu}\left(\varphi_n\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right) \to \bar{\mu}\left(\varphi\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right)
\]
whence
\[
\liminf_{n \to \infty} \varepsilon_n \geq \varepsilon,
\]
and consequently
\[
\lim_{n \to \infty} \varepsilon_n = \varepsilon.
\]

Assume now \( \varepsilon_{n-1} \geq \varepsilon \) as \( n \to \infty \). Then
\[
\bar{\mu}\left(\varphi_n\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right) \leq \bar{\mu}\left(\varphi\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right)
\]
that is to say,
\[
\varepsilon_n \leq \bar{\mu}\left(\varphi\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right)
\]
\( n \to \infty \). The sequence \( \varepsilon_n \) is bounded, for \( \bar{\mu}\left(\varphi\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right) \leq \varepsilon \). For any accumulation point of the sequence \( \varepsilon_n \) the equation \( \varepsilon_n = \bar{\mu}\left(\varphi\left(\varepsilon^{-1}\right) \varepsilon^{-1}\right) \) holds, and since, by 4.4, the equation \( \varepsilon = \bar{\mu}(\varepsilon) \varphi(\varepsilon^{-1}) \) is satisfied only for \( \varepsilon = \bar{\mu}(\varepsilon) \), we get
\[
\varepsilon_n \leq \bar{\mu}(\varepsilon) \varphi(\varepsilon^{-1})
\]
\( n \to \infty \). (b) Suppose \( \varepsilon > 0 \) and \( \varepsilon \in E \), \( \bar{\mu}(\varepsilon) > 0 \) given. Choose \( \lambda > 0 \) such that \( \varepsilon \lambda = \bar{\mu}(\varepsilon) \varphi(\varepsilon^{-1}) \), that is to say, \( \bar{\mu}(\varepsilon) = \varepsilon(\varphi(\varepsilon^{-1}))^{-1} \). Choose \( \varepsilon_n \) such that \( \varepsilon_n = \bar{\mu}(\varepsilon_n) \varphi(\varepsilon_n^{-1})^{-1} \lambda \). (b) implies \( \varepsilon_n = \varepsilon_n \varphi(\varepsilon_n^{-1}) \to \varphi(\varepsilon^{-1}) \) as \( n \to \infty \). In other words, for any \( u > 0 \) there exists a sequence \( \varepsilon_n \) such that \( \varepsilon_n \to 0, \varphi_n(\varepsilon_n) \to \varphi(\varepsilon^{-1}) \). If \( u \) belongs to \( (w', u') \), where \( 0 < u < u' \), are arbitrarily given, we get \( u' < u < u' \), \( \varphi_n(u') \leq \varphi_n(u) \leq \varphi_n(u) \) for sufficiently large \( n \), and it follows
\[
\varphi(u) = \liminf_{n \to \infty} \varphi_n(u') \quad \limsup_{n \to \infty} \varphi_n(u) = \varphi(u)
\]
Letting \( u \to u'' > 0 \) or \( u \to u'' + 0 \), we obtain
\[
\liminf_{n \to \infty} \varphi_n(u') \geq \varphi(u''), \quad \limsup_{n \to \infty} \varphi_n(u) \leq \varphi(u')
\]
and consequently
\[
\lim_{n \to \infty} \varphi_n(u) = \varphi(u) \quad \text{for } u > 0.
\]

5.8. (a) If \( \varphi \)-functions \( \varphi_n \) satisfy condition 5.3 (α), then the relation
\[
(\beta) \quad \lim_{n \to \infty} \sup \|\varphi\|_{\mu} \equiv 0 \quad \text{as } n \to \infty, \aleph \in X
\]
holds.

(b) If (α) is satisfied, then \( \varphi_n \) fulfills condition 5.3 (α)
We can assume \( \sup \|\varphi\|_{\mu} \equiv 0 \), for \( \sup \|\varphi\|_{\mu} = 0 \) for \( n = 1, 2, \ldots \) Suppose \( 0 < \varepsilon < \sup \|\varphi\|_{\mu} \). By 2.7.3 we have
\[
\|\varphi_n\|_{\mu} \leq \|\varphi\|_{\mu} \leq \sup \|\varphi\|_{\mu} \|\varphi\|_{\mu}
\]
where \( \varepsilon \in E \), \( \bar{\mu}(\varepsilon) > 0 \). Hence, by 5.3, the relation (α) holds.

(b) immediately follows by 5.3 and in virtue of the fact that the sup \( \lambda \geq \lambda \), when \( \lambda > 0 \), \( \bar{\mu}(\varepsilon) > 0 \).

5.9. Let \( \varphi_n \) be an \( s \)-convex \( \varphi \)-function for \( n = 1, 2, \ldots \)
(a) Under the assumption of 5.2 (a) the relation
\[
(\beta) \quad \|\varphi_n\|_{\mu} \leq \sup \|\varphi\|_{\mu}^{\infty} \quad \text{as } n \to \infty, \aleph \in X
\]
holds.
(b) If (**) is satisfied, and \( E \) fulfills the property \( (\mathcal{D}) \), then 5.2 (a) holds. To prove (a) we apply 5.4 (a) and the inequality
\[
\|F_\beta \|_{\mathcal{L}} \leq \|F_\alpha \|_{\mathcal{L}} \leq \|F_\alpha \|_{\mathcal{L}} \sup (\mathcal{A})^*;
\]
here \( \lambda, \epsilon \) have the same meaning as in the proof of 5.8 (a).
(b) follows immediately from 5.4 (b).

5.10. Let \( \varphi_n \) be a convex \( \varphi \)-function, satisfying (a), (\( \infty \)), for \( n = 1, 2, \ldots \)
(a) Under the assumption of 5.6 (**), the relation
\[
\|\chi \|_{\mathcal{L}} \to \sup |\varphi| \quad \text{as} \quad n \to \infty, \forall x \in X,
\]
holds.
(b) If \( E \) possesses the property \( (\mathcal{D}) \) and (**), \( \varphi_n \) satisfy condition 5.6 (**).

The proof of (a) follows by 5.6 and by the application of the inequality
\[
\lambda \|\varphi \|_{\mathcal{L}} \leq \|\varphi \|_{\mathcal{L}} \leq \|\varphi \|_{\mathcal{L}} \sup (\mathcal{A})^*;
\]
here \( \lambda, \epsilon \) have the same meaning as in the proof of 5.8 (a); (b) follows immediately from 5.6 (b).

6.1. In this section we are concerned, in the first place, with the following question:
On what conditions on \( \varphi \)-functions \( \varphi, \psi \) does the following inequality
\[
\|\varphi \|_{\mathcal{L}} \leq \|\psi \|_{\mathcal{L}} \quad \text{for all} \quad x \in X,
\]
hold. If for a pair \( \varphi, \psi \) the last written inequality is satisfied, then it will be said they possess the property \( (C, \psi, \varphi) \). Assuming \( \varphi, \psi \) to be convex \( \varphi \)-functions, and \( E \) (for a given \( \mathcal{M}(\cdot) \)) to fulfill the property \( (\mathcal{D}) \), we obtain for \( (C, \psi, \varphi) \), by 6.6 (b), and by setting \( x = x_0 \) the following necessary condition:
\[
\psi(u) \leq \varphi(u) \quad \text{for} \quad u \geq \psi(-1).
\]

On the other hand, the inequality
\[
\psi(u) \leq \varphi(u) \quad \text{for} \quad u \geq 0
\]
is obviously sufficient to have the property \( (C, \psi, \varphi) \) for any mean value. But the last mentioned inequality certainly does not give a necessary condition for the property \( (C, \psi, \varphi) \), for if \( \psi(u) = u^\beta, \varphi(u) = u^\alpha \), \( 1 \leq \beta < \alpha \), then \( (C, \psi, \varphi) \) is fulfilled.

Let us introduce the following notation: \( \omega(u) = \varphi(\psi(-u)) \).

6.2. Let us assume that \( \omega \) satisfies the following condition:
For any \( \epsilon > 0 \) there exists \( \lambda > 0 \) such that
\[
(\log(u) + (1 - \lambda)) \quad \text{for} \quad u \geq \epsilon.
\]

Under this assumption the property \( (C, \psi, \varphi) \) is fulfilled for any upper mean value.

It is known that \( \psi(1/n) \) implies \( \|\psi\|_{\mathcal{L}} \leq 1 \).
Denote by \( \lambda_n \) the number \( \lambda \) for which the inequality \( (\log(u) + (1 - \lambda)) \) is satisfied
when \( \epsilon = \psi(1/n) \), and set \( x_n = |x| \vee 1/n - 1/n \). By (**) we have
\[
\psi(|x_n|) \leq \lambda_n \psi(|x_n|) + (1 - \lambda_n),
\]
hence
\[
\psi(|x_n|) \leq \lambda_n \psi(|x_n|) + (1 - \lambda_n),
\]
and
\[\begin{align*}
\psi\left(\frac{\|\psi(|x_n|)\|}{\leq 1},
\end{align*}\]
for \( \psi\left(\frac{\|\psi(|x_n|)\|}{\leq 1} = \|\psi\|_{\mathcal{L}} \leq 1 \). But \( x_n \to \|x\|, \psi\left(\frac{\|\psi(|x_n|)\|}{\leq 1} \to \psi\left(\frac{\|\psi\|}{\leq 1} \right)\right)\)
as \( n \to \infty \), \( \psi\left(\frac{\|\psi\|}{\leq 1} \right) \to 1 \), by (***), we get \( \psi\left(\frac{\|\psi\|}{\leq 1} \right) \leq 1 \) and consequently \( \|\psi\|_{\mathcal{L}} \leq 1 \).

6.2.1. Under the same assumption on \( \omega \) as in 6.2 there holds the inequality
\[
\|\psi\|_{\mathcal{L}} \leq \|\varphi\|_{\mathcal{L}} \quad \text{for} \quad x \in X.
\]

6.3. Each of the following conditions is sufficient for convex \( \varphi \)-functions \( \varphi, \psi \) to fulfill the property \( (C, \psi, \varphi) \).

A. \( \psi(1) = \psi(1) = 1 \); there exists a \( \lambda > 0 \) such that \( \omega(u) \geq 1/\lambda \) for almost every \( u > 1 \), \( \omega(u) \leq 1/\lambda \) for almost every \( u < 1 \).

B. \( \omega \) is a convex function in \( (0, \infty) \), \( \omega(1) \geq 1 \), and \( \omega \) satisfies condition (a).

C. \( \omega \) satisfies condition (a), if \( \omega \) denotes the greatest convex function such that \( \omega(u) \geq \omega(u) \) for \( u > 0 \), then \( \omega(1) \geq 1 \).

A sufficient condition for \( \varphi \)-convex \( \varphi \)-functions \( \varphi, \psi \) to satisfy the inequality
\[
\|\psi\|_{\mathcal{L}} \leq \|\varphi\|_{\mathcal{L}} \quad \text{for} \quad x \in X,
\]
is \( A \), and under the assumption of local absolute continuity of \( \varphi, \psi \), \( B \) or \( C \), as well.

Ad A. Since \( \varphi(1) = \psi(1) = 1 \), we get \( \omega(1) = 1 \). Evidently the following conditions are equivalent:

(1) For \( \lambda > 0 \), \( 0 < \lambda < 1 \), the inequality \( u \leq \omega(u) + (1 - \lambda) \) for \( u \geq 0 \), holds;
(2) there hold the inequalities
\[\begin{align*}
(\log(u) + (1 - \lambda)) \quad \text{for} \quad u \geq 0; \quad 0 \leq u < 1; \\
(\log(u) + (1 - \lambda)) \quad \text{for} \quad u > 1,
\end{align*}\]
where \( 0 < \lambda < 1 \).
Since $\varphi'(u), \psi'(u)$ exist almost everywhere and $\varphi'(u) > 0$ if $u > 0$, the derivative $\varphi'(u)$ exist almost everywhere as well. Moreover $\varphi, \psi$ satisfy the Lipschitz condition in any finite interval so does $\omega$ in any interval $(a, u_0)$, where $0 < a$. From $A$ and $\omega(1) = 1$ the inequalities $(\beta)$ follow and this implies $6.3 (\ast)$, where $\lambda$ is independent of $\alpha$.

Ad B. From the integral representation

$$
\omega(u) = \int_0^u \omega'_s(t) dt,
$$

where $\omega'_s$ is non-decreasing in $(0, \infty)$ and by $\omega(1) \geq 1$ we get $\omega(u) - 1 \geq \omega'_s(1)(u - 1)$ for $u > 1$, $1 - \omega(u) \leq \omega'_s(\delta)(1 - u)$ for $0 < u < 1$. It is enough to prove that $\lambda^{-1} = \omega'_s(1) \leq 1$. Clearly

$$
1 \leq \omega(1) = \int_0^1 \omega'_s(t) dt \leq \omega'_s(1).
$$

Assuming $\omega'_s(1) = 1$ we must have

$$
\int_0^1 (\omega'_s(t) - \omega'_s(t)) dt = 0,
$$

$\omega'_s(t) = 1$ for $0 < t < 1$, $\omega(u) = u$ for $0 < u < 1$, which is contradictory to the fact that $\omega$ fulfills the condition $(\alpha)$. Hence $(\beta)$ is satisfied.

Assuming additionally that $\omega$ fulfills the condition $(\alpha_2)$ another proof can be given. It is instructive, because using the notion of the complementary function, and it runs the following lines.

There exists $u_0 > 0$ such that $\omega(u) \geq u$, where the Young's inequality $u \leq \omega(u) + \omega'_s(u)$ and $u \leq \omega'(u) + \omega'_s\omega(u)$, but $\omega(1) \geq 1$, hence $1 \geq u^{-1} + \omega'_s\omega(u)$ and it suffices to set $\lambda = u^{-1}$. Evidently $0 < \lambda < 1$, for $u_0 > 1$.

Ad C. In virtue of $\omega(u) \geq u$, $\omega$ satisfies the condition $(\alpha_3)$.

We have $u \leq \omega(u) + (1 - \lambda) \leq \omega(u) + (1 - \lambda)$, where $\lambda^{-1} = \omega'_s(1)$. Since $\omega(1) \geq 1$, by $A$, $0 < \lambda < 1$ follows.

It follows from 6.3 C that if $\omega$ satisfies $(\alpha_3)$ and $\omega(u) \geq u$, where $\varphi' > 0$, then the property $(\varphi, \psi, \omega)$ is satisfied. If $\varphi, \psi$ are $\varphi$-convex, then the analogous inequality $|\omega| \leq |\omega|^2$, for $x \in \mathbb{R}$, is satisfied too. Let us set $\varphi(u) = u^\gamma, \psi(u) = u^\alpha$, where $1 < \beta < \alpha$. The well-known theorem of the monotonic increase of the mean value $M(|\omega|)_{\lambda}$ in the interval $(1, \infty)$ of $\alpha$ is a direct consequence of the above remark. To obtain the monotony of $M(|\omega|)_{\lambda}$ for the interval $a(0, 1)$ let us remark that if $0 < \beta < \alpha < 1$, then $\varphi$ is $\varphi$-convex, $|\omega| \leq \omega^\beta$. For the function $\omega(u) = u^{\lambda}$ the condition 6.3 can be applied and by 6.2.1 we get $M(|\omega|)^\beta = M(|\omega|^\beta)$, or equivalently $M(|\omega|)^\beta \leq \omega^\beta$.}

6.4. The following property remains closely related to the property $(G, \varphi, \psi)$: A pair of convex $\varphi$-functions $\varphi, \psi$ is said to possess the property $(H, \varphi, \psi)$ if the inequality

$$(+) \quad M(x \gamma) \leq M(|x|)M(|\gamma|) \quad \text{for} \quad x, \gamma \in \mathbb{R}$$

is satisfied for any upper mean value. The inequality $(+)$ presents one of the possible types of Hölder's inequality. The other types of Hölder's inequality are known in the theory of Orlicz spaces [1] and they readily can be generalized to an arbitrary upper mean value; namely we have the following theorem:

6.4.1. For an arbitrary upper mean value $M(\cdot)$ there holds the following inequality:

$$(+) \quad M(xy) \leq M(|x|)M(|y|) \quad \text{for any} \quad x, y \in \mathbb{R}.$$ 

Indeed, for any $\lambda > 0$ we get, by the Young's inequality,

$$M(xy) \leq \lambda^{-1} M(xy) + \lambda^{-1} M(xy),$$

whence $M(xy) \leq M(|x|)M(|y|).$

In spite of inequality 6.4.1 (+), which is generally valid, the property $(H, \varphi, \psi)$ is only true under a special assumption on $\varphi$, and it is not directly deducible from 6.4.1 (+). The reason for this is that only the inequality $|\omega| \leq M(|\omega|)$ occurs. But, setting $\varphi(u) = u^\alpha, \alpha > 1$, we get $(H, \varphi, \varphi)$ (the classical Hölder's inequality), for this case

$$M(|x|) = M(|\omega|)^{1/\alpha}, \quad M(|x|) = M(|\omega|)^{1/\alpha}, \quad M(|y|) = M(|\omega|)^{1/\alpha}. $$

6.4.2. If a convex $\varphi$-function $\varphi$ satisfies the conditions $(\alpha_1)$, $(\alpha_2)$, $\varphi(1) = 1$, $\varphi(u) = \varphi(u)\varphi(u)$, $\varphi(u)$ is satisfied for $\varphi(u) = \varphi(u)$, and since $u \varphi \leq \varphi(u) \varphi(u)$, we obtain

$$|\omega| \leq \psi^{-1} \varphi + \varphi^{-1} \psi \varphi.$$ 

Let

$$|\omega| = M(|\omega|) = 1, \quad |\omega| = M(|\omega|) = 1;$$

substituting in the last inequality $u = |u|$, $u = |u|$ we get

$$M(xy) \leq \psi^{-1} M(|x|) + \varphi^{-1} \psi \varphi \quad \text{for} \quad x, y \in \mathbb{R},$$

which implies inequality 6.4 (+).
7. A set $U \subseteq \mathcal{F}$ is called $s$-convex, $0 < s \leq 1$, if for any $a, b$ which satisfy the conditions $a, b \geq 0$, $a^s + b^s = 1$, $x, y \in U$ implies $a^s x + b^s y \in U$. A linear topological Hausdorff space is called locally $s$-convex, if there is a base of $s$-convex neighbourhoods of 0 in it.

7.1. Let us assume $E$ fulfills the following property:

For a given natural $n$ and a positive $\eta$, for which $n \eta \leq 1$, there exist in $E$ $n$ disjoint sets $e_1, e_2, \ldots, e_n$ such that

\[ \bar{w}(e_i) \leq \eta, \quad n \bar{w} = \bar{w}(\bigcup_{i=1}^n e_i). \]

(*)

If the topology generated by the norm $\| \cdot \|_\eta$ in $\mathcal{F}$ is locally $s$-convex, then there is a $\phi$-function $\chi(u) = \psi(u^s)$, where $\psi$ is a convex $\phi$-function, for which

(\*)\[ \chi(\lambda u) \leq \phi(u^{\lambda}) \leq \chi(u^{\lambda}), \quad 0 < \lambda < 1, \quad u \geq u_0. \]

Choose in $\mathcal{F}$ an $s$-convex neighbourhood $U$ of zero and a $\delta > 0$ in such a manner that $\| u \|_\delta \leq \delta$ implies $\phi(x U)$ and $x U$ implies $\| u \|_\delta < 1$.

Given an $a$, $0 < a \leq 1$, let us denote by $n$ a non-negative integer for which

\[ 1 < na^s \leq 1. \]

Let us choose a $u$ satisfying the conditions

(\+)

$\phi(u) > \delta$, \hspace{1cm} a^s \phi(u) > \delta,$

and set $\eta = \delta \phi(u)\varphi^{-1}$. Since $\eta < \alpha^s$, $\eta \leq 1$, there exist a disjoint sets in $E$ for which the condition (*) holds. From the inequality

\[ \| u \|_\delta \leq \delta \phi(u) \varphi^{-1} \]

it follows $\| u \|_\delta < \delta$, for $\psi(u)\varphi^{-1}$ is strictly increasing. The elements $u_0^a$ belong to $U$ and by the $s$-convexity of $U$, $x = a u_0 + \ldots + a u_0 U$, for which $nsa^s \leq 1$. This implies

\[ \bar{w}(\varphi(t)) = \bar{w}(\varphi(a u_0 + \ldots + u_0 U)) = \varphi(a) \bar{w}(\bigcup_{i=1}^n e_i) \leq 1. \]

In view of (*) it follows

\[ sa^s \varphi(a) \leq 1, \]

whence, and by (+), we get

\[ na^s \phi(u) \varphi^{-1} \leq 2na^s. \]

We have proved the inequality

(\++)

\[ \phi(u) \leq 2 \delta^{-1} u^s \phi(u^s) \]

for all $a, u$ for which $0 < a < 1$ and (+) hold. From (\+++) it follows that $\phi(u) \geq c u^s$ for $u \geq u_0$, where $c$ is a positive constant, and $u_0$ sufficiently large. If this is not so, then one can find numbers $u_0$, such that $u_0 \to \infty$, $\phi(u_0^s) u_0^s \to 0$ as $n \to \infty$. Let us define $a_0$ by the requirement

\[ a_0 \phi(u_0^s) = \delta. \]

Since for sufficiently large $n$, $a_n < 1$, $\phi(u_n^s) > \delta$, so substituting in (\++) $a = a_0$, $u = u_0$, we obtain the inequality

\[ \phi(u_n^s) u_0^s \phi(u_0^s) u_0^s \geq \delta, \quad \phi(u_0^s) > \delta, \]

and by (\++)

\[ \phi(u_n^s) u_0^s \phi(u_0^s) u_0^s \geq 2 \delta^{-1} \phi(u_n^s) u_0^s \]

for $u_0 \geq u_0 \geq u_0$.

By a theorem in [4], 2.6.2, 2.7, the last inequality implies the existence of a $\phi$-function $\chi$ with the required properties.

7.2. If for a $\phi$-function $\phi$ there exists a $\phi$-function $\chi$ satisfying the inequalities 7.1 (**) and of the form $\chi(u) = \psi(u^s)$, where $\psi$ is a convex $\phi$-function, then the topology in $\mathcal{F}$, which is generated by the norm $\| \cdot \|_s$, is locally $s$-convex.

By 5.1 the convergence with respect to the norm $\| \cdot \|_s$ implies the convergence with respect to the norm $\| \cdot \|_s$, and conversely. But the topology which is generated by $\| \cdot \|_s$ is locally $s$-convex, for we can choose a base of neighbourhoods of zero composed of the following $s$-convex neighbourhoods

\[ U(e) = \{ x \in \mathcal{F} : \| x \|_s < e \}. \]

Let us conclude this section with the following remarks. Theorems 7.1, 7.2 generalize some results of [6], [9].

In section 6 one can find some example of set-algebras which satisfy the condition (\*) in 7.1. The condition 7.1 (**) implies, of course, the property (2) for a given $E$.

8. In this section some typical examples of spaces $X$ and subadditive mean values are given.

I. We write $I = (a, b)$, where $a$ and $b$ are finite.

(a) Let $X$ be the space of all real-valued and bounded functions in $I$. Then the class $E$ is the collection of subsets in $I$. We may define

\[
1) \ \bar{w}(a) = \sup_I |\varphi(t)|; \\
2) \ \bar{w}(a) = (b - a)^{-1} \int_a^b |\varphi(t)| \, dt.
\]
where \( \int \) ... means the Riemann upper integral. The mean value \( 1 \) is an extreme value, in this case \( \omega(\varepsilon) = 0 \) only for the empty set, \( \omega(\varepsilon) = 1 \) if \( \varepsilon \) is non-empty.

(b) Let \( X \) be the space of real-valued bounded and measurable functions in \((a, b)\). We may define

1) \( \omega(x) = \sup_s \{ x(s) \} \), where \( \sup_s \) denotes the essential supremum with respect to the ideal of Lebesgue-measurable sets with the measure 0;

2) \( \omega(x) = (b - a) \int_a^b |x(t)| \, dt \), where \( \int_a^b \) ... means the Lebesgue integral.

The mean value \( 1 \) is evidently extreme.

II. Let \( k(t, \tau) \) be a non-negative integrable function for any \( 0 < t, \tau \leq q \), where \( t, \tau \subseteq \mathbb{R} \), and either for \( t \) belonging to \( I_0 = \{ 0 < t < \tau \} \) or to \( I_\infty = \{ t, \tau \times \mathbb{R} \} \).

Let

\[
\int_0^t k(t, \tau) \, d\tau = 1 \quad \text{where} \quad (a) \quad \tau \in I_0, \quad (b) \quad \tau \in I_\infty.
\]

For \( X \) we choose the space of real-valued, bounded and measurable functions in \((0, t_0)\). We define

1) \( \omega(x) = \limsup_{t \to t_0} \int \{ k(t, \tau) |x(t)| \} \, d\tau \), where \( t_0 = 0 \), if \( k(t, \tau) \) is defined in \((0, t_0) \times \mathbb{R}, \tau_{\infty} = \infty \) if \( \tau \) is taken in \( I_\infty \);

2) \( \omega(x) = \sup_{t \in I_0} \int \{ k(t, \tau) |x(t)| \} \, d\tau \), where \( I_0 \) is either \( I_0 \) or \( I_\infty \).

The particularly important cases can be obtained setting \( k(t, \tau) = \tau^{-1} \) for \( 0 < t \leq \tau, k(t, \tau) = 0 \) for \( t > \tau \). Assuming \( t_0 = \infty, \tau \to 1 \), we obtain the following mean values:

1a) \( \omega(x) = \limsup_{t \to \infty} \tau^{-1} \int \{ x(t) \} \, dt \);

2a) \( \omega(x) = \sup_{t > 0} \tau^{-1} \int \{ x(t) \} \, dt \).

Assuming \( t_0 = 1, \tau \to 1 \) we get

1b) \( \omega(x) = \limsup_{t \to 1} \tau^{-1} \int \{ x(t) \} \, dt \);

2b) \( \omega(x) = \sup_{t < 1} \tau^{-1} \int \{ x(t) \} \, dt \).

III. Let \( X \) be the space of bounded sequences \( (t) \) of reals; then \( E \) is the class of all subsets of the collection of natural numbers. Let \( a_{\infty} \) be non-negative and let they satisfy the condition \( \sum a_{\infty} = 1 \) for any \( n \).

We define the following mean values

1) \( \omega(x) = \limsup_{n \to \infty} \sum_{a_{\infty}=0}^n \frac{1}{n} a_{\infty} |x_i| \);

2) \( \omega(x) = \sup_{n \in \mathbb{N}} \sum_{a_{\infty}=0}^n \frac{1}{n} a_{\infty} |x_i| \).

Setting \( a_{\infty} = 1/n \) for \( i = 1, 2, \ldots, n = 0 \) if \( i > n \), for \( n = 1, 2, \ldots \), we obtain the following mean values, which are of some importance, when investigating methods of the strongly summable sequences:

1a) \( \omega(x) = \limsup_{n \to \infty} \sum_{a_{\infty}=0}^n \frac{1}{n} a_{\infty} |x_i| \);

2a) \( \omega(x) = \sup_{n \in \mathbb{N}} \sum_{a_{\infty}=0}^n \frac{1}{n} a_{\infty} |x_i| \).

8.1. The classes \( E \) which correspond to the spaces \( X \) in I-III, and the mean values II(a), II(b), II(1a), II(2b), III(1a), III(2a), possess the property \( (\mathcal{D}) \). Let us consider, for example, the mean value II(2b). If \( e = (r_1, r_2) \), then \( \omega(e) = 1 - r_1 / r_2 \), which means that the values \( \omega(e) \) are dense in \((0, 1)\). In the case of III(2a), we obtain \( \omega(e) = 1 - p / q \), when \( e = (p, q, p \leq q < q) \) and consequently the property \( (\mathcal{D}) \) is fulfilled. Let us yet consider the mean value II(1a). Let \( 0 < r_1 < r_2 \) and choose the sets \( \epsilon_n = (r_1, r_1, r_2) \), where \( r_1 \) are positive integers such that \( r_1 < r_1 < r_2 < r_1 < r_2 < r_1 < r_2 \) for \( n = 1, 2, \ldots \) When we define \( \epsilon_n = 1 \) \( \epsilon_n \) we obtain the following inequalities

\[
\int_0^{r_1} x(t) \, dt \leq 1 - r_1 + \frac{1}{n} \quad \text{for} \quad r_1 < \tau \leq r_2 + r_2 + 1 / n,
\]

\[
1 - r_1 + \frac{1}{n} \leq \int_0^{r_2} x(t) \, dt \quad \text{if} \quad \tau = r_2 + r_2 + 1 / n,
\]

which implies

\[
\limsup_{n \to \infty} \tau^{-1} \int_0^{r_2} x(t) \, dt = 1 - r_1 / r_2,
\]

and the property \( (\mathcal{D}) \) is fulfilled. By similar arguments we can verify that property \( (\mathcal{D}) \) is fulfilled for \( E \), if the mean value is III(1a).

8.2. The set algebra \( E \) fulfills the property 7.1 (\( * \)), which is more general as the property \( (\mathcal{D}) \), if the corresponding mean value is II(1a),
II (2p), III (1a), III (2a) respectively. Let us consider the mean values \( \Pi (1a), \Pi (2a) \).

Define two increasing sequences of natural numbers \( l_r, k_r \) in such a manner that:

1) \( l_r \geq 2, \)
2) \( l_{r+1} > l_r + 1, k_{r+1} > l_r + k_r \) for \( r = 1, 2, \ldots \)
3) \( \varepsilon_r = \frac{k_r}{l_r + k_r} \to 1 \quad \text{as} \quad r \to \infty, \)
4) \( \frac{k_1 + k_2 + \ldots + k_{r-1}}{l_r} < 1 - \varepsilon_r \) for \( r = 2, 3, \ldots \).

For instance, we can choose \( l_r \) arbitrarily, but such that \( l_r \geq 2, \) \( l_{r+1} > l_r + 1, (l_{r+1} + 2l_1 + \ldots + (r-1)l_{r-1})l_r^{-1} < \frac{4}{3}(1+\varepsilon)^{-1} \) for \( r = 2, 3, \ldots \) and set \( k_r = r l_r. \)

Suppose \( 0 < \eta < \varepsilon_r \) and that, for a natural \( n, 1 \leq n \leq l_r \). Define \( \eta_n = (\eta + \varepsilon_r + k_r - 1, l_r + k_r) \) for \( k = 1, 2, \ldots, k_r. \) If \( \eta_n = 1 \), we decompose any \( \eta_n \) in \( n \) consecutive subintervals \( \eta_n^j \) of the length \( \eta_j \), if \( \eta_n < 1 \) in \( n+1 \) subintervals, where the first \( n \) are of the length \( \eta_j \), and the \( (n+1)^{th} \) is of length \( 1 - \eta_n \). Evidently distinct subintervals \( \eta_n^j \) are disjont, we define

\[ \eta_j = \sum_{k=1}^{n} \eta_n^j \quad \text{for} \quad j = 1, 2, \ldots, n, \quad \eta = \sum_{j=1}^{n} \eta_j. \]

If \( \eta_n \) for \( r = 1, 2, \ldots, k = 1, 2, \ldots, k_r \), then

\[ \frac{1}{r-1} \int_1^r x_r(x) \, dx \leq \frac{k_r}{l_r + k_r - 1} \eta \leq \frac{k_r}{l_r + k_r - 1} \eta = \varepsilon_r \eta, \]

and so this inequality is satisfied for \( \eta \leq \varepsilon_r \). We have also for \( \tau \) within \( L_r \) and \( L_r + k \)

\[ \frac{1}{r} \int_1^r x_r(x) \, dx \leq \frac{k_r + k_1 + \ldots + k_{r-1}}{l_r} \eta < (1 - \varepsilon_r) \eta, \]

and consequently

\[ \sup_{l_r < \eta < l_{r} + k_r} \frac{1}{r} \int_1^r x_r(x) \, dx \leq (1 - \varepsilon_r) \eta + \varepsilon_r \eta \leq \eta \]

for \( r = 1, 2, \ldots \).

\[ (+) \quad \sup_{r>0} \frac{1}{r} \int_1^r x_r(x) \, dx \leq \eta. \]

On the other hand, we have

\[ (+) \quad \frac{k_r}{l_r + k_r} \eta \leq \frac{1}{r} \int_1^r x_r(x) \, dx \leq \sup_{r>0} \frac{1}{r} \int_1^r x_r(x) \, dx, \]

and, by 3), \( k_r(1 - \varepsilon_r) \eta \to \eta. \) It follows from (+), (++), that

\[ \lim_{r \to \infty} \frac{1}{r} \int_1^r x_r(x) \, dx = \sup_{r>0} \frac{1}{r} \int_1^r x_r(x) \, dx = \eta \quad \text{for} \quad j = 1, 2, \ldots, n, \]

On the same way we can check that

\[ \lim_{r \to \infty} \frac{1}{r} \int_1^r x_r(x) \, dx = \sup_{r>0} \frac{1}{r} \int_1^r x_r(x) \, dx = \eta, \]

and it follows property 7.1 (*).

Similar arguments may be applied to prove that the mean values II (1a), III (2a) satisfy also property 7.1 (*).

References

Extensions of sequentially continuous linear functionals in inductive sequences of \( \mathcal{F} \)-spaces

par

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1. Introduction \( ^{1} \). These investigations was inspired a long time ago by a problem communicated to the author by L. Ehrenpreis. The problem concerned extensibility of sequentially continuous linear functionals defined on subspaces of Schwartz's spaces \( \mathcal{S}(\Omega) \) of infinitely differentiable functions with compact carriers contained in a fixed domain \( \Omega \) (cf. \( ^{3} \)). It can be easily verified that distributions from the domain of a partial differential operator on \( \mathcal{S}(\Omega) \) can always be considered as extensions of sequentially continuous functionals defined on the range of the adjoint differential operator acting on \( \mathcal{S}(\Omega) \). Hence, it becomes apparent that a necessary and sufficient condition for existence of such extensions must be closely connected with any set of conditions that are necessary and sufficient for the operator to map onto \( \mathcal{S}(\Omega) \). For convolution operators, including as a particular case differential operators with constant coefficients, such a set of conditions was given by Hörmander in \( ^{3} \).

Going one step further in generality, call \((\mathcal{F})\)-sequence any sequence \( \mathcal{X} \) of \( \mathcal{F} \)-spaces such that every linear space from \( \mathcal{X} \) is a subspace of the subsequent linear space from the sequence and that the identical injection of every \( \mathcal{F} \)-space from \( \mathcal{X} \) into the following one is continuous (cf. \( ^{12} \)).

Situation that necessitates using such a notion arises, for instance, when we discuss factor spaces of the Schwartz's \( (\mathcal{S}, \tau_{0}) \) space. Such factor spaces need not be \((\mathcal{F})\)-spaces any more though they always naturally decompose into \((\mathcal{F})\)-sequences.

Let \( \mathcal{X} \) denote the union of linear spaces from an \((\mathcal{F})\)-sequence \( \mathcal{X} \). A linear functional defined on a linear subspace of \( \mathcal{X} \) is called sequentially continuous if it is continuous in every \( \mathcal{F} \)-space from \( \mathcal{X} \). We formulate the general problem of extension as follows.

Given an \((\mathcal{F})\)-sequence \( \mathcal{X} \) find a natural condition for a linear subspace \( \mathcal{X}_{0} \) of \( \mathcal{X} \) defined above which is necessary and sufficient

\( ^{1} \) A substantial part of the results presented here was obtained when the author was at the Institute for Advanced Study in Princeton on the NSF Grant 0-14600.