Il s'ensuit que

$$x(0) = \sum_{j\in J} \int_{-\infty}^{0} g^{(j)}(s)\, dZ^{(j)}(s).$$

L'application de l'opérateur U_t donne la représentation en question.

COROLLAIRE. *$x(t)$ étant un processus stationnaire régulier à paramètre continu et continu en moyenne quadratique, on a*

$$(**) \qquad \hat{x}(t;\tau) = \sum_{j\in J} \int_{-\infty}^{t} g^{(j)}(t+\tau-s)\, dZ^{(j)}(s)$$

avec l'erreur de prédiction

$$\sigma_\tau^2 = \|x(t+\tau)-\hat{x}(t;\tau)\|^2 = \sum_{j\in J} \int_{0}^{\tau} |g^{(j)}(s)|^2 ds.$$

On obtient la décomposition de Wold en introduisant la mesure spectrale $Z^x(a, b)$ par rapport au processus $x(t)$ lui-même. Alors $x(s) \in L(Z^x; t)$, $s \leqslant t$, et enfin [2]

$$x(t) = \int_{-\infty}^{t} g(t-s)\, dZ^x(s), \qquad g(u) \in L_2.$$

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Pointwise convergence of distribution expansions

by

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The behaviour of Fourier series of distributions with respect to global properties has been extensively studied ([6], [5], [2]). The same is true for other orthonormal systems ([3], [7]). However, the introduction of the concept of value of distribution ([4], [8], [5]) has raised the question of how the orthonormal expansion of a distribution behaves locally. To this question and related ones for other orthonormal systems we address ourselves.

1. The theorems giving local criteria for pointwise convergence of expansions of functions with respect to Fourier series are well known. Also it is well known that Fourier series of distributions converge in the sense of distributions. It is clear (from the example of the δ distribution) that a distribution can be very well behaved locally and still not have its Fourier series convergent at any point. It is also clear (from the same example) that at some points the Fourier series of some distributions are $(C, 1)$ summable. Thus the question we pose first in this section is: which ones at which points?

First we need a few remarks. Throughout we use the $K_n^k(t, x)$ to denote the (C, k) kernel with respect to the orthonormal system $\{\varphi_n\}$ on the finite interval $[a, b]$. We use the concept of value of a distribution at a point in the form characterized by [4]. A distribution is integrable over the interval $[a, b]$ if its antiderivative has values at a and b. This differs with the definition in [5], but agrees with that used in [7].

THEOREM. *Let $\{\varphi_n\}$ be a constant preserving orthonormal system on $[a, b]$ in class C^{a-1} which for some $x \in (a, b)$ is uniformly bounded and in some neighborhood of x satisfies*

$$|\varphi_n^{(a-1)}(t)| \leqslant K_x n^{a-1}$$

and whose (C, k) kernel satisfies

$$|D_t^{k-1} K_n^k(t, x)| \leqslant \frac{M}{n|t-x|^{k+1}}$$

$$(t \in (a, b),\ t \neq x;\ n = 1, 2, \ldots;\ k = 1, \ldots, a).$$

where $K(t,x)$ is a C^∞ function on the set $[a, x-\delta] \cup [x+\delta, b]$, the hypothesis about the order of the distribution could have been dropped.

The hypothesis is satisfied by the trigonometric system for any integer $\alpha > 0$ (see Zygmund II [10], p. 60). Moreover, from this we see that the inequality also holds for certain eigenfunction expansions (see Titchmarsh I [7], p. 16.)

In fact, let $[\psi_n]$ be the set of normalized eigenfunctions of a regular Sturm-Liouville problem on $[a, b]$, which has 0 as an eigenvalue. Then we get the

COROLLARY. *Let f be a distribution integrable over $[a, b]$ with a value γ at $x_0 \in (a, b)$. Then the Sturm-Liouville expansion of f is C-summable at x_0 to γ.*

The expression "C-summable" in this case means (C, α)-summable for some $\alpha > 0$.

In the case of the trigonometric system, since each periodic distribution, using a slightly difference notion of integration, can be shown to be integrable (see [8] or [5]), the statement is much simplified:

Let f be a distribution of period 2π with a value γ at x_0. Then the trigonometric Fourier series of f is C-summable at x_0 to γ.

Many of the classical results in the theory of trigonometric series may be modified slightly to give us statements about Fourier series of distributions. The extension of the statement above to one sided values does not follow, however. A simple counterexample is the δ distribution which has both left and right hand values at 0 but whose Fourier series diverges to ∞ at 0 and hence is not C-summable. However we may add another hypothesis to get:

Let a distribution f of period 2π have left and right hand values at x_0 and be bounded at x_0. Then the trigonometric Fourier series of f is C-summable at x_0, to the average of the left and right values.

The boundedness of F at x_0 refers to the statement that $f(x_0+\lambda x)$ be bounded in the sense of distribution as $\lambda \to 0$. See Zieleźny [9] for the consequences of this definition.

The converse of the above theorems doesn't hold as is shown in both cases by the example of δ' whose expansion is C-summable at each point but which does not have a value at 0. A number of partial converses are possible, however. One is:

A trigonometric series which is C-summable on a set of positive measure is the Fourier series of a periodic distribution.

This follows from the fact (Zygmund I [10], p. 316) that such series have coefficients $O(n^k)$.

Another is:

A trigonometric series which, together with its conjugate series, is C-summable at a point x_0 is the Fourier series of a distribution which has a value at x_0.

We suppose the point to be $x_0 = 0$ and suppose both $\sum A_n(x)$ and its conjugate are C-summable to 0 there. Then

$$\sum_{n=0}^{\infty} A_n(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

is such that both a_n and b_n are C-summable to 0. We may assume both are (C, k) summable, k even. Then, since $a_n = o(n^k)$ and $b_n = o(n^k)$, the function F given by

$$F(x) = \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(in)^{k+2}}$$

is a continuous function such that $D^{k+2}F = f$ the distribution of which $\sum A_n(x)$ is the Fourier series. Then summing by parts $k+1$ times we get

$$\begin{aligned} F(x) &= (ix)^{k+2} \sum_n \frac{a_n \cos nx + b_n \sin nx}{(nx)^{k+2}} \\ &= (ix)^{k+2} \Big(\sum_n s_n^k \Delta^{k+1} \frac{\cos nx}{(nx)^{k+2}} + \sum_n \tilde{s}_n^k \Delta^{k+1} \frac{\sin nx}{(nx)^{k+2}} \Big) \\ &= (ix)^{k+2} \Big(\sum_n s_n^k \Delta^{k+1} \sum_{\nu=0}^{k/2} (-1)^\nu \frac{(nx)^{2\nu-k-2}}{2\nu!} + \\ &\qquad + \sum_n \tilde{s}_n^k \Delta^{k+1} \sum_{\nu=1}^{(k+1)/2} (-1)^\nu \frac{(nx)^{2\nu-k-3}}{(2\nu-1)!} \Big) + x^{k+2} R(x) \\ &= \sum_{\nu=0}^{k+1} a_\nu x^\nu + x^{k+2} R(x). \end{aligned}$$

(Note that s_n^k and $\tilde{s}_n^k$ are $o(n^k)$.) Since it may be shown that $R(x) \to 0$ as $x \to 0$ (see Zygmund II [10], p. 67), the continuous function

$$F(x) - \sum_{\nu=0}^{k+1} a_\nu x^\nu = \tilde{F}(x)$$

satisfies $\tilde{F}(x)/x^{k+2} \to 0$ as $x \to 0$, and we see that f has a value at 0.

The converse of this converse is also false since there are functions which are continuous whose conjugate Fourier series diverge to ∞ (see Zygmund I [10], p. 253).

2. Classical pointwise convergence theorems can usually be extended to those orthonormal systems other than the trigonometric which have the sum

$$K_n(t, x) = \sum_{\nu=0}^{n} \varphi_\nu(t)\varphi_\nu(x)$$

given by an expression similar to that for the trigonometric system. For example in the case of orthogonal polynomials one has the Christoffel--Darboux formula. Alexits [1] considers orthogonal systems of polynomial type for which

$$K_n(t, x) = \sum_{k=1}^{\nu} F_k(t, x) \sum_{i,j=-p}^{p} \gamma_{i,j,k}^{(n)} \varphi_{n+i}(t)\varphi_{n+j}(x),$$

where

$$F_k(t, x) = O_x\left(\frac{1}{t-x}\right) \text{ as } t \to x \quad \text{and} \quad \gamma_{i,j,k}^{(n)} = O(1) \text{ as } n \to \infty.$$

The natural extension of such systems is to systems whose $(C, 1)$ kernels $K_n^1(t, x)$ are of the form

$$(2.1) \qquad K_n^1(t, x) = \frac{1}{n+1} \sum_{k=1}^{m} F_k(t, x) \sum_{i,j=-p}^{p} \{\gamma_{i,j,k}^{(n)} \varphi_{n+i}(t)\varphi_{n+j}(x) + \sum_{\nu=0}^{n} \beta_{i,j,k}^{(\nu)} \varphi_{\nu+i}(t)\varphi_{\nu+j}(x)\},$$

where $F_k(t, x) = O_x(1/(t-x)^2)$; $D_t F_k$ is continuous for $t \neq x$ and $D_t F_k(t, x) = O_x(1/(t-x)^3)$; $\gamma_{i,j,k}^{(n)} = O(1)$ as $n \to \infty$; and $\beta_{i,j,k}^{(\nu)} = O(\nu^{-2})$ as $\nu \to \infty$.

We shall expect the derivatives of the φ_n to behave like the derivatives of Jacobi polynomials, namely

$$(2.2) \qquad \varphi_n'(t) = \sum_{j=-q}^{q} \delta_j^{(n)} a_j(t)\varphi_{n+j}(t),$$

where $a_j(t)$ is continuous in (a, b) and $\delta_j^{(n)} = O(n)$.

We observe that (2.1) is satisfied by a number of systems. Indeed, for an arbitrary orthonormal system of polynomials $[p_n]$, for example, we have

$$K_n^1(t, x) = \frac{1}{n+1}\frac{1}{(t-x)^2}\Big\{\sum_{\nu=0}^{n} 2\Big[\Big(\frac{a_{\nu+1}}{a_{\nu+2}}\Big)^2 - \Big(\frac{a_\nu}{a_{\nu+1}}\Big)^2\Big] p_{\nu+1}(x)p_{\nu+1}(t) + \sum_{\nu=0}^{n} \frac{a_\nu}{a_{\nu+1}}(\gamma_{\nu+1}-\gamma_\nu)[p_\nu(x)p_{\nu+1}(t)+p_{\nu+1}(x)p_\nu(t)] + 2\Big(\frac{a_0}{a_1}\Big)^2 p_0(x)p_0(t) + \frac{a_n}{a_{n+2}}p_n(x)p_{n+2}(t) - 2\Big(\frac{a_{n+1}}{a_{n+2}}\Big)^2 p_{n+1}(x)p_{n+1}(t) + \frac{a_n}{a_{n+2}}p_{n+2}(x)p_n(t)\Big\},$$

where a_n is the leading coefficient of p_n and γ_n is determined by the recurrence relation (see [1], p. 25)

$$(x-\gamma_n)p_n(x) = \frac{a_n}{a_{n+1}}p_{n+1}(x) + \frac{a_{n-1}}{a_n}p_{n-1}(x).$$

If the a_n and γ_n behave as they do in the case of the Legendre polynomials, e. g., $(a_n/a_{n+1} = n/\sqrt{n-\frac{1}{2}}\sqrt{n+\frac{1}{2}}$, $\gamma_n = 0)$ K_n^{-1} is of the form (2.1).

We will call the system one of „nice polynomial type" if it satisfies (2.1) and (2.2) and in addition

$$\sum_{\nu=0}^{n} \varphi_\nu^2(x) = O(n)$$

uniformly in interior intervals of (a, b). We have the

THEOREM. *Let $\{\varphi_n\}$ be a constant-preserving orthonormal system of nice polynomial type with respect to the weight function $\varrho > 0$ in (a, b), $\varrho \in C'[a, b]$. Then the expansion of a distribution f which is the global derivative of an L^1 function F and whose support lies in (a, b) is $(C, 2)$-summable to its value at each point in (a, b) where f has a value given by the local derivative $F'(x)$.*

The proof involves integrating by parts the integral expression for the difference between $\sigma_n^2(x)$ and the value γ of f at x. Then the theory of singular integrals is used to show that the integral

$$\int_{a+\delta}^{b-\delta} \Big(\frac{F(t)}{t-x} - \gamma\Big)(t-x)D_t\big(K_n^2(t, x)\varrho(t)\big)\,dt$$

converges to 0 as $n\to\infty$. Here δ is a number such that $[a+\delta,\ b-\delta]$ contains the support of f.

The singular integral considered will be the one with kernel $(t-x)\times\times D_tK_n^2(t,x)=\Phi_n(t,x)$. We need show that

(i) $\int\limits_J \Phi_n(t,x)\varrho(t)\,dt\to 0$ as $n\to\infty$ where $J=[\alpha,\beta]-[x-\eta,x+\eta]$ where $[\alpha,\beta]$ is an arbitrary subinterval of $[a+\delta,b-\delta]=I$,

(ii) $\int\limits_{x-\eta}^{x+\eta}\Phi_n(t,x)\varrho(t)\,dt\to 1$ as $n\to\infty$,

(iii) Φ_n is uniformly bounded in $I-[x-\eta,x+\eta]$,

(iv) $\int\limits_I|\Phi_n(t,x)|\varrho(t)\,dt$ is bounded.

The form taken by $\Phi_n(t,x)$ is dictated by (2.1) and (2.2) and may be written as

$$\frac{1}{n+2}\sum_{\nu=0}^{n}\frac{(\nu+1)}{(n+1)}(t-x)D_tK_\nu^1(t,x),$$

where $D_tK_n^1(t,x)$ is given by the expression

$$\sum_{k=1}^{m}F_k(t,x)\sum_{i,j=-p}^{p}\sum_{l=-q}^{q}\frac{\delta_l^{(n+i)}}{n+1}\gamma_{i,j,k}^{(n)}a_l(t)\varphi_{n+i+l}(t)\varphi_{n+j}(x)+ + \sum_{\nu=0}^{n}\frac{\delta_l^{(\nu+i)}}{n+1}\beta_{i,j,k}^{(\nu)}a_l(t)\varphi_{\nu+i+l}(t)\varphi_{\nu+i}(x)+K_n^1(t,x)\,O_x\left(\frac{1}{t-x}\right). \tag{2.3}$$

We first observe that (iii) is satisfied by virtue of the condition

$$\sum_{\nu=0}^{n}\varphi_\nu^2(x)=O(n)$$

which implies that

$$\sum_{\nu=0}^{n}(\nu+1)D_tK_\nu^1(t,x)=O(n^2)\quad\text{for}\quad |t-x|\geqslant\eta.$$

We observe also that (ii) follows from (i), the constant preserving property of $[\varphi_n]$, the integration by parts formula and the fact that $K_n^2(t,x)\to 0$ as $n\to\infty$ for $t\neq x$.

We next show that (iv) holds. We denote by $P(t,x)$ the characteristic function of the set on which Φ_n is positive and by $N(t,x)$ that of the set on which it is not positive. We then break the integral in (iv) up into $\int\limits_{x-1/n}^{x+1/n}$ and the integral over the set $[a+\delta,x-1/n]\cup[x+1/n,b-\delta]$.

From the first integral we have

$$\left|\int\limits_{x-1/n}^{x+1/n}P(t,x)(t-x)D_tK_n^2(t,x)\varrho(t)\,dt\right|^2$$

$$=\left|\sum_{j=-q}^{q}\int\limits_{x-1/n}^{x+1/n}P(t,x)(t-x)\sum_{\nu=0}^{n}\frac{A_{n-\nu}^2}{A_n^2}\delta_j^{(\nu)}a_j(t)\varphi_{\nu+j}(t)\varphi_\nu(x)\varrho(t)\,dt\right|^2$$

$$\leqslant(2q+1)\sum_{j=-q}^{q}\int\limits_{x-1/n}^{x+1/n}|P(t,x)(t-x)a_j(t)|^2\varrho(t)\,dt\times$$

$$\times\int\limits_a^b\left|\sum_{\nu=0}^{n}\delta_j^{(\nu)}\frac{A_{n-\nu}^2}{A_n^2}\varphi_{\nu+j}(t)\varphi_\nu(x)\right|^2\varrho(t)\,dt$$

$$\leqslant(2q+1)\sum_{j=-q}^{q}O_x(n^{-3})\sum_{\nu=0}^{n}O(\nu^2)\varphi_\nu^2(x)$$

$$\leqslant(2q+1)^2O_x(n^{-3})n^2O(n)=O_x(1).$$

The same procedure holds for $N(t,x)$ and thus the first integral is bounded. To show the other integral bounded we denote by H the function

$$H(t,x)=P(t,x)(t-x)F_k(t,x)\chi_n(t)a_j(t),$$

where χ_n is the characteristic function of $I-[x-1/n,x+1/n]$. Then that part of the integral contributed by the first line of (2.3) is given by

$$\sum_{i,j,k,l}\frac{1}{n+2}\sum_{\nu=0}^{n}\frac{\nu+1}{n+1}\left|\int\limits_a^b H(t,x)\gamma_{i,j,k}^{(\nu)}\frac{\delta_l^{(\nu+i)}}{\nu+1}\varphi_{\nu+i+l}(t)\varphi_{\nu+j}(x)\varrho(t)\,dt\right| \tag{2.4}$$

$$\leqslant\sum_{i,j,k,l}\frac{1}{n+2}\left\{\sum_{\nu=0}^{n}\left[\frac{\delta_l^{(\nu+i)}}{n+1}\gamma_{i,j,k}^{(\nu)}\varphi_{\nu+j}(x)\right]^2\right\}^{1/2}\times$$

$$\times\left\{\sum_{\nu=0}^{n}\left[\int\limits_a^b H(t,x)\varphi_{\nu+i+l}(t)\varrho(t)\,dt\right]^2\right\}^{1/2}$$

$$=\sum_{i,j,k,l}\frac{1}{n+2}\{O(n)\}^{1/2}\left\{\sum_{\nu=0}^{n}c_{\nu+i+l}^2(x)\right\}^{1/2},$$

where the $c_\nu(x)$ are the expansion coefficients of H. By Bessel's inequality we have

$$\sum_{\nu=0}^{n}c_\nu^2(x)\leqslant\int\limits_a^b H^2(t,x)\varrho(t)\,dt=O_x(n)$$

whence if follows that that part of the integral is bounded. The boundedness of that part given by the other lines follows similarly. Hence we get (iv).

We can get (i) by the simple expedient of taking χ to be the characteristic function of $I-[x-\eta, x+\eta]$ and repeating the last argument for the function

$$H(t, x) = (t-x)F_k(t, x)\chi(t)\alpha_j(t).$$

We then use the fact that $c_\nu(x) \to 0$ as $\nu \to \infty$ to deduce that

$$\sum_{\nu=0}^{n} c^2_{\nu+i+l}(x) = o_x(n).$$

We now may invoke a singular integral theorem (see [1], p. 257) to deduce that

$$\int_{a+\delta}^{b-\delta} \left(\frac{F(t)}{(t-x)} - \gamma\right)(t-x)D_t K^2_n(t, x)\varrho(t)\,dt$$

converges to 0 as $n \to \infty$ and, by a repetition of the argument, to show the same holds for the integral with ϱ replaced by ϱ'. Then another integration by parts leads us to the conclusion of the theorem.

The extension to values of higher order simply involves an extension to higher orders of complexity.

3. Classical convergence problems may often be phrased in terms of δ sequences, i. e. sequences of functions converging to the δ distribution. It is well known that the Dirichlet kernel and the Fejer kernel used in trigonometric Fourier series both form δ-sequences. These kernels are respectively the partial sums of the expansion of $\delta(x-t)$ and the $(C, 1)$ means of these partial sums. Therefore it might be expected that some statement about pointwise convergence of distribution expansions could be phrased in terms of the behaviour of the partial sums of the expansion of δ and its derivatives. The following theorem is one such statement:

THEOREM. *Let $\{\varphi_n\}$ be an orthonormal system on $[a, b]$, in class C^k such that for some $x \in (a, b)$, there is an α such that*

(i) $\delta_x, \delta'_x, \ldots, \delta^{(k)}_x$ has an expansion with respect to $\{\varphi_n\}$ dominately (C, α) summable to 0 on intervals $[A, B]$ not containing x;

(ii) $\{1\}$ has an expansion (C, α) summable at x to x.

(iii) $|(x-t)^{k+j}D^k_t K^\alpha_n(t, x)| \leqslant M$ on subintervals $J \subset [a, b]$ for some $j = -k, \ldots, 0, 1, \ldots$

Then the expansion of a distribution f of order $k-1$ integrable from a to b whose jth derivative or anti-derivative has a value γ at x of order $k-3+j$ is (C, α)-summable to γ.

The (C, α) means of the expansion of f are given by

$$\sigma^\alpha_n(x) = \int_a^b f(t)K^\alpha_n(t, x)\,dt.$$

We may integrate this expression by parts k times to get

$$(-1)^k \int_a^b f^{(-k)}(t)D^k_t K^\alpha_n(t, x)\,dt +$$

$$+[(-1)^{k-1}f^{(-k)}(t)D^{k-1}_t K^\alpha_n(t, x) + \ldots + f^{(-1)}(t)K^\alpha_n(t, x)]^b_a.$$

The integrated terms converge to 0 as $n \to \infty$ since they constitute linear combinations of the (C, α) means of the expansion of δ and its first $k-1$ derivatives evaluated at a and b. By the hypothesis, there is a continuous function F such that $D^kF = f$ and, by changing F by adding a polynomial of degree $k-1$ if necessary, we can suppose that

$$\frac{F(t)}{(t-x)^{k+j}} \to \frac{\gamma}{(k+j)!} \quad \text{as} \quad t \to x.$$

Hence we may write the integral as

$$\int_a^b F(t)D^k_t K^\alpha_n(t, x)\,dt = \int_a^{x-\varepsilon_1} + \int_{x-\varepsilon_1}^{x+\varepsilon_2} + \int_{x+\varepsilon_2}^{b}$$

where ε_1 and ε_2 are to be determined. It is clear that the first and last integral converge to 0 as $n \to \infty$. We may write the middle integral as

$$\int_{x-\varepsilon_1}^{x+\varepsilon_2} \frac{F(t)}{(t-x)^{k+j}}(t-x)^{k+j}D^k_t K^\alpha_n(t, x)\,dt$$

$$= \int_{x-\varepsilon_1}^{x+\varepsilon_2} \left(\frac{F(t)}{(t-x)^{k+j}}\right) - \frac{\gamma}{(k+j)!}(t-x)^{k+j}D^k_t K^\alpha_n(t, x)\,dt +$$

$$+ \frac{\gamma}{(k+j)!}\int_{x-\varepsilon_1}^{x+\varepsilon_2}(t-x)^{k+j}D^k_t K^\alpha_n(t, x)\,dt.$$

Using (iii) of the hypothesis, it is clear that the first integral can be made small by taking ε_1 and ε_2 small enough. The theorem will be proved if

$$\int_{x-\varepsilon_1}^{x+\varepsilon_2} \frac{(t-x)^{k+j}}{(k+j)!} D_t^k K_n^a(t, x)\, dt \to 1 \quad \text{as} \quad k \to \infty.$$

Again integrating by parts k times we find that

$$\int_{x-\varepsilon_1}^{x+\varepsilon_2} \frac{(t-x)^{j}}{j!} K_n^a(t, x)\, dt \to 1 \quad \text{as} \quad n \to \infty,$$

which it does by hypothesis (iii).

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Nicht verbesserbare Strukturbedingungen

von

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Einleitung. Das Haarsche System ist im Intervall $[0,1]$ folgenderweise definiert: $\chi_0^{(0)}(x) = 1$ und für $n = 0, 1, \ldots$ und $k = 1, 2, \ldots, 2^n$

$$\chi_n^{(k)}(x) = \begin{cases} \sqrt{2^n} & \text{für} \quad x \in \left(\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{für} \quad x \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right), \\ 0 & \text{sonst.} \end{cases}$$

Wir setzen

$$\chi_0^{(0)}(x) = \chi_1(x) \quad \text{und} \quad \chi_n^{(k)}(x) = \chi_m(x),$$

wobei $m = 2^n + k$ $(n = 0, 1, \ldots;\ k = 1, 2, \ldots, 2^n)$ ist.

Sei $f(x)$ eine $L[0,1]$-integrierbare Funktion mit der folgenden Entwicklung

$$f(x) \sim \sum_{m=1}^{\infty} c_m \chi_m(x).$$

Kürzlich haben Ciesielski und Musielak [1], Uljanov [4] und Golubov [2] u. a. für die Konvergenz der Reihen von der Form

$$\sum_{m=1}^{\infty} |c_m|^\beta m^\delta \quad (\beta > 0) \tag{1}$$

hinreichende Strukturbedingungen und für $\sum_{m=k}^{l} |c_m|^\beta m^\delta$ in verschiedenen Spezialfällen Größenordnungen gegeben.

In dieser Arbeit geben wir zuerst für die Konvergenz der Reihe $\sum |c_m|^\beta \lambda(m)$ eine hinreichende Strukturbedingung, woraus fast alle bekannten Ergebnisse, die sich auf die Behauptungen bezüglich der Reihen von der Form (1) beziehen, als Korollare folgen. Unsere Behauptungen leiten wir aus dem allgemeinen Hilfssatz II ab, den wir durch Anwendung der Ergebnisse von Golubov [2] beweisen.