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Continuous mappings induced by isometries of spaces of continuous function

by

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Suppose that X and Y are two compact Hausdorff spaces and $T\colon C(X)\to C(Y)$ is a linear isometry. $C(X)$ and $C(Y)$ denote the spaces of all continuous complex-valued functions on X and Y, respectively (1). If T is onto, then it induces a homeomorphism from Y onto X, by the well-known Banach-Stone theorem ([4], p. 442). There are two typical cases when such isometries are not onto, namely isometries onto subrings induced by continuous mappings from Y onto X and those induced by Borsuk's simultaneous extensions (cf. [2]).

Gęba and Semadeni [5] have proved that if T is an isotonic linear isometry (T is *isotonic* if and only if $f\geqslant g$ is equivalent to $Tf\geqslant Tg$, $f,g\in C(X)$), then there exists a closed subset Y_0 of Y such that $Tf(y) = f(\varphi(y))$ for any $f\in C(X)$ and $y\in Y_0$. In other words, any such isotonic linear isometrical embedding is obtained by superposition of a simultaneous extension and an embedding induced by a continuous mapping.

A very close theorem has independently been established by Bauer [1] in a quite different form, as a criterion of solvability of an abstract Dirichlet problem and this, in turn, can be expressed as a characterization of infinite dimensional simplices with closed boundaries (see Choquet and Meyer [3]). If the image of T, i. e., the set $E=\{Tf\colon f\in C(X)\}$, separates Y and contains the unit of $C(Y)$, then Y_0 is the Šilov boundary of E.

The purpose of this paper is to prove an analogous theorem about any isometry (without assumption that it is isotonic).

First we shall introduce some notions. Let

$$S_x = \{f\in C(X)\colon \|f\|=1 \text{ and } |f(x)|=1\},\qquad x\in X;$$
$$R_y = \{g\in C(Y)\colon \|g\|=1 \text{ and } |g(y)|=1\},\qquad y\in Y;$$
$$Q_x = \{y\in Y\colon T(S_x)\subset R_y\},\qquad x\in X;$$
$$Y_0 = \bigcup_{x\in X} Q_x;$$
$$\Gamma = \{z\in\mathscr{C}\colon |z|=1\},\ \text{where } \mathscr{C} \text{ is the complex plane.}$$

(1) The proof of the theorem below is also valid if $C(X)$ and $C(Y)$ are the spaces of real-valued continuous functions.

It is obvious that

$$Q_x = \{y\in Y\colon \bigwedge_{f\in C(X)} [(\|f\|=1)\wedge(|f(x)|=1)\Rightarrow |Tf(y)|=1]\} = \bigcap_{f\in S_x} (Tf)^{-1}(\Gamma).$$

THEOREM. *Let X and Y be two compact Hausdorff spaces. Let $T\colon C(X)\to C(Y)$ be a linear isometry. Then the set Y_0 is closed and the relation $\varphi\subset Y\times X$ defined by*

$$(y,x)\in\varphi \text{ if and only if } y\in Q_x$$

is a continuous function $\varphi\colon Y_0\to X$ from Y_0 onto X, and there exists a function $\alpha\in C(Y)$ such that

$$\|\alpha\|=1,\quad |\alpha(y)|=1 \quad \text{for} \quad y\in Y_0$$

and

$$Tf(y) = f(\varphi(y))\alpha(y) \quad \text{for all } y\in Y_0 \text{ and } f\in C(X).$$

Proof. We shall prove the theorem in a number of steps.

(i) If $f(x)=0$, then $Tf(y)=0$ for y in Q_x and f in $C(X)$.

Indeed, we may assume that $\|f\|=1$. Suppose that $f(x)=0$ and $Tf(y)\neq 0$. Let

$$g_z = z\Big[\min\Big(1+\mathrm{Re}(\bar z f),\ \sqrt{1-(\mathrm{Im}(\bar z f))^2}\Big)+i\cdot\mathrm{Im}(\bar z f)\Big]$$

where $z\in\Gamma$. Then $g_z(x)=z$ and $|g_z(x)|=\|g_z\|=1$, $|g_z(x)-f(x)|=1=\|g_z-f\|$ (as $\|g_z-f\|=\|\bar z g_z-\bar z f\|$ and if we put $g'=\bar z g_z$, $f'=\bar z f$, then it is obvious that $\|g'-f'\|\leqslant 1$). Hence $g_z\in S_x$ and $g_z-f\in S_x$, and

$$|Tg_z(y)| = |Tg_z(y)-Tf(y)| = 1 \quad \text{for} \quad z\in\Gamma.$$

Furthermore

$$2 = |g_z(x)-g_{-z}(x)| \leqslant \|g_z-g_{-z}\| \leqslant \|g_z\|+\|g_{-z}\| = 2,$$

and hence

$$\tfrac{1}{2}(g_z-g_{-z})\in S_x \quad \text{and} \quad \|Tg_z(y)-Tg_{-z}(y)\| = 2.$$

This means that $k\colon\Gamma\to\Gamma$ defined by $k(z)=Tg_z(y)$ is a continuous function such that $k(-z)=-k(z)$. Since the set $k(\Gamma)$ is self-antipodal and connected, it must coincide with Γ, and hence k maps Γ onto Γ.

Thus $Tg_z(y)=Tf(y)/|Tf(y)|$ for some $z\in\Gamma$ and this contradicts the equation

$$|Tg_z(y)| = |Tg_z(y)-Tf(y)| = 1.$$

(ii) $Q_x\cap Q_{x'}=\emptyset$ for $x\neq x'$.

Indeed, there exists $f\in C(X)$ such that $0\leqslant f\leqslant 1$, $f(x)=1$, $f(x')=0$. If $y\in Q_x$, then $|Tf(y)|=1$ and if $y\in Q_{x'}$, then, from (i), $|Tf(y)|=0$. Thus there exists no point y in $Q_x\cap Q_{x'}$.

(iii) $Q_x\neq\emptyset$ for $x\in X$.

Let x be fixed ($x\in X$). Consider any finite set of functions $f_1,f_2,\ldots,f_n\in S_x$ and define

$$h = \sum_{i=1}^{n}\overline{f_i(x)}\cdot f_i.$$

Then $h\in C(X)$ and $|h(x)|=n$ whence $\|h\|=\|Th\|=n$ and there exists $y\in Y$ such that

$$|Th(y)| = \Big|\sum_{i=1}^{n}\overline{f_i(x)}\cdot Tf_i(y)\Big| = n$$

whence $|Tf_i(y)|=1$ for $i=1,2,\ldots,n$ and $y\in(Tf_i)^{-1}(\Gamma)$, $i=1,2,\ldots,n$. Thus, the family of all sets $(Tf)^{-1}(\Gamma)$, $f\in S_x$, has the finite intersection property and Q_x is non-empty by compactness of Y.

Thus, (ii) and (iii) imply that φ is a function from Y_0 onto X and $Q_x=\varphi^{-1}(x)$.

(iv) If $f\in C(X)$, $x\in X$ and $x=\varphi(y)$, then $|Tf(y)|\geqslant|f(x)|$.

Indeed, we may assume that $\|f\|=1$ and $f(x)\neq 0$. If $|f(x)|=1$, then $|Tf(y)|=1$ (because $y\in Q_x$), so let us consider the remaining case $0<|f(x)|<1$. We put

$$g(x') = \frac{f(x')}{\max(|f(x)|,|f(x')|)} \quad \text{for} \quad x'\in X.$$

Then $g\in C(X)$, $|g(x)|=1$ and $\|g\|=1$, whence $g\in S_x$, $|Tg(y)|=1$, and $|g(x)-f(x)|=\|g-f\|=1-|f(x)|$. Hence, since

$$\frac{g-f}{\|g-f\|}\in S_x,$$

we obtain $|Tg(y)-Tf(y)|=\|Tg-Tf\|=1-|f(x)|$. Consequently,

$$|Tf(y)| \geqslant |Tg(y)|-|Tg(y)-Tf(y)| = |f(x)|.$$

(v) $\varphi\colon Y_0\to X$ is a continuous mapping and Y_0 is closed in Y.

We shall show that if $A=\bar A\subset X$, then the set

$$\varphi^{-1}(A) = \bigcup_{x\in A} Q_x$$

is closed in Y. Indeed, if $y_0\in Y\setminus\varphi^{-1}(A)$, then for any point $x\in A$ there exists a function $f_x\in S_x$ such that $|Tf_x(y_0)|<1-\varepsilon_x$ for a certain $\varepsilon_x>0$. We put $U_x=\{x'\in X\colon |f_x(x')|>1-\varepsilon_x\}$,

$$V_x = \{y\in Y\colon |Tf_x(y)|>1-\varepsilon_x\},\qquad G_x = \{y\in Y\colon |Tf_x(y)|<1-\varepsilon_x\}.$$

All sets defined above are open and $\bigcup_{x\in A} U_x \supset A = \bar{A}$. Hence there exists a finite cover $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ of the set A, where $x_1, x_2, \ldots, x_n \in A$. If $y \in \varphi^{-1}(U^x)$, then for $x' = \varphi(y) \in U_x$ we have $|f_x(x')| > 1-\varepsilon_x$ and, from (iv), $Tf_x(y) > 1-\varepsilon_{x'}$, hence $y \in V_x$. Thus $V_x \supset \varphi^{-1}(U_x)$ for $x \in A$, whence

$$\bigcup_{i=1}^{n} V_{x_i} \supset \bigcup_{i=1}^{n} \varphi^{-1}(U_{x_i}) = \varphi^{-1}\Big(\bigcup_{i=1}^{n} U_{x_i}\Big) \supset \varphi^{-1}(A).$$

Now, $\bigcap_{i=1}^{n} G_{x_i} \cap \bigcup_{i=1}^{n} V_{x_i} = \emptyset$ (because $G_x \cap V_x = \emptyset$) and

$$\bigcap_{i=1}^{n} G_{x_i} \cap \varphi^{-1}(A) = \emptyset.$$

Since $\bigcap_{i=1}^{n} G_{x_i}$ is a neighbourhood of the point y_0, $\varphi^{-1}(A)$ is a closed set in Y. In particular $Y_0 = \varphi^{-1}(X)$ is closed in Y.

(vi) $f'(x)\cdot Tf(y) = f(x)\cdot Tf'(y)$ for $f, f' \in C(X)$, $x \in X$, $y \in Q_x$.

Indeed, we consider the function $g = f'(x)\cdot f - f(x)\cdot f'$ (x is fixed). Then $g(x) = 0$ and $Tg(y) = 0$ (by (i)) and this yields (vi).

Now, let $\alpha(y) = T1(y)$, where $y \in Y$ and the function $1 \in C(X)$ is defined by $1(x') = 1$ for $x' \in X$.

Then, since $1 \in S_x$, $|\alpha(y)| = 1$ for $y \in Q_x$, $x \in X$, and from (vi), $Tf(y) = f(\varphi(y))\cdot\alpha(y)$ for $y \in Y_0$, q. e. d.

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Sur une représentation de la prédiction d'un processus stationnaire régulier

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En partant de la définition de la prédiction dans [4] on en donne dans ce travail les représentations (*) et (**), analogues à la représentation de la prédiction linéaire basée sur la décomposition de Wold. On profite, dans le cas discret, des propriétés de l'espace de Hilbert exposées dans [1], et dans le cas continu la construction est semblable à celle contenue dans [2].

Soit $x(t)$, $-\infty < t < \infty$, un processus strictement stationnaire sur l'espace $(\Omega = \{\omega\}, F, P)$ et $F_{-\infty}^{t}$ un corps borelien engendré par les événements $\{\omega\colon x(t_1) \in K_1, \ldots, x(t_k) \in K_k\}$ pour tout $t_1 \leqslant t, \ldots, t_k \leqslant t$ et $K_1, \ldots, K_k$ (éléments du corps borelien du plan complexe). Soit ensuite H un espace de Hilbert dont les éléments sont les variables aléatoires y sur (Ω, F, P) à dispersion finie: $\int_\Omega |y(\omega)|^2 dP < \infty$; produit scalaire: $(x, y) = E\{x\bar{y}\} = \int_\Omega x(\omega)\overline{y(\omega)}\,dP$. Nous supposerons dorénavant que le processus $x(t)$ est à dispersion finie $E\{|x(t)|^2\} = E\{|x(0)|^2\} < \infty$. Désignons par $H_{-\infty}^{t}$ le sous-espace H, dont les éléments sont les variables aléatoires à dispersion finie et mesurables sur $F_{-\infty}^{t}$. L'opérateur $U_\tau x(t) = x(t+\tau)$ peut être isométriquement élargi sur tout l'espace H, de façon que U_t, $-\infty < t < \infty$, soit le groupe des opérateurs unitaires. L'espace H sera toujours séparable dans le cas d'un processus $x(t)$ à paramètre discret. Dans le cas d'un paramètre continu, pour que l'espace H soit séparable, il suffit que le processus $x(t)$ soit continu en moyenne quadratique [5].

On définit la prédiction $\hat{x}(t, \tau)$ du processus $x(t)$ dans l'instant $t+\tau$, relativement au passé jusqu'à l'instant t, comme l'espérance mathématique conditionnelle: $\hat{x}(t; \tau) = E\{x(t+\tau) | F_{-\infty}^{t}\}$ [4]. Considérant d'après [4] le processus $x(t)$ comme une courbe dans l'espace H on réduit la prédiction à la projection $\hat{x}(t, \tau) = P_{H_{-\infty}^{t}}[x(t+\tau)]$.

Le processus $x(t)$ sera dit *régulier* si le corps borelien $\bigcap_t F_{-\infty}^{t}$ est