et cette série converge uniformément dans tout intervalle fermé $[0, x_0]$. La série précédente converge uniformément dans tout intervalle $[0, x_0/\omega]$. Mais elle converge aussi pour tout $|\lambda| < \omega|\lambda_0|$.

Travaux cités

[1] J. Mikusiński, *Operational calculus*, New York — London — Paris 1959.

[2] C. Ryll-Nardzewski, *Sur la convergence des séries de puissances de l'opérateur différentiel*, Studia Math. 13 (1953), p. 37-40.

Reçu par la Rédaction le 3. 10. 1964

STUDIA MATHEMATICA, T. XXVI. (1966)

On integrals of functions with values in a complete linear metric space

by

D. PRZEWORSKA-ROLEWICZ and S. ROLEWICZ (Warszawa)

In paper [5] S. Mazur and W. Orlicz gave a definition of a Riemann integral of a function $x(t)$ determined on a set in a Euclidean space with values in a complete linear metric space X (F-space in Banach's terminology [3]). In that paper it was shown that the space X is locally convex if and only if each continuous function $x(t)$ is integrable.

In this note some conditions of the existence of a Riemann integral will be given. This will permit us to extend the integral method of Banach algebras [4], [5] to the theory of locally bounded algebras [11], [12]. As a particular case, the existence of analytic functions of many elements will be proved.

At the end of the paper we will show what difficulties arise concerning the definition of the Lebesgue-Bochner integral in the case where X is not a locally convex space.

1. Let L be a set in the n-dimensional Euclidean space E^n. And let $|\cdot|$ be some measure determined on L. We will assume that the measure of the whole L is finite.

By a *partition* Δ we shall mean a decomposition of the set L into a union of closed sets $L = \bigcup_{i=1}^{n} L_i$ such that $|L_i \cap L_j| = 0$ for $i \neq j$. We will write $\Delta = (L_1, \ldots, L_n)$. We say that the sequence of partitions $\Delta^j = (L_1^j, \ldots, L_n^j)$ is *normal* if the largest diameter of L_i^j tends to 0:

$$\lim_{j\to\infty} \sup_{1\leqslant i\leqslant n_j} \sup_{t,t_1\in L_i^j} |t-t_1| = 0.$$

Let X be a complete linear metric space. Let $x(t)$ be a function determined on L with values in X. Let $\Delta^j = (L_1^j, \ldots, L_{n_j}^j)$ be a normal sequence of partitions. Let $t_i^j \in L_i^j$. We write

$$S(x, \Delta^j, t_i^j) = \sum_{i=1}^{n_j} x(t_i^j)|L_i^j|.$$

If for each normal sequence of partitions Δ^j and each choice of points t_i^j there is a limit of $S(x, \Delta^j, t_i^j)$, then this limit is called a *Riemann integral of the function $x(t)$ on L.* We will denote it by

$$\int_L x(t)\,dt.$$

In the same way as in the classical considerations we can prove that the integral is independent of the choice of a normal sequence of partitions and of the choice of points t_i^j.

A function $x(t)$ is called *integrable* if the integral $\int_L x(t)\,dt$ exists.

A function $x(t)$ which is not integrable will be called *non-integrable.*

The Riemann integral possesses the same arithmetic properties as the Riemann integral of real-valued functions. In particular case, if $x(t)$ is a simple function, i. e.

$$x(t) = \sum_{i=1}^{n} x_i \chi_{L_i},$$

where $x_i \in X$, χ_{L_i} are characteristic functions of sets L_i whose boundaries are of measure 0, then $x(t)$ is integrable.

THEOREM 1. *Let X be a linear metric space with norm $\|\;\|$* (1). *Let $x(t)$ be a function determined on L with values in X. If for each $\varepsilon > 0$ there is an integrable function $x_\varepsilon(t)$ such that*

$$\|S(x, \Delta, t_n) - S(x_\varepsilon, \Delta, t_n)\| = \|S(x - x_\varepsilon, \Delta, t_n)\| < \varepsilon$$

for an arbitrary partition Δ and for an arbitrary choice of t_i, then $x(t)$ is integrable.

Proof. Let $\Delta^i = (L_1^i, \ldots, L_{n_i}^i)$ be a normal sequence of partitions. Let $t_n^i \in L_n^i$. The function $x_\varepsilon(t)$ is integrable, whence there is an i_0 such that for $i, j > i_0$

$$\|S(x_\varepsilon, \Delta^i, t_n^i) - S(x_\varepsilon, \Delta^j, t_n^j)\| < \varepsilon.$$

Therefore

$$\|S(x, \Delta^i, t_n^i) - S(x, \Delta^j, t_n^j)\| \leqslant \|S(x, \Delta^i, t_n^i) - S(x_\varepsilon, \Delta^i, t_n^i)\| +$$
$$+ \|S(x_\varepsilon, \Delta^i, t_n^i) - S(x_\varepsilon, \Delta^j, t_n^j)\| + \|S(x_\varepsilon, \Delta^j, t_n^j) - S(x, \Delta^j, t_n^j)\| < 3\varepsilon.$$

Hence, the fact that ε is arbitrary implies that $x(t)$ is integrable.

(1) By the *norm* we mean a non-negative function $\|\;\|$ satisfying the following conditions: 1) $\|x\| = 0$ if $x = 0$; 2) $\|x+y\| < \|x\| + \|y\|$ (see [3], ch. III).

COROLLARY. *Let X be a complete linear metric space with monotonic norm, i. e. such that the function $\|tx\|$ is not decreasing for $t > 0$. Let*

$$x(t) = \sum_{i=1}^{\infty} f_i(t) x_i,$$

where $f_i(t)$ are scalar functions uniformly bounded, $|f_i(t)| < M$, and x_i are elements of X. If

$$\sum_{i=1}^{\infty} \|x_i\| < +\infty,$$

then the function $x(t)$ is integrable.

Proof. Without loss of generality we can assume that M is an integer. Let ε be an arbitrary positive number. There is an n_ε such that

$$\sum_{i=n_\varepsilon+1}^{\infty} \|x_i\| < \frac{\varepsilon}{M}.$$

Let

$$x_\varepsilon(t) = \sum_{i=1}^{n_\varepsilon} f_i(t) x_i.$$

Obviously $x(t)$ as a finite sum of integrable functions is integrable. On the other hand, for an arbitrary partition $\Delta = (L_1, \ldots, L_n)$ and for a system of t_n, $t_i \in L_i$

$$\|S(x - x_\varepsilon, \Delta, t_n)\| = \Big\| S\Big(\sum_{i=n_\varepsilon+1}^{\infty} f_i(t) x_i, \Delta, t_n\Big)\Big\|$$
$$\leqslant \sum_{i=n_\varepsilon+1}^{\infty} \|S(f_i x_i, \Delta, t_n)\| \leqslant \sum_{i=n_\varepsilon+1}^{\infty} \|M x_i\| \leqslant M \sum_{i=n_\varepsilon+1}^{\infty} \|x_i\| < \varepsilon,$$

q. e. d.

2. Let X be a locally bounded space with a p-homogenous norm, $0 < p \leqslant 1$, i. e. such a norm $\|\;\|$ that $\|tx\| = |t|^p \|x\|$ (see [1], [8]). We say that a function $x(t)$ is *analytic in a domain Ω* contained in the n-dimensional real (or complex) Euclidean space if for each point $t_0 \in \Omega$ there is a neighbourhood $U \subset \Omega$ such that $x(t)$ possesses an expansion in the series

$$x(t) = \sum_{i_1, \ldots, i_n = 0}^{\infty} (t_1 - t_1^0)^{i_1} \ldots (t_n - t_n^0)^{i_n} x_{i_1, \ldots, i_n}$$

convergent in U, $x_{i_1, \ldots, i_n}$ are elements of X, $t = (t_1, \ldots, t_n)$, $t_0 = (t_1^0, \ldots, t_n^0)$. Obviously an analytic function of n complex variables can be considered as an analytic function of $2n$ real variables.

PROPOSITION 1. *Let X be a locally bounded space with p-homogenous norm $\|\ \|$. Let Ω be a domain of the n-dimensional real (or complex) Euclidean space. Let $x(t)$ be an analytic function in Ω. Let L be defined as in section 1 and compact, $L \subset \Omega$. Then there exists an integral $\int_L x(t)\,dt$.*

Proof. Let the series

$$x(t) = \sum_{i_1,\dots,i_n=0}^{\infty} (t_1-t_1^0)^{i_1}\dots(t_n-t_n^0)^{i_n} x_{i_1,\dots,i_n}$$

converge at the point $t' = (t_1', \dots, t_n')$. Then

$$\sum_{i_1,\dots,i_n=0}^{\infty} \|y_{i_1,\dots,i_n}\| < +\infty,$$

where

$$y_{i_1,\dots,i_n} = (t_1'-t_1^0)^{i_1}\dots(t_n'-t_n^0)^{i_n}\frac{1}{2^{i_1+\dots+i_n}} x_{i_1,\dots,i_n}.$$

Moreover, if $|t_k-t_k^0| < \frac{1}{2}|t_k'-t_k^0|$, then

$$x(t) = \sum_{i_1,\dots,i_n=0}^{\infty} f_{i_1,\dots,i_n}(t)\,y_{i_1,\dots,i_n}, \tag{1}$$

where

$$f_{i_1,\dots,i_n}(t) = \prod_{k=1}^{n} 2^{i_k}\frac{(t_k-t_k^0)^{i_k}}{(t_k'-t_k^0)^{i_k}} \quad \text{and} \quad |f_{i_1,\dots,i_n}| < 1.$$

Hence for each point t_0 there is a neighbourhood U in which $x(t)$ can be represented in the form given in Corollary 1 and

$$|f_{i_1,\dots,i_n}(t)| < 1, \qquad \sum_{i_1,\dots,i_n=0}^{\infty} \|y_{i_1,\dots,i_n}\| < +\infty.$$

Corollary 1 implies that there exists an integral $\int_{U\cap L} x(t)\,dt$. But L is compact and we can cover L by a finite system of neighbourhoods U_i $(i = 1, 2, \dots, l)$ satisfying the property described above. Then the integral

$$\int_L x(t)\,dt = \sum_{i=1}^{l} \int_{L\cap(U_i\setminus U_{i-1})} x(t)\,dt$$

exists, q. e. d.

Let X be a locally bounded commutative algebra over the field of complex numbers. We will denote the unit of this algebra by e. A linear

functional f is called *multiplicative-linear* if $f(xy) = f(x)f(y)$. The set of all multiplicative-linear non-zero functionals is denoted by $\mathfrak{M}$.

The following set of complex numbers is called the *spectrum* of an element x:

$$\sigma(x) = \{f(x): f\in\mathfrak{M}\}.$$

If $\lambda\notin\sigma(x)$, then there exists an $(x-\lambda e)^{-1}$. A *radical* of the algebra X (rad X) is the set

$$\operatorname{rad} X = \{x\in X:\ \sigma(x) = \{0\}\} \qquad \text{(see [10], [11]).}$$

The radical is a closed ideal as the intersection of maximal ideals. The set

$$\sigma(x_1, \dots, x_n) = \{(f(x_1), \dots, f(x_n)): f\in\mathfrak{M}\}$$

contained in the n-dimensional Euclidean space is called the *joint spectrum* of elements $x_1, \dots, x_n$.

The main method of investigations of Banach algebras are Riemann integrals of some analytic functions. So far the locally bounded algebras have been investigated by other methods [10], [11], [12], [13]. Proposition 1 permits us to return to the integral method. As a particular case, we will show the existence of analytic functions of many elements. It has not been known hitherto (see [14]).

Let $\Phi(z_1, \dots, z_n)$ be an analytic function of n-variables determined on some domain $\Omega \supset \sigma(x_1, \dots, x_n)$. If there is such an element y that

$$f(y) = \Phi(f(x_1), \dots, f(y_n))$$

for each $f\in\mathfrak{M}$, then we say that y is a *value* of the analytic function Φ on the points $x_1, \dots, x_n$ and we write

$$y = \Phi(x_1, \dots, x_n).$$

The element y is uniquely determined only with respect to an element of the radical. Indeed, if $r\in\operatorname{rad}X$, then $f(y+r) = f(y)$ for $f\in\mathfrak{M}$ and, conversely, if $f(y) = f(y_0)$ for all $f\in\mathfrak{M}$, then $f(y-y_0) = 0$ and $r = y-y_0\in\operatorname{rad}X$.

THEOREM 2. *Let $x_1, \dots, x_n$ be elements of a locally bounded algebra X. Let $\Phi(z_1, \dots, z_n)$ be an analytic function determined on domain U containing the spectrum $\sigma(x_1, \dots, x_n)$. Then there exists a $\Phi(x_1, \dots, x_n)$.*

Proof. The proof is the same as for Banach algebras (Arens, Calderon [2], Walbroeck [9]). At the beginning we consider the case where $x_1, \dots, x_n$ are generators of the algebra X. We show that there is a domain of Weil W, i. e. a domain

$$W=\{\zeta=(\zeta_1,\ldots,\zeta_n):|P_i(\zeta_1,\ldots,\zeta_n)|<1;\ i=1,2,\ldots,N\},$$

where $P_i(\zeta_1,\ldots,\zeta_n)$ are polynomials, such that $\sigma(x_1,\ldots,x_n)\subset W\subset\Omega$.

By the Weil formula each analytic function $f(\zeta_1,\ldots,\zeta_n)$ determined in W can be represented on $\sigma(x_1,\ldots,x_n)$ by the formula

$$f(\tau_1,\ldots,\tau_n)=\frac{1}{(2\pi i)^n}\sum_{(i_1,\ldots,i_n)}\int\limits_{S_{i_1,\ldots,i_n}}\frac{D_{i_1,\ldots,i_n}f(\zeta_1,\ldots,\zeta_n)\,d\zeta_1\ldots d\zeta_n}{\prod\limits_{\nu=1}^{n}[P_{i_\nu}(\zeta_1,\ldots,\zeta_n)-P_{i_\nu}(\tau_1,\ldots,\tau_n)]},$$

where the summation is extended on all systems of $1=i_1<\ldots<i_n=N$

$$S_{i_1,\ldots,i_n}=\bigcap_{k=1}^{n}\{\zeta\colon |P_{i_k}(\zeta_1,\ldots,\zeta_n)|=1\}$$

with some determined orientation, $D_{i_1,\ldots,i_n}$ are some polynomials of $\zeta_1,\ldots,\zeta_n$ with coefficients which are polynomials of $\tau_1,\ldots,\tau_n$.

Let

(2) $\Phi(x_1,\ldots,x_n)$

$$=\frac{1}{(2\pi i)^n}\sum_{(i_1,\ldots,i_n)}\int\limits_{S_{i_1,\ldots,i_n}}\frac{D_{i_1,\ldots,i_n}\Phi(\zeta_1,\ldots,\zeta_n)\,d\zeta_1\ldots d\zeta_n}{\prod\limits_{\nu=1}^{n}[P_{i_\nu}(\zeta_1,\ldots,\zeta_n)e-P_{i_\nu}(x_1,\ldots,x_n)]}$$

We will prove that the integrals presented on the right side exist. We fix a choice $i_1,\ldots,i_n$. We consider the integral

$$\int\limits_{S_{i_1,\ldots,i_n}}(\ldots)\,d\zeta_1\ldots d\zeta_n.$$

For each i_ν the spectrum of the element $P_{i_\nu}(x_1,\ldots,x_n)$ is a unit disc. Therefore by Żelazko's theorem [10] the series

$$\sum_{k=0}^{\infty}\left(\frac{P_{i_\nu}(x_1,\ldots,x_n)}{P_{i_\nu}(\zeta_1,\ldots,\zeta_n)}\right)^k$$

is absolutely convergent for all $(\zeta_1,\ldots,\zeta_n)\in S_{i_1,\ldots,i_n}$. But P_{i_ν} is polynomial and $P_{i_\nu}(\zeta_1^0,\ldots,\zeta_n^0)=1$ for each point $\zeta^0=(\zeta_1^0,\ldots,\zeta_n^0)\in S_{i_1,\ldots,i_n}$. Hence the expression

$$\frac{1}{P_{i_\nu}(\zeta_1,\ldots,\zeta_n)}$$

possesses an absolutely convergent expansion in the neighbourhood of ζ^0. Therefore for each ζ^0 there is such a neighbourhood U that the expression

$$[P_{i_\nu}(\zeta_1,\ldots,\zeta_n)-P_{i_\nu}(x_1,\ldots,x_n)]^{-1}$$

possesses an analytic expansion in U. The expression

$$\frac{D_{i_1,\ldots,i_n}(\zeta_1-\zeta_n)\Phi(\zeta_1,\ldots,\zeta_n)}{\prod\limits_{\nu=1}^{n}[P_{i_\nu}(\zeta_1,\ldots,\zeta_n)-P_{i_\nu}(x_1,\ldots,x_n)]}$$

is an analytic function on $S_{i_1,\ldots,i_n}$ because it is the product of functions analytic on $S_{i_1,\ldots,i_n}$.

Hence Proposition 1 implies that the integrals on the right side of formula (2) exist. Further considerations are identical to those in the case of a Banach algebra, q. e. d.

It is easy to check that

Remark. Theorem 2 can be extended to complete algebras in which a topology is given by a sequence of submultiplicative p-homogeneous pseudonorms.

For $n=1$ Theorem 2 was proved by an other method by Żelazko [11], [12]. In general the assumption of analyticity cannot be replaced by the assumption of differentiability. This follows from

PROPOSITION 2. *There is a function $x(t,s)$ determined on the square $[0,1]\times[0,1]$ with values on L^p $(0<p<\frac{1}{2})$ which is non-integrable and possesses both partial derivatives equal to zero.*

Proof. If a function $y(t,s)$ satisfies the Lipschitz inequality, then both partial derivatives are equal to zero. Indeed,

$$\left\|\frac{y(t+h,s)-y(t,s)}{h}\right\|\leqslant\frac{h}{h^p}=h^{1-p}\to 0,$$

$$\left\|\frac{y(t,s+h)-y(t,s)}{h}\right\|\leqslant\frac{h}{h^p}=h^{1-p}\to 0.$$

Hence it is enough to construct a non-integrable function satisfying the Lipschitz inequality.

We determine a function $x_n(t,s)$ in the following way. We divide the square into n^2 squares dividing each side into n equal segments. We order all small squares. In the i-th small square $1\leqslant i\leqslant n^2$ we determine a function $x_n(t,s)$ as follows:

$$x_n(t,s)=\left(\frac{n}{2}\right)^{1/p}\chi_{[a_i,b_i]},\qquad a_i=\frac{i-1}{n^2};\qquad b_i=\frac{i}{n^2}-\frac{2}{n}\max(|t-t_i|,|s-s_i|),$$

where (t_i,s_i) is the centre of the i-th square.

The function $x_n(t,s)$ satisfies the Lipschitz inequality with constant 1, i. e.

$$\|x_n(t,s)-x_n(t_0,s_0)\|\leqslant\max(|t-t_0|,|s-s_0|).$$

Indeed, if (t, s) and (t_0, s_0) are in the same i-th square, then

$$\|x_n(t, s) - x_n(t_0, s_0)\| \leqslant \left\| \left(\frac{n}{2}\right)^{1/p} \chi_{\left[\frac{i}{n^2} - \frac{2}{n}\max(|t-t_i|, |s-s_i|), \frac{i}{n^2} - \frac{2}{n}\max(|t_0-t_i|\, s_0-s_i|)\right]} \right\|$$

$$\leqslant \frac{n}{2} \cdot \frac{2}{n} \max(|t-t_0|, |s-s_0|);$$

if (t, s) is in the i-th square and (t_0, s_0) is in the k-th square ($i \neq k$), then

$$\|x_n(t, s) - x_n(t_0, s_0)\|$$

$$\leqslant \left\| \left(\frac{n}{2}\right)^{1/p} \chi_{[a_i, b_i]} \right\| + \left\| \left(\frac{n}{p}\right)^{1/p} \chi_{\left[\frac{k}{n^2}, \frac{k}{n^2} - \frac{2}{n}\max(|t_0-t_k|, |s_0-s_k|)\right]} \right\|$$

$$\leqslant \frac{n}{2} \left\| \chi_{\left[0, \frac{2}{n}\varrho_i(t,s)\right]} \right\| + \frac{n}{2} \left\| \chi_{\left[0, \frac{2}{n}\varrho_k(t,s)\right]} \right\| \leqslant \max(|t-t_0|, |s-s_0|)$$

(by $\varrho_i(t, s)$ we denote the distance of the point (t, s) from the boundary of the i-th square).

Each function $x_n(t, s)$ is integrable. We do not prove this, because if there is a non-integrable $x_n(t, s)$, then this $x_n(t, s)$ satisfies the theorem.

Let Δ^n be a partition of the square $[0, 1] \times [0, 1]$ into n^2 squares obtained by a division of the sides into n equal segments. Let (t_i^n, s_i^n) be the centres of these squares; then

$$S(x_n, \Delta^n, (t_i, s_i)) = \left(\frac{1}{2}\right)^{1/p} n^{(1/p-2)} \chi_{[0,1]}.$$

Hence if $n \to \infty$, then $\| S(x_n, \Delta^n, (t_i^n, s_i)) \| \to \infty$.

We choose a subsequence n_k such that

$$(*) \quad \| S(x_{n_k}, \Delta^{n_k}, (t_i^{n_k}, s_i^{n_k})) \| \geqslant 4^k \| S(x_{n_{k-1}}, \Delta^i(t_i^l, s_i^l)) \|, \qquad l = 1, 2, \ldots,$$

and n_k/n_{k-1} is an even integer. This is possible because $x_n(t, s)$ is integrable. Let

$$x(t, s) = \sum_{k=1}^{\infty} \frac{1}{2^k} x_{n_k}(t, s).$$

The function $x(t, s)$ satisfies the Lipschitz inequality. Indeed,

$$\|x(t, s) - x(t_0, s_0)\| \leqslant \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^p \|x_{n_k}(t, s) - x_{n_k}(t_0, s_0)\|$$

$$\leqslant \left(\sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^p \right) \max(|t-t_0|, |s-s_0|).$$

Therefore $x(t, s)$ has both partial derivatives equal to zero. On the other hand,

$$\| S(x, \Delta^{n_k}, (t_i^{n_k}, s_i^{n_k})) \|$$

$$\geqslant \| S(x_{n_k}, \Delta^{n_k}, (t_i^{n_k}, s_i^{n_k})) \| - \left\| S\left(\sum_{j=1}^{k-1} x_{n_j}, \Delta^{n_k}, (t_i^{n_k}, s_i^{n_k})\right) \right\| -$$

$$- \left\| S\left(\sum_{j=k+1}^{\infty} x_{n_j}, \Delta^{n_k}, (t_i^{n_k}, s_i^{n_k})\right) \right\| \geqslant 4^k \left(1 - \frac{k-1}{4^k}\right) > 2^k,$$

as follows from $(*)$ and the fact that $x_{n_j}(t_i^{n_k}, s_i^{n_k}) = 0$ for $j > k$ (n_k/n_{k-1} is an even integer).

Hence $x(t, s)$ is not integrable, q. e. d.

We do not know whether it is possible to construct such an example for a function determined on a segment with values in L^p. Neither do we know if there exists a non-integrable function satisfying the Lipschitz inequality on a square with values in a space L^p, $1/2 \leqslant p < 1$.

3. In the preceding considerations we considered the Riemann integrals of function with values in a complete linear metric space X. Now we will show why there are difficulties regarding the definition of the Lebesgue integral if X is not a locally convex space.

In a locally convex linear metric spaces we may consider the so called Bochner integral. It is an analogue of the Lebesgue integral.

A countable-valued function

$$y(t) = \sum_{i=1}^{\infty} y_i \chi_{E_i}(t),$$

where E_i are disjoint measurable sets, is called *Bochner-integrable* (Hille [6], p. 79) if the series $\sum_{i=1}^{\infty} y_i |E_i|$ is convergent. The sum of this series is called the *Bochner integral* of function $y(t)$ and denoted by (B) $\int_L y(t)\,dt$.

A function $x(t)$ is called *Bochner-integrable* if it is the limit of an almost uniformly convergent (²) sequence of countable-valued Bochner-integrable function $x_n(t)$ such that $\|x_n(t)\| < g(t)$, where $g(t)$ is a measurable function. Then there is a limit

$$\lim_{n\to\infty} \int_L x_n(t)\,dt$$

and we will call it the *Bochner integral* of the function $x(t)$ and we will denote it by (B) $\int_L x(t)\,dt$.

(²) We say that the sequence $x_n(t)$ is *almost uniformly convergent* to $x(t)$ if for an arbitrary $\varepsilon > 0$ and an arbitrary neighbourhood U of zero there is a set L_ε, $|L_\varepsilon| < \varepsilon$, such that $x_n(t) - x(t) \in U$ for $t \notin L_\varepsilon$.

The uniqueness of the Bochner integral of function $x(t)$ follows from the fact that if a sequence $x_n(t)$ tends almost uniformly to 0 and $\|x_n(t)\| < g(t)$, where $g(t)$ is an integrable function, then

$$\lim_{n\to\infty} (\mathrm{B}) \int_L x_n(t)\,dt = 0.$$

If the space X is not locally convex, then this is not true, as is shown by the following trivial

PROPOSITION 3. *If the linear metric space X is not locally convex, then there is a sequence of finite-valued functions $x_n(t)$ uniformly tending to* 0 *and such that the sequence $\int_L x_n(t)\,dt$ is not convergent to* 0.

Let $x_n(t)$ tend almost uniformly to 0. Suppose that there is a limit

$$\lim_{n\to\infty} \int_L x_n(t)\,dt = y.$$

If f is a linear continuous functional, then

$$\lim_{n\to\infty} f\big(x_n(t)\big)\,dt = f(y).$$

But $|f(x_n(t))| < g(y)\|f\|$ and $f(x_n(t))$ tends to 0 almost uniformly. Therefore $f(y) = 0$. If in the space X there is a total family of functionals, then $y = 0$.

If there are no linear functionals on X, then the situation may be different. For example, let $X = L^p[0,1]$, $0 < p < 1$. Let

$$x_n(t) = n\chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} \quad \text{if} \quad \frac{k-1}{n} \leqslant t \leqslant \frac{k}{n}, \qquad k = 1, 2, \ldots, n.$$

For each t, $\|x_n(t)\| = n^{p-1}$; hence $x_n(t)$ tends uniformly to 0. On the other hand,

$$(\mathrm{B}) \int_0^1 x_n(t)\,dt = \frac{1}{n} \sum_{k=1}^{n} n\chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} = \chi_{[0,1]}.$$

References

[1] T. Aoki, *Locally bounded linear topological spaces*, Proc. Imp. Ac. Tokyo 18 (1942), p. 588-594.

[2] R. Arens and A. P. Calderón, *Analytic functions on several Banach algebra elements*, Ann. of Math. 2 (1955), p. 204-216.

[3] S. Banach, *Théorie des opérations linéaires*, Warszawa 1933.

[4] I. M. Gelfand, *Normierte Ringe*, Mathem. Sborn. 9(51)(1941), p. 3-66.

[5] И. М. Гельфанд, Д. А. Райков и Г. Е. Шилов, *Коммутативные нормированные кольца*, Москва 1960.

[6] E. Hille and R. S. Philips, *Functional analysis and semigroups*, Coll. Publ. Amer. Math. Soc. 31, Providence 1957.

[7] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires I*, Stud. Math. 10 (1948), p. 184-208; *II* 13 (1953), p. 137-179.

[8] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Pol. Sci. 5 (1957), p. 479-484.

[9] L. Waelbroeck, *Le calcul symbolique dans les algèbres commutatives*, Journ. Math. Pures Appl. 33,2 (1954), p. 147-186.

[10] W. Żelazko, *On the radicals of p-normed algebras*, Stud. Math. 21 (1962), p. 203-206.

[11] — *On the locally bounded and m-convex topological algebras*, ibidem 19 (1960), p. 333-356.

[12] — *Analytic functions in p-normed algebras*, ibidem 21 (1962), p. 345-350.

[13] — *Metric generalizations of Banach algebras*, Rozprawy Matem. 47 (1965).

[14] — *On the decomposition of a commutative p-normed algebra into a direct sum of ideals*, Coll. Math. 10 (1963), p. 57-60.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Reçu par la Rédaction le 24. 10. 1964