D'après [1], p. 234, l'opérateur $\tilde{w} = \frac{\tilde{q}}{\tilde{p}}$ est un logarithme droit dans l'intervalle $0 \leqslant t \leqslant 2T$ avec le nombre caractéristique α.

D'après de théorème cité au commencement de cette note, il s'ensuit, pour $\overline{T} = T$, que $w = w_0 - \alpha s$.

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STUDIA MATHEMATICA, T. XXV. (1964)

The norm of a discrete singular transform

by

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1. Introduction. In [3], Calderon and Zygmund proved that certain discrete singular transforms bear a striking resemblance to the better known continuous ones. The theorems concerning the l^2 or L^2 norm were similar, and in the one-dimensional case the discrete and continuous analogues have the same norm. It was natural to conjecture that this was true in higher dimensions. This paper shows that this is not the case and presents the first serious divergence between the discrete and continuous theory. Incidentally, it produces an amusing summation formula.

The discrete transforms considered in [3] are of the type

$$\tilde{a}_j = \sum_k{}' \frac{\Omega(k)}{|k|^n} a_{j-k}. \tag{1}$$

The points j and k are of the form $m_1 e_1 + m_2 e_2 + \ldots + m_n e_n$, the m_i being integers and the e_i a fixed basis for n-dimensional Euclidean space. The summation is over all such points except the origin. The a_i and $\Omega(k)$ are real or complex valued, $\Omega(h) = \Omega(k/|k|)$, the integral of Ω on the unit sphere is zero, and Ω's modulus of continuity satisfies the Dini condition. The principal result ([3], p. 268) was that the l^2 norm of (1) is the essential least upper bound of the modulus of the function with Fourier series

$$\sum_k{}' \frac{\Omega(k)}{|k|^n} e^{2\pi i(k\cdot x)}. \tag{2}$$

The continuous version of this theorem concerns the singular integral in n dimensions of the form

$$\lim_{\varepsilon\to 0} \int\limits_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy, \tag{3}$$

where Ω has the properties given above ([2], p. 88-91). This transform has L^2 norm equal to the essential least upper bound of the modulus of

$$\lim_{\varepsilon\to 0} \int\limits_{\varepsilon>|y|<1/\varepsilon} \frac{\Omega(y)}{|y|^n} e^{2\pi i(y\cdot x)}\,dy. \tag{4}$$

The one-dimensional transforms

$$\lim_{\varepsilon\to 0}\int\limits_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy \quad\text{and}\quad \sum_k{}' \frac{a_{j-k}}{k}$$

both have norm π and the functions obtained from them by use of (2) and (4) resemble each other. Both have a jump discontinuity at the origin with one-sided limits of π and $-\pi$.

Analogous two-dimensional transforms are

$$\lim_{\varepsilon\to 0}\int\limits_{|y|>\varepsilon}\frac{f(x-y)}{y^2}\,dy \tag{5}$$

where x and y are complex and the integration extends over the complex plane and the transform

$$\sum_k{}' \frac{a_{j-k}}{k^2} \tag{6}$$

where j and k are complex integers. The corresponding functions given by (2) and (4) again both have discontinuities at the origin and in both cases the modulus of the functions approaches π at the origin. Like the one-dimensional case the function corresponding to the integral transform has constant modulus while the other does not. The modulus of the latter function, however, takes on values larger than π so that (5) and (6) do not have the same norm. The proof of this and the determination of the l^2 norm of (6) are the main part of this paper.

The function $\varphi(x, y)$ with Fourier series

$$\sum_{m,n}{}' \frac{e^{2\pi i(mx+ny)}}{(m+in)^2} \tag{7}$$

is the function corresponding to (6). In a slightly more general form it occurs in the work of Kronecker on elliptic functions. It was the subject of considerable investigation by Maier in [4], [5], and [6]. There, again in a slightly more general form, it was known as $t_2(x, y)$. It is expressible in terms of elliptic theta functions.

2. The value of $\varphi(\frac{1}{2}, 0)$. We get now

THEOREM 1. *We have*

$$\varphi(\tfrac{1}{2}, 0) = -\left(\int\limits_0^{\pi/2}\frac{dt}{\sqrt{1-\frac{1}{2}\sin^2 t}}\right)^2 = -3.437604\ldots$$

where $\varphi(x, y)$ is the function with the Fourier series (7).

To obtain this expression, use can be made of a formula proved by Maier ([6], p. 766) for $\varphi(x, y)$, or in his terminology $t_2(x, y)$ with periods 1 and i. His formula shows that

$$\varphi(\tfrac{1}{2}, 0) = -\frac{\sqrt{g_2}}{4} \tag{8}$$

where

$$g_2 = 60\sum_{m,n}{}' \frac{1}{(m+in)^4}$$

is the usual constant for the Weierstrass p function with periods 1 and i. Appell ([1], p. 70) states that if a Weierstrass p function has one real period, 2ω, and one purely imaginary period, $2\omega'$, then

$$2\omega = 2\int\limits_{e_1}^{\infty}\frac{dx}{\sqrt{4x^3-g_2x-g_3}}, \tag{9}$$

where

$$g_2 = 60\sum_{m,n}{}' \frac{1}{(2\omega m+2\omega' n)^4},\qquad g_3 = 140\sum_{m,n}{}' \frac{1}{(2\omega m+2\omega' n)^6}$$

and e_1 is the largest real root of $4x^3-g_2x-g_3 = 0$. Since $2\omega = 1$ and $2\omega' = i$, $g_3 = 0$ and $e_1 = \frac{1}{2}\sqrt{g_2}$. Using these facts, (9) reduces to

$$1 = \int\limits_{\frac{1}{2}\sqrt{g_2}}^{\infty}\frac{2dx}{\sqrt{4x^3-g_2x}}. \tag{10}$$

With the substitution $x = \sqrt{g_2}/2\cos^2 t$, (10) becomes

$$1 = \int\limits_0^{\pi/2}\frac{2dt}{\sqrt[4]{g_2}\sqrt{1-\frac{1}{2}\sin^2 t}}.$$

Solving for g_2 and substituting in (8) gives the desired result.

3. The norm of the transform. We prove now

THEOREM 2. *The expression*

$$\left(\int\limits_0^{\pi/2}\frac{dt}{\sqrt{1-\frac{1}{2}\sin^2 t}}\right)^2$$

is the maximum value of $|\varphi(x, y)|$ and the l^2 norm of the transform (6).

Because of theorem 1 and the Calderon-Zygmund result stated previously, it is sufficient to prove the first part of this theorem.

The Fourier series (7) for $\varphi(x, y)$ may be written formally as

$$(11)\qquad \sum_{m,n}{}' \frac{e^{2\pi i(mx+ny)}}{(m+in)^2} = \sum_{n}{}' \Big[\sum_{m} \frac{e^{2\pi i(mx+ny)}}{(m+in)^2}\Big] + \sum_{m}{}' \frac{e^{2\pi imx}}{m^2}.$$

Finding the Fourier series for x, x^2, $e^{2\pi nx}$, and $xe^{2\pi nx}$ on $0 < x < 1$, and combining the results shows that $\sum\limits_{m} e^{2\pi imx}/(m+in)^2$ is the Fourier series for

$$\frac{\pi^2 e^{2\pi nx}}{\sinh^2 n\pi}\,(-1+2xe^{-n\pi}\sinh n\pi)$$

and that $\sum\limits_{m}{}' e^{2\pi imx}/m^2$ is the Fourier series for $2\pi^2(x^2-x+\frac{1}{6})$. Applying these to (11) and setting $x+iy = z$ indicates that

$$(12)\quad \varphi(x, y) = \pi^2\Big(2x^2-2x+\frac{1}{3}+\sum_{n=1}^{\infty}\Big[\frac{-2\cosh 2\pi nz}{\sinh^2 n\pi}-\frac{4x\sinh n\pi(1-2z)}{\sinh n\pi}\Big]\Big).$$

That (12) actually holds for $0 < x < 1$ follows easily by computing the Fourier coefficients of the expression on the right.

Using the identity $(\sinh a)(\sinh b) = \frac{1}{2}[\cosh(a+b)-\cosh(a-b)]$, formula (12) may be expressed as

$$(13)\quad \varphi(x, y) = \pi^2\Big(2x^2-2x+\frac{1}{3}-\sum_{n=1}^{\infty}\frac{(2-2x)\cosh 2\pi nz+2x\cosh 2\pi n(1-z)}{\sinh^2 n\pi}\Big)$$

for $0 < x < 1$.

Now $\varphi(x, y)$ has period one in both variables. By inspection of the original Fourier series it is also clear that $\varphi(x, y) = -\varphi(-y, x)$. From (13) it follows that $\varphi(x, y) = \varphi(x, -y)$. Because of these facts it will be sufficient to show that $-\varphi(\frac{1}{2}, 0)$ is the maximum value of $|\varphi(x, y)|$ in the triangle with vertices $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0)$.

Since $\cosh 2\pi n(x+iy) = \cosh 2\pi nx\cos 2\pi ny + i\sinh 2\pi nx\sin 2\pi ny$, it follows that $|\cosh 2\pi nz| \leqslant \cosh 2\pi nx$. Similarly, we have $|\cosh 2\pi n(z-1)| \leqslant \cosh 2\pi n(x-1)$. Then using (13) it is clear that $|\varphi(x, y)| \leqslant -\varphi(x, 0)$ for $\frac{1}{6}(3-\sqrt{3}) \leqslant x \leqslant \frac{1}{2}$. Taking a partial derivative of (13) and putting $(x, y) = (\frac{1}{2}, 0)$ shows that $-\varphi(x, 0)$ has a vanishing derivative at $x = \frac{1}{2}$. That the second derivative is negative can be shown by some simple estimations since the series in question converges very rapidly. Therefore, $-\varphi(\frac{1}{2}, 0)$ is a local maximum of $-\varphi(x, 0)$. Again by simple but tedious estimating the second derivative of $-\varphi(x, 0)$ may be shown to be negative for $.4 \leqslant x \leqslant .5$. For $\frac{1}{6}(3-\sqrt{3}) \leqslant x \leqslant .4$ estimates of the value of $\varphi(x, 0)$ using (13) are sufficient to show that $|\varphi(x, y)|$ is less than $-\varphi(\frac{1}{2}, 0)$.

For the rest of the triangle under consideration, that is, for $0 < y < x < \frac{1}{6}(3-\sqrt{3})$, the consideration is complicated by the fact that $|\varphi(x, y)|$ is not necessarily less than $|\varphi(x, 0)|$. To obtain the necessary inequality the expression may be split into three parts with each to be treated separately. The first part consists of the polynomial part of (13) with $4x\pi^2/(e^{2\pi z}-1)$ subtracted. The second part is the first half of the summation. The third part is the second half of the summation plus $4x\pi^2/(e^{2\pi z}-1)$. Again by straightforward but tedious estimating the desired inequality is obtained.

4. A summation formula. A great many summation formulas could be produced by equating the expression produced by (13) for various points to the known values of φ. The most peculiar and least obvious one is obtained by attempting to evaluate $\sum' 1/(m+in)^4$ directly. Summing on m by methods resembling those used in the derivation of (12) produces

$$(14)\qquad \sum{}' \frac{1}{(m+in)^4} = \frac{\pi^4}{45}+\pi^4\sum_{n=1}^{\infty}\frac{6+4\sinh^2 n\pi}{3\sinh^4 n\pi}.$$

$\sum\limits_{n=1} 1/\sinh^2 n\pi$ can be computed by considereing $\lim\limits_{x\to 0+}\varphi(x, 0)$. Maier ([4], p. 103) showed that this limit is $-\pi$. Using (13) and observing that

$$\lim_{x\to 0+}\sum_{n=1}^{\infty}\frac{2x\cosh 2\pi n(1-x)}{\sinh^2 n\pi} = \lim_{x\to 0+}\sum_{n=1}^{\infty}\frac{xe^{2\pi n(1-x)}}{\frac{1}{4}e^{2n\pi}} = \frac{2}{\pi}$$

leads to the fact that

$$\sum_{n=1}^{\infty}\frac{1}{\sinh^2 n\pi} = \frac{\pi-3}{6\pi}.$$

Using this result in (14) and the value obtained in theorem 1 produces

$$\sum_{n=1}^{\infty}\frac{1}{\sinh^4 n\pi} = \frac{2}{15}\Big(\frac{1}{\pi}\int_0^{\pi/2}\frac{dt}{\sqrt{1-\frac{1}{2}\sin^2 t}}\Big)^4+\frac{1}{3\pi}-\frac{11}{90}.$$

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Total and partial differentiability in L^p

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1. A function $f(x) = f(x_1, x_2, \ldots, x_n)$ defined in the neighborhood of a point $x^0 = (x_1^0, \ldots, x_n^0)$ and of the class L^p there, $1 \leqslant p \leqslant \infty$, is said to have a k-th differential in L^p at x^0 if there is a polynomial $P(t) = P(t_1, \ldots, t_n)$ of degree k (or less) such that

$$\left\{\frac{1}{\varrho^n}\int\limits_{|t|\leqslant\varrho}|f(x^0+t)-P(t)|^p dt\right\}^{1/p} = o(\varrho^k) \qquad (\varrho \to 0).$$

If $p = \infty$, the expression on the left is to be interpreted, of course, as $\operatorname{ess\,sup}|f(x^0+t)-P(t)|$ for $|t| \leqslant \varrho$. The definition has been introduced in [1]. The domain of integration $|t| \leqslant \varrho$ can clearly be replaced by a cube containing the origin and of side tending to 0.

The main result of the present paper is the following

THEOREM 1. *Let $f(x) = f(x_1, \ldots, x_n)$ belong to L^p, $1 \leqslant p \leqslant \infty$, over the unit cube*

$$(\mathrm{Q}_0) \qquad 0 \leqslant x_j \leqslant 1 \quad (j = 1, 2, \ldots, n),$$

and suppose that at each point x of a set $E \subset Q_0$ the function f has a k-th differential in L^p. Let m be a fixed integer satisfying $1 \leqslant m < n$. Then at almost all points $x \in E$ the function f has a k-th differential in L^p with respect to the variable $x' = (x_1, x_2, \ldots, x_m)$.

The sets and functions that occur in the proof below are all Lebesgue measurable, even if it is not stated explicitly (the proofs of measurability, when needed, are routine). The cubes will be always closed cubes. We may restrict our argument to the case $1 \leqslant p < \infty$, since if $p = \infty$ it is not difficult to see that the function f^* which coincides with f at the points of set Z where $f(x)$ is the derivative of its indefinite integral and elsewhere is defined by the condition $f(x_0) = \limsup f(x)$ for x tending to x_0 through Z, satisfies the relation $f^*(x_0+t) - P(t) = o(|t|^k)$, and it is enough to observe that the m-dimensional measure of the intersection of the complement of T with almost all subspaces $x_{m+1} = \text{const}, \ldots, x_n = \text{const}$, is 0.