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Smoothness and differentiability in L_p

by

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1. A measurable function $f: I_0 \to R$, $I_0 = [0,1]$, R reals, will be called *L_p-symmetric*, *L_p-smooth*, if for each $x \in I_0^0$, $I_0^0 = (0,1)$,

$$(1) \qquad \left\{\frac{1}{h}\int_0^h |\Delta^2 f(x,t)|^p\,dt\right\}^{1/p} = o(1),\ o(h), \quad \text{as} \quad h \to 0,$$

respectively, where $\Delta^2 f(x,t) = f(x+t)+f(x-t)-2f(x)$. Throughout this paper p will be $\geqslant 1$. The well-known notions of symmetry and smoothness given by

$$(2) \qquad \Delta^2 f(x,t) = o(1),\ o(h), \quad \text{as} \quad h \to 0,$$

respectively, can be viewed as the $p = \infty$ versions of (1). The question arises whether certain of the results for (2) are also true for (1) with perhaps estimating some of the inequalities in the metric of L_p.

In particular, it is known that a measurable smooth function has a derivative on a set which is of the power of the continuum in each interval [4, 10]. In [2], A. P. Calderon and A. Zygmund introduced the notion of L_p-differentiability. We say that f has at x_0 a first *L_p-derivative* provided there is a linear polynomial $a_0 + a_1 t$ such that

$$(3) \qquad \left\{\frac{1}{2h}\int_{-h}^h |f(x_0+t)-a_0-a_1 t|^p dt\right\}^{1/p} = o(h), \quad \text{as} \quad h \to 0.$$

The polynomial $a_0 + a_1 t$ is unique, and we write $a_1 = f'_{L_p}(x_0)$. One of the results that we obtain shows that L_p-smoothness implies L_p-differentiability on a set which is of the power of the continuum in each interval. That this may be the case was noted by A. Zygmund as the author learned in a conversation with E. M. Stein. We will first prove that the theorem is true for continuous functions, and then we will show that a measurable L_p-smooth function is continuous on a dense open set. We will show that this is the best possible continuity property for an L_p-smooth function and that in the case $p = \infty$ a substantial improvement is possible;

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in fact, the set of points of discontinuity of a measurable smooth function is nowhere dense and countable. Finally, we will include some results on the Darboux property of f'_{L_p} analogous to those in [4, 10], and we will conclude this paper with the Baire classification of an L_p-symmetric function.

2. In this paragraph we will collect some definitions and results which will be needed in the sequel. We say that a function $f\colon I_0\to R$ has L as an *approximate limit* at $x_0\in I_0$ if for each $\varepsilon>0$, the set $E_\varepsilon=\{x\colon |f(x)-L|\geqslant\varepsilon\}$ has x_0 as a point of dispersion, i. e., $|E_\varepsilon\cap I|=o(|I|)$ as $I\to x_0$, $x_0\in I$. Approximate continuity and approximate differentiation are defined in an evident manner. We denote the approximate derivative of f at x_0 by $f'_{\rm ap}(x_0)$. For all this and more information we refer the reader to [5].

We have occasion to use the following results.

Lemma 1. *Let $f\colon I_0\to R$ be measurable and let $p\geqslant 1$. Assume that for each $x\in I_0^0$ there is $h=h_x>0$ such that*

$$\int_0^h |f(x+t)+f(x-t)|^p dt<\infty .$$

Then $|f|^p$ is integrable in a neighborhood of almost every point of I_0.

For $p=2$, this is lemma 13 in [8], and the proof given there needs only obvious changes.

Lemma 2. *Let $\sum a_n<\infty$, $a_n\geqslant 0$, $n=1,2,\ldots$, and let $f_n\colon I_0\to R$, $n=1,2,\ldots$, be a sequence of continuous functions with the property that for each $x\in I_0$ there is $N(x)>0$ such that $f_n(x)\leqslant a_n$, $n\geqslant N(x)$. Then each interval contains a subinterval on which $\sum f_n(x)$ converges uniformly.*

For the proof see [1]. The following lemma is easy to verify:

Lemma 3. *Let $f\colon I_0\to R$ be non-decreasing and let E be a measurable subset of I_0. Then $\int_0^e f\leqslant\int_E f$, where $e=|E|$.*

3. We say that a function $f\colon I_0\to R$ *is approximately smooth* on I_0 if for each $x\in I_0^0$ the set $E_\varepsilon=\{h\colon |\Delta^2 f(x,h)|\geqslant\varepsilon|h|\}$ has 0 as a point of dispersion for each $\varepsilon>0$. We let $M(f)$ denote the set of points $x_0\in I_0^0$ such that for some $m\in R$, $f(x)-mx$ has a local extremum at x_0.

Lemma 4. *Let $f\colon I_0\to R$ be continuous and approximately smooth on I_0. Then $f'_{\rm ap}$ exists on $M=M(f)$ and M is of the power of the continuum in each interval.*

Proof. The proof is essentially the same as the one in [12], p. 43, for 3.3, and we include it for completeness. Let $x_0\in M$ and choose $m\in R$ such that $g(x)=f(x)-mx$ has a local extremum, say, maximum at x_0. We will show that $g'_{\rm ap}(x_0)=0$. Let $\varepsilon>0$ be given, and let $E_\varepsilon=\{h\colon |\Delta^2 g(x_0,h)|\geqslant\varepsilon|h|\}$, $A_\varepsilon=\{h\colon |g(x_0+h)-g(x_0)|\geqslant\varepsilon|h|\}$. Since 0 is a point of dispersion of E_ε, it suffices to show that $A_\varepsilon\cap[-\delta,\delta]\subset E_\varepsilon$ for some $\delta>0$. Since $g(x_0)$ is a local maximum, there is $\delta>0$ such that $g(x_0\pm h)-g(x_0)\leqslant 0$, $|h|\leqslant\delta$. It follows that

$$\left|\frac{g(x_0+h)-g(x_0)}{h}\right|\leqslant\left|\frac{\Delta^2 g(x_0,h)}{h}\right|,\qquad h\leqslant\delta,$$

establishing the desired inclusion and also $f'_{\rm ap}(x_0)=m$. Thus, if $0\leqslant\alpha<\beta\leqslant 1$ and $m(\alpha,\beta)=[f(\beta)-f(\alpha)]/(\beta-\alpha)$, there is a point $x_0\in(\alpha,\beta)$ at which $f'_{\rm ap}(x_0)=m(\alpha,\beta)$. Unless f is linear, in which case the lemma is obvious, the collection of distinct $m(\alpha,\beta)$, and hence M, is of the power of the continuum.

Remark. Under the hypothesis of lemma 4, if $E=\{x\colon f'_{\rm ap}(x)\text{ exists}\}$, then $f'_{\rm ap}$ has the Darboux property on E. The proof is not difficult, and we shall treat this in connection with L_p-derivatives in a later section.

Lemma 5. *Let $f\colon I_0\to R$ be measurable and L_p-smooth on I_0. Then f is approximately smooth on I_0.*

Proof. Let $x_0\in I_0^0$, and let $\varepsilon>0$ be given. We have to show that the set $E_\varepsilon=\{t\colon |\Delta^2 f(x_0,t)|\geqslant\varepsilon|t|\}$ has 0 as a point of dispersion. Let $E_h=E_\varepsilon\cap[0,h]$. By lemma 3,

$$\left\{\frac{1}{h}\int_0^h|\Delta^2 f(x_0,t)|^p dt\right\}^{1/p}\geqslant\left\{\frac{1}{h}\int_{E_h}\varepsilon^p t^p dt\right\}^{1/p}$$
$$\geqslant\left\{\frac{1}{h}\frac{\varepsilon^p}{p+1}|E_h|^{p+1}\right\}^{1/p}=\left\{\frac{\varepsilon^p}{p+1}\frac{|E_h|}{h}\right\}^{1/p}\cdot|E_h|,$$

and, by hypothesis, this is $o(h)$. Hence $|E_h|=o(h)$, and the proof is complete.

Theorem 1. *Let $f\colon I_0\to R$ be continuous and L_p-smooth on I_0. Then the set S of points x at which $f'_{L_p}(x)$ exists contains $M=M(f)$, and hence is of the power of the continuum in each interval.*

Proof. We will show that for $x_0\in M$,

$$\left\{\frac{1}{2h}\int_{-h}^{h}|f(x_0+t)-f(x_0)-f'_{\rm ap}(x_0)\cdot t|^p dt\right\}^{1/p}=o(h).$$

Since $x_0\in M$, there is $m\in R$ such that $g(x)=f(x)-mx$ has a local extremum at x_0. By lemma 4, $f'_{\rm ap}(x_0)=m$, and for $|t|$ sufficiently small $|f(x_0+t)-f(x_0)-f'_{\rm ap}(x_0)t|=|g(x_0+t)-g(x_0)|\leqslant|\Delta^2 g(x_0,t)|=|\Delta^2 f(x_0,t)|$. Hence

$$\left\{\frac{1}{2h}\int_{-h}^{h}|f(x_0+t)-f(x_0)-f'_{\rm ap}(x_0)t|^p dt\right\}^{1/p}=o(h).$$

4. We remove now the continuity hypothesis of theorem 1. For this purpose we establish the following lemma:

LEMMA 6. *Let $f: I_0 \to R$ be integrable. If f is L_p-smooth on I_0, then f is continuous on a dense open set in I_0.*

Proof. If we let

$$F(x) = \int_0^x f(t)\,dt,$$

we obtain

$$\frac{F(x+h)-F(x-h)}{2h} - f(x) = \frac{1}{2h}\int_0^h \Delta^2 f(x,t)\,dt.$$

Since $p \geqslant 1$, we have the inequality

$$\frac{1}{h}\int_0^h |\Delta^2 f(x,t)|\,dt \leqslant \left\{\frac{1}{h}\int_0^h |\Delta^2 f(x,t)|^p dt\right\}^{1/p}$$

from which

$$\left|\frac{F(x+h)-F(x-h)}{2h} - f(x)\right| = o(h), \quad \text{as} \quad h \to 0.$$

Hence there is $\delta_x > 0$ such that

$$\left|\frac{F(x+h)-F(x-h)}{2h} - f(x)\right| < h, \quad 0 < h < \delta_x.$$

From now on the proof parallels the one in [1]. Let I be a closed interval in I_0^0, and let $h_n \to 0$, $h_n > 0$, such that $\sum h_n < \infty$, $h_n > h_{n+1}$, and $x \pm h_n \in I_0^0$ for every $x \in I$. For $x \in I$, let $\varphi_n(x) = [F(x+h_n) - F(x-h_n)]/2h_n$ and observe that there is $N(x) > 0$ such that $|\varphi_n(x) - f(x)| < h_n$, and hence $|\varphi_n(x) - \varphi_{n+1}(x)| < 2h_n$, $n \geqslant N(x)$. Application of lemma 2 to $\varphi_1(x) + \sum[\varphi_{n+1}(x) - \varphi_n(x)] = f(x)$ completes the proof.

THEOREM 2. *Let $f: I_0 \to R$ be measurable and L_p-smooth on I_0^0. Then* (i) *f is continuous on a dense open set in I_0, and* (ii) *the set of points at which f has a first L_p-derivative is of the power of the continuum in each interval.*

Proof. We only need to verify (i), since (ii) follows from (i) in view of theorem 1. To prove (i), we first observe that for each $x \in I_0^0$ there is $h = h_x > 0$ such that

$$\int_0^h |\Delta^2 f(x,t)|^p dt < \infty.$$

Hence

$$\int_0^h |f(x+t)+f(x-t)|^p dt < \infty.$$

By lemma 1, f is integrable in a neighborhood of almost every point of I_0. If we invoke lemma 6 to any such neighborhood, we obtain the desired dense open set.

The set of L_p-differentiability in theorem 2 can be a set of measure zero. Let $f: I_0 \to R$ be continuous and smooth such that f' exists only on a set of measure zero ([12], p. 206). It is well-known that the set of points at which f'_{ap} exists is also of measure zero [3]. Thus the function f will provide the desired example for L_p-differentiability if we can show that the first L_p-derivative is equal to the approximate derivative a. e.

LEMMA 7. *Let $f: I_0 \to R$ be measurable. Then at almost all points x at which $f'_{L_p}(x)$ exists, $f'_{L_p}(x) = f'_{\text{ap}}(x)$.*

Proof. Let x be a point of L_p-differentiability of f. Then

$$\left\{\frac{1}{2h}\int_{-h}^{h} |f(x+t) - a_0(x) - a_1(x)t|^p dt\right\}^{1/p} = o(h),$$

where $a_1(x) = f'_{L_p}(x)$. Using an argument as in lemma 5, we see that, for each $\varepsilon > 0$, the set $E_\varepsilon = \{t: |f(x+t) - a_0(x) - a_1(x)t| \geqslant \varepsilon|t|\}$ has 0 as a point of dispersion. If x also belongs to the Lebesgue set of f, then $a_0(x) = f(x)$, and hence $a_1(x) = f'_{\text{ap}}(x)$.

Remark. If in lemma 7 measurability is replaced by continuity, then, as is seen from the proof, at every point x at which $f'_{L_p}(x)$ exists, $f'_{L_p}(x) = f'_{\text{ap}}(x)$.

5. In this section we will show that the first L_p-derivative of an L_p-smooth function behaves with regard to the Darboux property as the first derivative of a smooth function (see [4, 11]). Let us recall that a function f defined on a set E has the Darboux property on E if for $a < b$ in E, f assumes all values between $f(a)$, $f(b)$ on $(a, b) \cap E$.

THEOREM 3. *Let $f: I_0 \to R$ be continuous and L_p-smooth, and let $S = \{x: f'_{L_p}(x)$ exists$\}$. Then f'_{L_p} has the Darboux property on S.*

Proof. Let us set $g(x) = f'_{L_p}(x)$, $x \in S$, and let us note that in view of the continuity of f, $g(x) = f'_{\text{ap}}(x)$, $x \in S$. We only need to show that, if $a < b$, $a, b \in S$, and $g(a) < 0 < g(b)$ then there is a point $\xi \in (a, b)$ such that $g(\xi) = 0$. We may assume that $f(b) \geqslant f(a)$. If $f(b) = f(a)$, we infer from lemma 4 that there is a point $\xi \in (a, b) \cap M$ at which $f'_{\text{ap}}(\xi) = 0$. By theorem 1, $M \subset S$ and $f'_{\text{ap}}(\xi) = g(\xi)$. If $f(b) > f(a)$, we have in view of $f'_{\text{ap}}(b) > 0$, $f'_{\text{ap}}(a) < 0$, a point $c \in (a, b)$ such that $f(c) = f(a)$. As before, there is a point $\xi \in (a, c)$ such that $f'_{\text{ap}}(\xi) = g(\xi) = 0$.

We remark that *continuity* cannot be replaced by *measurability* in theorem 3. The example of a measurable smooth function whose derivative assumes only the values 0 and 1 (see [4]) also provides the desired example for L_p-smoothness. However, the same situation as in [4] prevails.

THEOREM 4. *Let $f\colon I_0 \to R$ be measurable and L_p-smooth, and let $S = \{x\colon f'_{L_p}(x)$ exists$\}$, $E = \{x\colon f'(x)$ exists$\}$. If $|E \cap I| < |I|$ for every interval $I \subset I_0$, then f'_{L_p} has the Darboux property on S.*

Proof. Using theorem 1, it is seen from the proof of lemma 4 (mean value theorem) that, if $f'_{L_p}(x) \geqslant 0$, $x \in I \cap S$, then f is non-decreasing on I and hence f' exists a. e. on I. The proof is now the same as the one given for theorem 7 in [4].

6. We will show that (i) of theorem 2 cannot be improved for $p < \infty$ and that in the case $p = \infty$ an improvement is possible. We denote the closure of A by $\bar{A}$.

LEMMA 8. *Let $a < b$ and let $p > 1$. Then there exists $A \subset (a, b)$ such that* (i) *A is a union of disjoint closed intervals,* (ii) $\{a\} = \bar{A} - A$ (iii) $|A \cap J| = o(|J|^p)$ *as $J \to a$, $a \in J$.*

The proof is clear.

LEMMA 9. *Let C be a closed nowhere dense subset of I_0, let $G_i = (a_i, b_i)$, $i = 1, 2, \ldots$, be its complementary intervals, and let $p \geqslant 1$. Then there exists $A \subset \bigcup G_i$ such that* (i) *A is a union of disjoint closed intervals,* (ii) $C \subset \bar{A} - A$, (iii) $|A \cap J| = o(|J|^p)$, *as $J \to c$, $c \in J$, for any $c \in C$.*

Proof. Let $A_i \subset G_i$ be as in lemma 8. There exists $\delta_i > 0$ such that $|J| \leqslant \delta_i$, $a_i \in J$, implies $|A_i \cap J| < 2^{-i}|J|^p$.

Let $A_i^* = A_i \cap [a_i, a_i + \delta_i]$, and let $A = \bigcup A_i^*$.

Let $c \in C$ and let $J_n \to c$, $J_n = [\alpha_n, \beta_n]$. Then, if $a_i < \beta_i$ and $J_n^* = [a_i, a_i + \delta_i] \cap J_n$,

$$\frac{|A_i^* \cap J_n|}{|J_n|^p} = \frac{|A_i \cap J_n^*|}{|J_n^*|^p} \cdot \frac{|J_n^*|^p}{|J_n|^p} < \frac{1}{2^i},$$

and, if $a_i = \beta_n$, $|A_i^* \cap J_n| = 0$. Let $\varepsilon > 0$ be given, and choose n_0 so that $2^{-n_0+1} < \varepsilon/2$. For $n \geqslant N$,

$$|A_i^* \cap J_n| < \frac{\varepsilon}{2n_0} |J_n|^p, \qquad i = 1, \ldots, n_0.$$

Thus, for $n \geqslant N$,

$$\frac{|A \cap J_n|}{|J_n|^p} = \sum_{i=1}^{n_0-1} \frac{|A_i^* \cap J_n|}{|J_n|^p} + \sum_{i=n_0}^{\infty} \frac{|A_i^* \cap J_n|}{|J_n|^p} < \varepsilon.$$

THEOREM 5. *Let C be a nowhere dense and closed subset of I_0, and let $p \geqslant 1$. Then there exists a bounded measurable $f\colon I_0 \to R$ such that* (i) *f is L_p-smooth,* (ii) $C = \{x\colon f$ not continuous at $x\}$.

Proof. Let A be the set constructed in lemma 9 with p replaced by $p+1$. Define $f(x) = 0$, $x \in I_0 - A$, and on each component $[a, b]$ of A let f be differentiable on A with $f'(a) = f'(b) = 0$, $f(\frac{1}{2}(a+b)) = 1$, and $|f| \leqslant 1$. We only need to show that f is L_p-smooth, and this is clear at $x \notin C$. If $x \in C$, then, letting $A_0 = \{a - x\colon a \in A\}$,

$$\left\{\frac{1}{h}\int_0^h |\Delta^2 f(x, t)|^p dt\right\}^{1/p} \leqslant \left\{\frac{1}{h}\int_0^h |f(x+t)|^p dt\right\}^{1/p} + \left\{\frac{1}{h}\int_0^h |f(x-t)|^p dt\right\}^{1/p}$$
$$\leqslant \left\{\frac{|A_0 \cap [0, h]|}{h}\right\}^{1/p} + \left\{\frac{|A_0 \cap [-h, 0]|}{h}\right\}^{1/p} = o(h).$$

7. We have already observed that the set $D(f)$ of points of discontinuity of a measurable smooth ($p = \infty$) function is nowhere dense. We will show that $D(f)$ is also countable. The main idea of the proof is taken from Sierpiński [6] which contains a similar theorem concerning first symmetric differences.

Let $f\colon I_0 \to R$, and for $x \in I_0^0$ let

$$\Delta_* f(x, t, \tau) = \Delta^2 f(x, t+\tau) - \Delta^2 f(x, t).$$

In the notation of [12], p. 50, this is $2[\varphi_x(t+\tau) - \varphi_x(t)]$, an expression which also appears in Lebesgue's convergence test for Fourier series.

THEOREM 6. *Let $f\colon I_0 \to R$ and let $D = D(f)$. If*

$$\limsup_{\tau \to 0} |\Delta_* f(x, t, \tau)| = o(t) \quad \text{as} \quad t \to 0$$

for each $x \in I_0^0$, then either $D = I_0$ or else D is countable.

Proof. Let us assume that $D \neq I_0$ and that D is uncountable. Let $\alpha \in I_0 - D$. Then either $D \cap [0, \alpha]$ or $D \cap [\alpha, 1]$ is uncountable, say $D \cap [\alpha, 1] = E$ is uncountable.

We let $\omega(x) = \limsup |f(z) - f(x)|$ as $z \to x$. There exists $\sigma > 0$ such that

$$E(\sigma) = \{x\colon x \in E \text{ and } \omega(x) > \sigma(1 + x - \alpha)\}$$

is uncountable. Let $c = \sup\{\gamma\colon \gamma \geqslant \alpha$ and $E(\sigma) \cap [\alpha, \gamma]$ is countable$\}$. Then it follows that (i) $c > \alpha$, (ii) $H = \{x\colon \omega(x) > \sigma(1 + x - \alpha),\ \alpha \leqslant x \leqslant c\}$ is countable, and (iii) for every $d > 0$, $A_d = \{x\colon \omega(x) > \sigma(1 + x - \alpha),\ c \leqslant x \leqslant c + d\}$ is uncountable.

Let

$$\varphi(x, t) = \limsup_{\tau \to 0} |\Delta_* f(x, t, \tau)|.$$

There is $0 < \delta < c-a$ such that $0 \leqslant \varphi(c, t) < 2\sigma \cdot t$, $0 < t \leqslant \delta$. Since $f(c+t+\tau)-f(c+t) = \Delta_* f(c, t, \tau) - [f(c-t-\tau)-f(c-t)]$ we have $\omega(c+t) \leqslant 2\sigma \cdot t + \omega(c-t)$. For $c+t \in A_\delta$, we obtain $\omega(c+t) > \sigma(1+c+t-a)$ which, with the inequality just established, gives

$$\omega(c-t) > \sigma(1+c+t-a) - 2\sigma t = \sigma(1+c-t-a).$$

This implies that $c-t \in H$. Since there are uncountably many t for which $c+t \in A_\delta$, we have that H is uncountable, a contradiction.

COROLLARY 1. *If $f: I_0 \to R$ is measurable and smooth, then $D(f)$ is countable.*

8. The previous theorem has an L_p-version. For $f: I_0 \to R$ measurable, we let

$$D_p = \left\{x\colon x \in I_0^0 \text{ such that } \int_{-h}^{h} |f(x+t)-f(x)|^p dt \neq o(h) \text{ as } h \to +0\right\},$$

and we call D_p the set of *L_p-discontinuities* of f. We also set

$$\omega_p(x) = \limsup_{h\to+0} \left\{\frac{1}{2h}\int_{-h}^{h} |f(x+t)-f(x)|^p dt\right\}^{1/p},$$

and we observe that $D_p = \{x\colon \omega_p(x) > 0\}$.

THEOREM 7. *Let $f: I_0 \to R$ be measurable and assume that*

$$\limsup_{h\to 0} \left\{\frac{1}{h}\int_0^h |\Delta_* f(x, t, \tau)|^p d_\tau\right\}^{1/p} = o(t),$$

for each $x \in I_0^0$. If $\omega_p(x)$ is continuous at some point, then D_p is countable.

Proof. Let $a \in I_0$ be a point of continuity of $\omega_p(x)$. Hence $\omega_p(a) < \infty$ and $|f|^p$ is integrable in a neighborhood N of a. Consequently, $\omega_p(x) = 0$ a. e. in N, and therefore, $\omega_p(a) = 0$. The proof is now the same as the one for theorem 6 except for estimating certian inequalities in the metric of L_p.

Remark. It is not known to the writer whether L_p-smoothness implies countability of D_p.

9. In this section we will show that measurable L_p-symmetric functions are in the first Baire class. This is known for the case $p = \infty$ [4].

Let $f: I_0 \to R$ be measurable, and let

$$A_p = \left\{x\colon \varepsilon > 0 \quad \text{implies} \quad \int_{x-\varepsilon}^{x+\varepsilon} |f(t)|^p dt = \infty\right\}.$$

It is clear that A_p is closed.

LEMMA 10. *Let $f: I_0 \to R$ be measurable and assume that*

$$\int_0^h |\Delta^2 f(x, t)|^p dt = O(1), \qquad x \in I_0^0.$$

If $x_0 \in A_p$, then

$$\int_0^h |f(x_0+t)-f(x_0-t)|^p dt = \infty, \qquad h > 0.$$

Proof. The proof follows from the inequality

$$2|f(x_0)-f(x_0+t)| \leqslant |f(x_0+t)-f(x_0-t)| + |\Delta^2 f(x_0, t)|$$

and

$$\int_{-h}^{h} |f(x_0+t)-f(x_0)|^p dt = \infty.$$

LEMMA 11. *Let $f: I_0 \to R$ be measurable and assume that*

$$\int_0^h |\Delta^2 f(x, t)|^p dt = O(1), \qquad x \in I_0^0.$$

Then A_p is countable.

Proof. By lemma 1 we know that A_p is nowhere dense. If we suppose that A_p is uncountable, we can write $A_p = P \cup N$, where P is perfect, N is countable, and $P \cap N = 0$. Let (a, b) be a complementary interval of P with $b \in P \cap I_0^0$. Then there exists $0 < 2\delta < b-a$ such that, if $0 \leqslant |a| \leqslant \delta$, $0 < h \leqslant \delta$, then

$$\int_a^{a+h} |\Delta^2 f(b, t)|^p dt < M,$$

for some constant $M > 0$. Let $\gamma \in P \cap (b, b+\delta)$ and let $\gamma_0 = 2b-\gamma$. Since $|f(\gamma+t)-f(\gamma-t)| \leqslant |f(\gamma_0+t)+f(\gamma-t)-2f(b)| + |f(\gamma_0-t)+f(\gamma+t) -2f(b)| + |f(\gamma_0+t)-f(\gamma_0-t)|$, we have

$$\left\{\int_0^h |f(\gamma+t)-f(\gamma-t)|^p dt\right\}^{1/p} \leqslant \left\{\int_{\gamma_0-b}^{\gamma_0-b+h} |\Delta^2 f(b, u)|^p du\right\}^{1/p} + \\ + \left\{\int_{\gamma-b}^{\gamma-b+h} |\Delta^2 f(b, u)|^p dt\right\}^{1/p} + \left\{\int_0^h |f(\gamma_0+t)-f(\gamma_0-t)|^p dt\right\}^{1/p}.$$

If $0 < h \leqslant \delta$ we see from lemma 10 that $\gamma_0 \in A_p$. Since $P \cap (b, b+\delta)$ is uncountable, the set $A_p \cap (a, b)$ is uncountable, contradicting $A_p \cap (a, b) \subset N$.

Remark. Lemma 11 implies that the exceptional set in lemma 1 is countable.

THEOREM 8. *Let $f: I_0 \to R$ be measurable, and assume that*

$$\left\{\frac{1}{h}\int_0^h |\Delta^2 f(x,t)|^p dt\right\}^{1/p} = o(1), \qquad x \in I_0^0.$$

Then f is in the first Baire class.

Proof. We first assume that $f \in L_1$. If we let $F(x) = \int_0^x f(t)\,dt$, it follows readily that (note $p \geqslant 1$)

$$\left|\frac{F(x+h)-F(x-h)}{2h} - f(x)\right| \leqslant \frac{1}{2h}\int_0^h |\Delta^2 f(x,t)|\,dt$$

$$\leqslant \frac{1}{2}\left\{\frac{1}{h}\int_0^h |\Delta^2 f(x,t)|^p dt\right\}^{1/p} = o(1).$$

Since F is continuous, the proof for the case $f \in L_1$ is complete.

Next, we assume that f is integrable in a neighborhood of every point of I_0^0. Let $\{I_n\}$ be a sequence of closed intervals such that $I_n \subset I_{n+1}$ and $\bigcup I_n = I_0^0$. Since $f \in L_1(I_n)$, we infer that $f|I_n$ is in the first Baire class on I_n. It readily follows that f is in the first Baire class on I_0.

Since A_p is closed and countable, the general case is an immediate consequence of what has already been established.

Remark. There exists a bounded approximately continuous function $f: I_0 \to R$ for which $\{x: f(x) = 0\}$ is dense and of measure zero [10]. This function is L_p-symmetric and discontinuous a. e., even though, as seen from theorem 8, it is continuous on a dense G_δ-set. It is known that in the case $p = \infty$ one obtains continuity a. e. [8].

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