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w*-bases and bw*-bases in Banach spaces

by

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1. Introduction. Let X be a linear space endowed with a locally convex topology τ. A sequence $\{M_i\}$ of non-trivial subspaces in X is a *τ-basis* of subspaces for X if and only if corresponding to each $x \in X$ there is a unique sequence $\{x_i\}$, $x_i \in M_i$, such that

$$x = \lim_n \sum_{i=1}^{n} x_i,$$

convergence in the topology τ. Corresponding to a basis of subspaces $\{M_i\}$ is a sequence of orthogonal projections $\{E_i\}$ ($E_i^2 = E_i$ and $E_iE_j = 0$ if $i \neq j$) defined by $E_i(x) = x_i$ if $x = \sum_{i=1}^{\infty} x_j$, $x_j \in M_j$. If each E_i is continuous the basis is called a *τ-Schauder basis* of subspaces (τ-Sbos). This concept was first systematically studied (independently) by Mazur and McArthur, although the notion essentially dates back to Grinblyum [8]. We remark that a τ-Schauder basis of one-dimensional subspaces coincides with the notion of τ-Schauder basis of vectors (τ-Sbov). (A very good discussion of basis of vectors in linear topological spaces can be found in [1].)

In this paper we study τ-Schauder bases of subspaces where τ is the norm topology on a Banach space X or the w* or bounded w*-topology on X^* or X^{**}. In speaking of the "τ-Sbos $\{M_i, E_i\}$" we shall mean the basis $\{M_i\}$ and the associated projections. If the norm topology is under consideration we drop the prefix and speak of Schauder bases of subspaces (Sbos) and Schauder bases of vectors (Sbov).

A Sbos $\{M_i, E_i\}$ is *shrinking* if $\{R(E_i^*)\}$ where $R(E_i^*)$ denotes the range of the adjoint of E_i, is a Sbos for X^*. In the one-dimensional case

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this notion of shrinking coincides with that usually given ([9], p. 522, Thm. 3, [5], p. 70, Lemma 1).

After a few elementary considerations concerning w*-continuous linear operators of a conjugate Banach space X^* into itself, it is shown, generalizing a result of Singer ([12], p. 80, Thm. 3), that a Banach space X has a Sbos if and only if X^* has a w*-Sbos. This theorem (3.1), actually shows how to construct a Sbos for X (a w*-Sbos for X^*) if one is given a w*-Sbos for X^* (a Sbos for X). It is also shown that if X^* has a w*-Sbos which is also a Sbos, then X must have a shrinking Sbos.

The final part of the paper is devoted to showing that the notions of w*- and bounded w*-Sbos are equivalent.

2. Notation and remarks. In the sequel X will denote an infinite dimensional Banach space, T will denote a w*-continuous operator from X^* into itself, T^* the adjoint of T and J the canonical map of X into X^{**}. Also, $R(E)$ denotes the range of a linear operator E and if $\{E_i\}$ is a sequence of linear operators on X into X then $[\bigcup_{i=1}^{\infty} R(E_i)]$ denotes the closed linear span of $\bigcup_{i=1}^{\infty} R(E_i)$.

The following readily established remarks are of fundamental importance to our work.

(2.1) *The set $T^*J(X)$ is contained in $J(X)$* (1).

(2.2) *The w*-continuous linear operator T is norm-continuous on X^* and hence T^* is norm-continuous on X^{**}.*

(2.3) *If P is a w*-continuous projection on X^* then $E = J^{-1}P^*J$ is a norm-continuous projection on X.*

The following lemma will be used without proof. It is a generalization of a theorem of Banach ([2], p. 107, Thms. 2 and 3).

LEMMA 2.4. *If $\{E_i\}$ is a sequence of continuous, non-trivial, orthogonal projections of a Banach space X into itself and if*

$$\Big\{\sum_{i=1}^{n} E_i(x)\Big\}_{n=1}^{\infty}$$

is bounded for each $x \in X$ then $\{R(E_i)\}_{i=1}^{\infty}$ is a Sbos for $[\bigcup_{i=1}^{\infty} R(E_i)]$ and $\{R(E_i^*)\}$ is a Sbos for $[\bigcup_{i=1}^{\infty} R(E_i^*)]$.

3. w*-Schauder bases of subspaces in X^*. We prove now

THEOREM 3.1. *If $\{M_i, E_i\}$ is a Sbos for X then $\{R(E_i^*), E_i^*\}$ is a w*-Sbos for X^*. Conversely, if $\{N_i, P_i\}$ is a w*-Sbos for X^* then $\{R(E_i), E_i\}$, where $E_i = J^{-1}P_i^*J$, is a Sbos for X.*

(1) See [4], IV.1, section 2, Proposition 1.

Proof. Let $f \in X^*$ and $x \in X$. Then

$$\sum_{i=1}^{n} \big(E_i^*(f)\big)(x) = \sum_{i=1}^{n} f\big(E_i(x)\big) \to f(x),$$

whence $\{\sum_{i=1}^{n} E_i^*(f)\}$ converges to f in the w*-topology. Suppose $\{\sum_{i=1}^{n} f_i\}$, $f_i \in R(E_i^*)$, is w*-convergent to f. Then, for fixed $j \in \omega$ and $x \in X$,

$$\big(E_j^*(f)\big)(x) = f\big(E_j(x)\big) = \sum_{i=1}^{\infty} f_i\big(E_j(x)\big) = f_j(x),$$

since $E_i^*E_j^* = 0$ and $f_j \in R(E_j^*)$. Thus the expression is unique. It is well known that the adjoints of norm-continuous linear operators are w*-continuous and thus $\{R(E_i^*), E_i^*)\}$ is a w*-Sbos for X^*.

Now suppose $\{N_i, P_i\}$ is a w*-Sbos for X^*. By (2.3), $E_i = J^{-1}P_i^*J$ is a norm-continuous projection on X, for each i. Since $\{P_i\}$, and hence $\{P_i^*\}$, is a sequence of orthogonal projections, it follows that $\{E_i\}$ is a sequence of continuous orthogonal projections. By hypothesis, for each $x \in X$ and $f \in X^*$, we have

$$f(x) = \sum_{i=1}^{\infty} \big(P_i(f)\big)(x).$$

By (2.1) there is a $Y_i \in X$ such that $P_i^*J(x) = J(Y_i)$. Thus $E_i(x) = J^{-1}P_i^*J(x) = Y_i$. Hence

$$f(x) = \sum_{i=1}^{\infty} \big(P_i(f)\big)(x) = \sum_{i=1}^{\infty} \big(P_i^*J(x)\big)(f) = \sum_{i=1}^{\infty} f\big(E_i(x)\big),$$

for every $f \in X^*$ and $x \in X$. Thus

$$\sup_n \Big\|\sum_{i=1}^{n} E_i(x)\Big\| < +\infty \quad \text{for all} \quad x \in X.$$

Hence by Lemma 2.4, $\{R(E_i)\}$ is a Sbos for $X_0 = [\bigcup_{i=1}^{\infty} R(E_i)]$. Suppose there is an $x \in X \setminus X_0$. Then there is an $f \in X^*$ such that $f(x) = 1$ and $f(X_0) = 0$. Thus

$$1 = f(x) = \lim_n \sum_{i=1}^{n} f\big(E_i(x)\big) = 0$$

since $E_i(x) \in X_0$ for each $i \in \omega$. Hence $X = X_0$ and this completes the proof.

THEOREM 3.2. *If $\{N_i, P_i\}$ is a w*-Sbos for X^*, then $\{N_i P_i\}$ is a Sbos for $[\bigcup_{i=1}^{\infty} R(P_i)]$.*

Proof. By Theorem 3.1, $\{R(J^{-1}P_i^*J)\}$ is a Sbos for X and hence, by Lemma 2.4, $\{R([J^{-1}P_i^*J]^*)\}$ is a Sbos for $[\bigcup_{i=1}^{\infty} R([J^{-1}P_i^*J]^*)]$. We show that $(J^{-1}P_i^*J)^* = P_i$, for each $i \in \omega$. Let $x \in X$ and $f \in X^*$. By (2.1) there is a $Y_i \in X$ such that $P_i^*J(x) = J(Y_i)$. Thus $([J^{-1}P_i^*J]^*(f))(x) = f(J^{-1}P_i^*J(x)) = f(Y_i) = (P_i^*J(x))(f) = (P_i(f))(x)$ and it follows that $(J^{-1}P_i^*J)^* = P_i$, completing the proof of the theorem.

As a special case of Theorem 3.2 we see that a w*-Sbov, $\{f_i\}$, for X^* is, in the terminology of Bessaga and Pełczyński [3], a basic sequence of vectors.

That a Sbos for X^* need not be a w*-Sbos for X^* is easily seen by letting $\{M_i, E_i\}$ be the Sbos formed from Gelbaum's non-retro basis for l^1 (see [12], p. 76). On the other hand, there are spaces which have Schauder bases of subspaces which are also w*-Schauder bases of subspaces. For example, the Sbos formed by the unit vectors in l^1 is also a w*-Sbos for l^1. Indeed, it follows from Theorem 3.1 that if X has a shrinking Sbos $\{M_i, E_i\}$, then $\{R(E_i^*), E_i^*\}$ is both a Sbos and a w*-Sbos for X^*. Conversely, if $\{N_i, P_i\}$ is both a Sbos and a w*-Sbos for X^*, then $\{R(J^{-1}P_i^*J)\}$ is a shrinking Sbos for X; for, by Theorem 3.1, $\{R(J^{-1}P_i^*J)\}$ is a Sbos for X and in the proof of Theorem 3.2 it was shown that $(J^{-1}P_i^*J)^* = P_i$. Thus, by Lemma 2.4, $\{R(J^{-1}P_i^*J)\}$ is shrinking.

The unit vectors $\{e_i\}$ form a w*-Sbos for (m) and thus there are conjugate spaces with w*-Sbov which cannot have Sbov. Consider the converse question:

(*) If X^* has a Sbov, does X^* have a w*-Sbov?

We observe that a negative answer to (*) (which is unlikely) would yield a negative answer to the Schauder basis problem. On the other hand, an affirmative answer to (*), together with Theorem 3.1, would yield an affirmative answer to a question of Karlin ([10], p. 984).

4. w*-Schauder bases of subspaces in X^{**}. In view of Theorem 3.2, the following question is of interest:

If X^{**} has a w^*-Sbos $\{N_i, P_i\}$, under what conditions will $[\bigcup_{i=1}^{\infty} N_i] = J(X)$?

A partial answer is given in the following theorem:

THEOREM 4.1. *Suppose X^{**} has a w*-Sbos $\{M_i, E_i\}$ and let E_i' be the restriction of E_i to $J(X)$. If $\{R(E_i')\}$ is a Sbos for $J(X)$, then $\{R(J^{-1}E_i'J)\}$ is a shrinking Sbos for X.*

Conversely, if X has a shrinking Sbos $\{N_i, D_i\}$, then $\{R(D_i^{**})\}$ is a w-Sbos for X^{**} and if P_i is the restriction of D_i^{**} to $J(X)$, $\{R(P_i)\}$ is a Sbos for $J(X)$* (²).

(²) Day [5], p. 71, proved the second assertion of the theorem in the case where each N_i is one-dimensional.

Proof. Let J_* be the canonical map of X^* into X^{***}. If $\{M_i, E_i\}$ is a w*-Sbos for X^{**} and $\{R(E_i')\}$ is a Sbos for $J(X)$ then, by Theorem 3.1, $\{R(J_*^{-1}E_i^*J_*)\}$ is a Sbos for X^*. Clearly, $\{R(J^{-1}E_i'J)\}$ is a Sbos for X. Thus, it suffices to show that $(J^{-1}E_i'J)^* = J_*^{-1}E_i^*J_*$. Since $\{R(E_i')\}$ is a basis of subspaces for $J(X)$, $E_i'J(X) \subset J(X)$ for each $i \in \omega$; thus given $x \in X$, there is a $Y_i \in X$ such that $E_i'J(x) = J(Y_i)$. By (2.1), $E_i^*J_*(X^*) \subset J_*(X^*)$; thus, for each $f \in X^*$ there is a $g_i \in X^*$ such that $E_i^*J_*(f) = J_*(g_i)$. Hence $((J^{-1}E_i'J)^*(f))(x) = f(J^{-1}E_i'J(x)) = f(Y_i) = (J_*(f))(E_iJ(x)) = (E_i^*J_*(f))(J(x)) = g_i(x) = (J_*^{-1}E_i^*J_*(f))(x)$. It follows that $(J^{-1}E_i'J)^* = J_*^{-1}E_i^*J_*$, and so $\{R(J^{-1}E_i'J)\}$ is a shrinking Sbos for X.

Now if $\{N_i, D_i\}$ is a shrinking Sbos for X then $\{R(D_i^*)\}$ is a Sbos for X^* and hence, by Theorem 3.1, $\{R(D_i^{**})\}$ is a w*-Sbos for X^{**}. Clearly, $\{R(JD_iJ^{-1})\}$ is a Sbos for $J(X)$. Thus we need only show that $JD_iJ^{-1} = P_i$ for each $i \in \omega$. Let $x \in X$ and $f \in X^*$. Then $(P_iJ(x))(f) = (D_i^{**}J(x))(f) = f(D_i(x)) = ((JD_iJ^{-1})(J(x)))(f)$, i. e., $P_i = JD_iJ^{-1}$.

In particular, if $\{x_i\}$ is a w*-Sbov for X^{**} such that each $x_i \in J(X)$, then $\{J^{-1}x_i\}$ is a shrinking Sbov for X.

We remark that the second conjugate space of a space with a Sbov need not have a w*-Sbov. For example l^1 has a Sbov but since (m) has no Sbov, $(m)^*$ has no w*-Sbov. On the other hand, an affirmative answer to (*) together with Theorem 3.1 would imply that if X^{**} has a w*-Sbov, then X must have a Sbov.

5. Bounded w*-Schauder bases of subspaces. Recall that the bounded w*-topology (bw*-topology) for X^* is the strongest topology which coincides with the w*-topology on each set $aS^* = \{f \in X^* \mid \|f\| \leqslant a\}$.

In infinite dimensional spaces, the bw*-topology is different from the w*-topology ([5], p. 43). However, as we show in this section, the notions of w*- and bw*-Schauder bases of subspaces are equivalent.

Dieudonné [5] has shown that a basic system of neighborhoods of the origin in the bw*-topology of X^* consists of the sets $\{f \in X^* \mid |f(x_i)| < 1\}$ where $\{x_i\}$ is a sequence of elements of X converging to zero in norm. Thus X^*, with its bw*-topology, is a locally convex linear topological space.

It is known ([7], p. 428) that a linear functional F on X^* is continuous in the w*-topology if and only if it is continuous in the bw*-topology. From these facts we derive the following lemma:

LEMMA 5.1. *Let T be a linear operator from X^* into itself. Then, T is w*-continuous if and only if T is bw*-continuous.*

Proof. Suppose T is w*-continuous. Let $\{x_i\}$ be an arbitrary sequence in X converging to zero in norm and let

$$U = \{g \in X^* \mid |g(x_i)| < 1\}.$$

By (2.1), $T^*J(X) \subset J(X)$ and it follows that $J^{-1}T^*J$ is a norm-continuous linear operator on X into X. Let $V = \{f \in X^* \mid |f(J^{-1}T^*J(x_i))| < 1\}$ where $\{x_i\}$ is the sequence in X defining U. Again by (2.1), there is a $Y_i \in X$ such that $T^*J(x_i) = J(Y_i)$. Thus if $f \in V$, then $|T(f)(x_i)| = |(T^*J(x_i))(f)| = |f(J^{-1}T^*J(x_i))| < 1$, i. e., $T(f) \in U$ and so T is bw*-continuous.

Conversely, suppose T is bw*-continuous. Let $U = U(0; x_1, \ldots, x_n; \varepsilon)$ be an arbitrary basic w*-neighborhood of 0. Since T is bw*-continuous, $J(x_i)T$ is a bw*-continuous linear functional on X^* for $i = 1, \ldots, n$, and hence a w*-continuous linear functional on X^*.

Thus for the $\varepsilon > 0$ defining U and for fixed j, $1 \leqslant j \leqslant n$, there is a w*-neighborhood of 0, U_j, such that $f \in U_j$ implies $|T(f)(x_j)| < \varepsilon$. Let $U_0 = \bigcap_{j=1}^{n} U_j$. Then, $f \in U_0$ implies $T(f) \in U$ and so T is w*-continuous on X^*.

The next lemma was communicated to the author by Professor R. D. McWilliams.

LEMMA 5.2. *Let $\{g_n\}$ be a sequence in X^* and let $f \in X^*$. Then,* w*-$\lim_n g_n = f$ *if and only if* bw*-$\lim_n g_n = f$.

Proof. If w*-$\lim_n g_n = f$, then $\{\|g_n\|\}$ is bounded and so there is an $a > 0$ such that $\|f\| < a$ and $\|g_n\| < a$ for all $n \in \omega$. Let U be an arbitrary bw*-neighborhood of f. Then $U \cap aS^*$ is relatively w*-open in aS^*. Hence there is a w*-neighborhood V such that $f \in V$ and $U \cap aS^* = V \cap aS^*$. Since w*-$\lim_n g_n = f$ there is an $N \in \omega$ such that $n \geqslant N$ implies $g_n \in V \cap aS^*$, whence $g_n \in U$ and so bw*-$\lim_n g_n = f$.

By definition, the bw*-topology is a topology stronger than the w*-topology. Thus if bw*-$\lim_n g_n = f$, it follows that w*-$\lim_n g_n = f$.

THEOREM 5.3. *A sequence of non-trivial subspaces $\{N_i\}$ in X^* is a w*-Sbos for X^* if and only if $\{N_i\}$ is a bw*-Sbos for X^*.*

Proof. Suppose $\{N_i\}$ is a w*-Sbos for X^*. Let $\{P_i\}$ be the associated sequence of orthogonal projections. Then each P_i is w*-continuous and if $f \in X^*$,

$$f = \text{w*-}\lim_n \sum_{i=1}^{n} P_i(f)$$

and the expansion is unique. By Lemma 5.1, each P_i is bw*-continuous and by Lemma 5.2,

$$f = \text{bw*-}\lim_n \sum_{i=1}^{n} P_i(f).$$

It follows, again from Lemma 5.2, that the expansion is unique and so $\{N_i\}$ is a bw*-Sbos for X^*. The converse follows from Lemmas 5.1 and 5.2 in the same manner.

COROLLARY 5.4. *If $\{M_i, E_i\}$ is a Sbos for X, then $\{R(E_i^*)\}$ is a bw*-Sbos for X^*.*

In his thesis, Ruckle [11], p. 28, showed that the notions of weak- and norm-Schauder bases of subspaces are equivalent. His result, together with Theorem 5.3, suggests the following problem:

If X is a linear topological space with two locally convex topologies τ_1 and τ_2 and if (X, τ_1) and (X, τ_2) have the same continuous linear functionals, is a sequence of non-trivial subspaces $\{N_i\}$ in X a τ_1-Sbos for X if and only if it is a τ_2-Sbos for X?

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