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Two examples in the theory of topological linear spaces

by

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Singer [7] recently had occasion to consider the following three properties of a topological linear space (1):

(B) Each closed linear subspace can be strictly separated from each point of its complement by means of a closed hyperplane.

(C) Each symmetric closed convex subset can be strictly separated from each point of its complement by means of a closed hyperplane.

(D) Each closed convex subset can be strictly separated from each point of its complement by means of a closed hyperplane.

Obviously (D) $\Rightarrow$ (C) $\Rightarrow$ (B), and Singer asked whether the reverse implications are also valid. We show here that they are not. The example showing that (C) does not imply (D) is very simple, but the one showing that (B) does not imply (C) is more complicated and depends on the continuum hypothesis. Both constructions are based on the following situation.

(1) Suppose that L is the space S of all measurable functions on $[0,1]$, topologized by means of the metric

$$\varrho(x,y)=\int_0^1\frac{|x(t)-y(t)|}{1+|x(t)-y(t)|}\,dt,$$

or, equivalently, by means of convergence in measure. Alternatively, suppose that L is the space $L^p[0,1]$ with $0<p<1$, topologized by means of the quasinorm

$$\|x\|=\Big(\int_0^1|x(t)|^p\,dt\Big)^{1/p}\ (^2).$$

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(1) All of the linear spaces in this note are over the real number field $\mathfrak{R}$. A *linear topology* for a linear space is one for which both vector addition and scalar multiplication are jointly continuous. A *topological linear space* is a linear space with an associated linear topology which satisfies the T_2 separation axiom.

(2) Actually, the elements of L are equivalence classes of functions.

In each case, let P be the positive cone consisting of all $x \in L$ such that $x(t) \geqslant 0$ for almost all $t \in [0, 1]$, and let K be the symmetric set consisting of all $x \in L$ such that $|x(t)| \leqslant 1$ for almost all $t \in [0, 1]$. Let M be the linear hull of K. Then L is a separable complete metrizable topological linear space, P and K are closed convex subsets of L, and M is a dense linear subspace of L. If a linear form on L is continuous or is non-negative everywhere on P, then the form is identically zero.

All of the facts stated in (1) are well known or follow readily from the relevant definitions. See [1] and [5] for discussion of continuous linear forms on L, and [3] for non-negative linear forms.

In showing that (C) does not imply (D), we employ also the following consequence of the Hahn-Banach theorem:

(2) If a topological linear space is such that every linear form is continuous, then the space has property (C).

THEOREM. *There exists a topological linear space E which has property* (C) *and yet contains a proper closed convex cone P which cannot be separated from any point of $E \sim P$ by means of a hyperplane.*

Proof. Let L and P be as in (1) and let τ' be the original topology for L. Let τ'' be a linear topology for L such that every linear form on L is τ''-continuous. Finally, let $E = (L, \tau)$, where τ is the supremum of the topologies τ' and τ''. Use (2) to show that E has the property (C).

In showing that (B) does not imply (C), we employ the continuum hypothesis in conjunction with the following three facts.

(3) In an arbitrary topological linear space, each closed linear subspace of finite deficiency can be strictly separated from each point of its complement by means of a closed hyperplane.

(4) Suppose that τ_0 and τ_+ are two linear topologies for a linear space L, that the space (L, τ_0) is of the second category and admits no non-zero continuous linear form, and that the space (L, τ_+) is locally convex. Let τ be the supremum of the topologies τ_0 and τ_+. Then every τ-open convex subset of L is also τ_+-open.

(5) If Z is a linear subspace of infinite deficiency in a separable metrizable topological linear space M, then there are a linear subspace Z' of M and a linearly independent countable dense subset B of M such that $Z \subset Z'$ and each point of M admits a unique expression as the sum of a point of Z' and a linear combination of B.

Remark (3) follows at once from consideration of the canonical mapping onto the finite-dimensional quotient space. The proposition (4) is proved in [4], and (5) follows from reasoning in [6].

THEOREM. *There exists a topological linear space E which has property* (B) *and yet contains a symmetric closed convex proper subset K which cannot be separated from any point of $E \sim K$ by means of a closed hyperplane.*

Proof. Let L, K and M be as in (1), and let τ_0 denote the original topology for L. Let Ω denote the set of all countable ordinal numbers $\geqslant 1$. In a manner which will be described below, we are going to define a transfinite sequence

$$f_1, f_2, \ldots, f_\beta, \ldots \qquad (\beta \in \Omega)$$

of linear forms on L. When $\beta \in \Omega$ and f_α has been selected for $1 \leqslant \alpha \leqslant \beta$, we define τ_β as the coarsest topology for L which is finer than τ_0 and which renders continuous each of the linear forms f_α $(1 \leqslant \alpha \leqslant \beta)$. Clearly there is a countable basis for the class of all τ_β-open subsets of L, and hence the space (L, τ_β) is a separable metrizable topological linear space.

Having defined τ_β, we shall denote by $\mathscr{S}_\beta$ the set of all ordered pairs (Y, x) such that Y is a τ_β-closed linear subspace of infinite deficiency in L and x is a point of $L \sim Y$. Since the space (L, τ_β) is separable and metrizable, the cardinality of $\mathscr{S}_\beta$ must be that of the continuum, and thus the continuum hypothesis guarantees the existence of a biunique mapping $\sigma(\beta, \cdot)$ of Ω onto $\mathscr{S}_\beta$. Defining $Y(\beta, \gamma)$ and $x(\beta, \gamma)$ by the condition that $\sigma(\beta, \gamma) = \big(Y(\beta, \gamma),\, x(\beta, \gamma)\big)$, we have

$$\mathscr{S}_\beta = \big\{\big(Y(\beta, \gamma), x(\beta, \gamma)\big) : \gamma \in \Omega\big\}.$$

Now let the product $\Omega \times \Omega$ be well-ordered in such a way that $(0, 0)$ is the first element of $\Omega \times \Omega$ and each element of $\Omega \times \Omega$ has only countably many predecessors. Let $\zeta(0, 0) = 1$, and having defined ζ for all the predecessors of a certain element (μ, ν) of $\Omega \times \Omega$, let $\zeta(\mu, \nu)$ be the smallest $\sigma \in \Omega$ such that $\sigma > \mu$ and $\sigma > \zeta(\mu', \nu')$ whenever (μ', ν') is a predecessor of (μ, ν). Then ζ is a biunique mapping of $\Omega \times \Omega$ into Ω such that ζ is isotone with respect to the given well-ordering of $\Omega \times \Omega$ and the natural well-ordering of Ω, and in addition $\zeta(\mu, \nu) > \mu$ for all $(\mu, \nu) \in \Omega \times \Omega$. The function ζ will be used in defining the linear forms f_β.

The forms f_β are to be selected so that the following three conditions are satisfied for all $\beta \in \{0\} \cup \Omega$:

(i) $fK = \mathfrak{R}$ whenever f is a non-zero linear combination of $\{f_\alpha : 1 \leqslant \alpha \leqslant \beta\}$;

(ii) the linear subspace M is τ_β-dense in L;

(iii) if Y is a τ_β-closed linear subspace of infinite deficiency in L, then the intersection $Y \cap M$ is of infinite deficiency in M.

Note that when $\beta = 0$, (i) is vacuously satisfied and (ii) is part of (1). Note also that for all β, (ii) implies (iii) and hence it will suffice to check (i) and (ii). Indeed, suppose that $Y \cap M$ is of finite deficiency in L, whence there is a finite-dimensional linear subspace F of L such that $M \subset Y+F$. With Y closed and F finite-dimensional, it follows that $Y+F$ is closed, whence (by (ii)) $Y+F = L$ and Y is of finite deficiency. The contradiction shows that (ii) implies (iii).

Now we are ready, at last, to select the linear forms f_β. We shall begin with f_1, recalling that $\zeta(0,0)=1$. The set $Y(0,0)$ is a τ_0-closed linear subspace of infinite deficiency in L, and $x(0,0)$ is a point of $L \sim \sim Y(0,0)$. Let $\mathfrak{Q}$ denote the set of all rational numbers. With the aid of (iii) and (5) we can verify the existence of a linear subspace Y_1 of L and an indexed family $\{B_1^q\colon q \in \mathfrak{Q}\}$ of pairwise disjoint τ_0-dense subsets of M such that $Y(0,0) \subset Y_1$, each set B_1^q intersects K, and each point of L admits a unique expression as the sum of a point of Y_1, a multiple of $x(0,0)$, and a linear combination of $\bigcup_{q\in\mathfrak{Q}} B_1^q$. (Thus the set $\{x\} \cup \bigcup_{q\in\mathfrak{Q}} B_1^q$ is a Hamel basis for a linear subspace supplementary to Y_1 in L). The linear form f_1 is defined by the requirements that $f_1 = 0$ on Y_1, $f_1(x(0,0)) = 1$, and $f_1 = q$ on B_1^q. Since each set B_1^q intersects K, we have $fK \supset \mathfrak{Q}$ and hence $fK = \mathfrak{R}$ because K is convex. Thus condition (i) is satisfied when $\beta = 1$. To establish condition (ii), we note that a basis for the non-empty τ_1-open subsets of L is the family of all sets of the form $G \cap f^{-1}]a,b[$, where $a<b$ and G is a non-empty τ_0-open subset of L. Consider an arbitrary set of this sort, and choose $q \in]a,b[\cap \mathfrak{Q}$. Since M is τ_0-dense in L, the intersection $G \cap M$ is a τ_0-open subset of M, and then since B_1^q is a τ_0-dense subset of M there is a point of $G \cap M$ at which f_1 has the value $q \in]a,b[$. It follows that M is τ_1-dense in L, and condition (ii) is satisfied when $\beta = 1$.

Now suppose that $2 \leqslant \gamma \in \Omega$ and that the linear forms f_β have been defined for $1 \leqslant \beta < \gamma$ so that conditions (i), (ii) and (iii) are satisfied in addition to the following conditions:

(iv) if the ordinal number β is not in the range of the function ζ, then the linear form f_β is identically zero;

(v) if $\zeta(\mu,\nu) = \beta$ (whence $\mu < \beta$, $Y(\mu,\nu)$ is a τ_μ-closed linear subspace of infinite deficiency in L, and $x(\mu,\nu)$ is a point of $L \sim Y(\mu,\nu)$), then $f_\beta = 0$ on $Y(\mu,\nu)$ and $f_\beta(x(\mu,\nu)) = 1$.

We will show that a linear form f_γ can be selected in such a way that the conditions (i)-(v) are all satisfied when $\beta = \gamma$ as well. It will then follow by transfinite induction that the entire transfinite sequence f_β $(\beta \in \Omega)$ can be defined so as to satisfy conditions (i)-(v).

If γ is not in the range of the function ζ, let the linear form f_γ be identically zero. Then the necessary verification is trivial. Suppose, on the other hand, that $\gamma = \zeta(\mu,\nu)$ with $\mu < \gamma$. Then the selection of f_γ is very similar to that of f_1, with the roles of $Y(0,0)$ and $x(0,0)$ being played by $Y(\mu,\nu)$ and $x(\mu,\nu)$ respectively. However, some additional care is necessary in connection with condition (i), and a slight change in the discussion of (ii) is required in case γ is a limit ordinal.

Let τ_γ^- denote the supremum of the topologies τ_β for $\beta < \gamma$. It follows from the inductive hypothesis (condition (ii)) that M is τ_γ^--dense in L. With the aid of (iii) and (5) it is possible to produce a linear subspace Y_γ of L and an indexed family $\{B_\gamma^q\colon q \in \mathfrak{Q}\}$ of pairwise disjoint τ_γ^--dense subsets of M such that $Y(\mu,\nu) \subset Y_\gamma$, each set B_γ^q intersects K, and each point of L admits a unique expression as the sum of a point of Y_γ, a multiple of $x(\mu,\nu)$, and a linear combination of $\bigcup_{q\in\mathfrak{Q}} B_\gamma^q$. For each $q \in \mathfrak{Q}$, select a single point $b_\gamma^q \in B_\gamma^q \cap K$, whence of course $B_\gamma^q \sim \{b_\gamma^q\}$ is still a τ_γ^--dense subset of M. Let the members of $\mathfrak{Q}$ be arranged in a sequence $q_1, q_2, \ldots$ and let the set of all ordinals $< \gamma$ be arranged in a sequence $\beta_1, \beta_2, \ldots$ (³), For $n = 1, 2, \ldots$, let

$$r_n = (-1)^n n^2 \max\left(1, |f_{\beta_1}(b_\gamma^{q_1})|, \ldots, |f_{\beta_n}(b_\gamma^{q_n})|\right).$$

Finally, let the linear form f_γ be defined by the requirements that $f_\gamma = 0$ on Y_γ, $f_\gamma(x(\mu,\nu)) = 1$, $f_\gamma = q$ on $B_\gamma^q \sim \{b_\gamma^q\}$, and $f_\gamma(b_\gamma^{q_n}) = r_n$. The conditions (ii)-(v) are clearly satisfied by f_γ, and condition (i) with $\beta = \gamma$ follows from the fact that on suitable subsequences of the sequence $b_\gamma^{q_1}, b_\gamma^{q_2}, \ldots$, the function f_γ converges to ∞ or $-\infty$ more rapidly than does any linear combination of the functions f_{β_1}. Thus the transfinite sequence f_β $(\beta \in \Omega)$ can be constructed as desired.

Now let τ_Ω denote the supremum of all the topologies τ_β for $\beta \in \Omega$, and let $E = (L, \tau_\Omega)$. Clearly E is a topological linear space and K is a symmetric closed convex proper subset of E. Since the space (L, τ_0) is complete metric and admits no non-zero continuous linear form, it follows from (4) that every continuous linear form on E is a linear combination of $\{f_\beta\colon \beta \in \Omega\}$. Then condition (i) implies that every continuous linear form on E maps the set K onto the entire real number field $\mathfrak{R}$ and hence K cannot be separated from any point of $E \sim K$ by means of a closed hyperplane. It remains only to show that the space E has property (B); that is, that if Y is a closed linear subspace of E and x is a point of $E \sim Y$, then Y can be separated from x by means of a closed hyperplane. For each $\beta \in \Omega$, let Y^β denote the τ_β-closure of Y. Then $Y = \bigcap_{\beta\in\Omega} Y^\beta$, and hence there exists $\beta \in \Omega$ such that $x \notin Y^\beta$. If Y^β is of finite deficiency in L, we deduce from (3) that Y^β (and hence Y) can be strictly separated from x by means of a τ_β-closed (and hence τ_Ω-closed) hyperplane. If Y^β is of infinite deficiency then $(Y^\beta, x) = \sigma(\beta, \gamma)$ for some $\gamma \in \Omega$, and then $f_{\zeta(\beta,\gamma)}$ is a linear form defining a $\tau_{\zeta(\beta,\gamma)}$-closed (and hence τ_Ω-closed) hyperplane which strictly separates Y from x. This completes the proof.

(³) These are ordinary sequences, of order type ω.

It is unknown whether (B) implies (C) or (C) implies (D) under the hypothesis that the space in question is the congugate space E^* of a locally convex space E, in the aw^*-topology for E^* (see [2] and [7]).

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