On generalized derivatives

by

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In the theory of trigonometric series the Riemann derivative plays an important rôle. Repeated formal integration of a Cesàro summable trigonometric series provides an expression which treated by the Riemann derivative yields the sum of the series. This fact leads naturally to a definition of generalized integrals by means of which expressions, analogous to the classical Fourier formulae, are obtained for the coefficients of the trigonometric series. Basic work is due to Denjoy [2]. For pertinent developments see [1], [4], [5], [6], [7] and [8].

In this paper we consider a derivative which is a further generalization of the generalized derivative considered by Denjoy [3]. The existence of Denjoy's n-th generalized derivative implies that of our $(n-1)$-st derivative under the conditions of our theorem.

Let f be a real-valued function over D, a subset of the reals. For any integer $n>0$ and any set $P\subset D$ consisting of $n+1$ distinct points $p_1, \ldots, p_{n+1})$, the n-th divided difference of f corresponding to P, $\Delta_n(f:P)$, is given by

$$\Delta_n(f:P) = \Delta_n(f:p_1, \ldots, p_{n+1}) = \sum_{j=1}^{n+1} f(p_j)/\{(z-p_1)\ldots(z-p_{n+1})\}'_{z=p_j},$$

the prime denoting ordinary differentiation.

Let now D be the interval (a, b) and f be continuous over D. Denote by E the set consisting of $n+1$ points $x_1, \ldots, x_{n+1}$. If

$$\lim_{h\to 0} n!\,\Delta_n(f: x+x_1h, \ldots, x+x_{n+1}h)$$

exists and is finite, it is called, following Denjoy [3], the *n-th E-generalized derivative* of f at x, and denoted by $f_{n,E}(x)$. If $f_{n,E}(x)$ is independent of the choice of $E\subset S$ for a subset S of the reals, we will call it the *n-th S-generalized derivative* and denote it by $f_{n,S}(x)$. When E consists of the points $-n, \ldots, 2j-n, \ldots, n$, for $j=0, \ldots, n$, then $f_{n,E}(x)$ becomes the *n-th Riemann derivative* $D^nf(x)$.

Let f and E be as above. Denote by A any n-tuple of points of E.

THEOREM. *Suppose that $f_{n,E}(x)$ exists. If $f_{n-1,A}(x)$ exists, then $f_{n-1,E}(x)$ exists and they are equal.*

Proof. Suppose that A and B are any two distinct subsets of E consisting of n points. Since they must have $n-1$ points in common, a suitable permutation of the elements of E will reduce this to the case in which A consists of the last n points of E and B consists of the first n points of E. Then, the theorem follows from the relation

$$(n-1)!\,\Delta_{n-1}(f\colon x+x_2h,\ldots,x+x_{n+1}h)-\\ -(n-1)!\,\Delta_{n-1}(f\colon x+x_1h,\ldots,x+x_nh)\\ = hn^{-1}(x_{n+1}-x_1)\big(n!\,\Delta_n(f\colon x+x_1h,\ldots,x+x_{n+1}h)\big).$$

The following three corollaries are direct consequences of the above-mentioned theorem.

COROLLARY 1. *If $D^2f(x)$ and $D^1f(x)$ exist, then $f'(x)$ exists also and $f'(x)=D^1f(x)$.*

This corollary is essentially Lemma 4.1 in [6].

COROLLARY 2. *Suppose that $f_{n,S}(x)$ exists and A is any n-tuple of points of S. If $f_{n-1,A}(x)$ exists, then $f_{n-1,S}(x)$ exists and they are equal.*

Remark. Let B be any n-tuple of points of S, $b\in B-A$ and $a\in A-B$. Since $f_{n-1,A}(x)$ and $f_{n,A\cup\{b\}}(x)$ exist by hypothesis, the theorem given above implies that $f_{n-1,(A\cup\{b\})-\{a\}}(x)$ exists and is equal to $f_{n-1,A}(x)$. Hence, in order to show Corollary 2, it suffices to apply the above-mentioned argument $n-m$ times $(0\leqslant m\leqslant n)$, where m is the number of points of $A\cap B$.

COROLLARY 3. *Suppose that $f_{n,S}(x)$ exists. Let S_{n-i+1} be any $(n-i+1)$-tuple of points of S $(1\leqslant i\leqslant n-1)$. If, for i fixed, $f_{n-1,S_n}(x)$, $f_{n-2,S_{n-1}}(x)$, $\ldots$, $f_{n-i,S_{n-i+1}}(x)$ exist, then $f_{n-1,S}(x)$, $f_{n-2,S}(x)$, $\ldots$, $f_{n-i,S}(x)$ exist also, and they are equal respectively to $f_{n-1,S_n}(x)$, $f_{n-2,S_{n-1}}(x)$, $\ldots$, $f_{n-i,S_{n-i+1}}(x)$.*

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