Concerning function algebras*

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1. Introduction. According to Morera's theorem, the collection of functions which are continuous on the closure of the open unit disk U, and such that $\int_C f dz = 0$ for each f in the collection and for each simple closed rectifiable curve C contained in U, is the same as the set of functions that are continuous on $\overline{U}$, and analytic at each point of U. This collection forms a Banach algebra under supremum norm and the usual pointwise operations.

Suppose now that instead of taking the usual line integral, that we integrate with respect to a continuous function of bounded variation. Specifically, let g be continuous on $\overline{U}$, and of bounded variation on each simple closed rectifiable curve C contained in U. We denote by $A(g)$, the collection of all functions which are continuous on $\overline{U}$, and such that $\int_C f dg = 0$ for each simple closed rectifiable curve contained in U. The collection $A(g)$ will always be a Banach space under supremum norm, and the usual pointwise operations. It is not known, however, whether or not $A(g)$ is always an algebra. The study of sufficient conditions for $A(g)$ to be an algebra, will be one of our main concerns in this paper.

Our first result characterizes the integrator g as being a continuous function on $\overline{U}$ that satisfies a uniform Lipschitz condition on each compact subset of U. The characterization plays an important role in the results that follow.

An important subset of $A(g)$, denoted by $L(g)$, is the collection of those functions in $A(g)$ which satisfy a uniform Lipschitz condition on each compact subset of U. It is shown in Section 3, that the product $A(g)\cdot L(g)$ is contained in $A(g)$. In particular, $L(g)$ is an algebra contained in $A(g)$. In some cases $L(g)$ is all of $A(g)$, or at least is a dense subset of

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$A(g)$. For these cases it follows that $A(g)$ is an algebra. An example is given to show that $L(g)$ need not always be dense in $A(g)$.

Special attention is given in Section 4 to the case in which g is locally one-to-one at a point in U. In a neighborhood in which g is one-to-one, each function in $A(g)$ can be represented as an analytic function of g by a formula analogous to the Cauchy integral formula. The proof of this theorem is somewhat involved and depends on theorems on boundary values of analytic functions.

If the restriction of g to U is a light open map, then g is locally one-to-one in U, except at isolated points. In this case, the collection $A(g)$ is an algebra, and, moreover, it can be identified with some analytic function algebra.

It is not always the case that $A(g)$ will have a structure resembling that of an analytic function algebra. If we let $g(z) = R_e z$, then $A(g)$ can be identified with the algebra of all continuous functions on the interval $[-1, +1]$.

2. Characterization of the integrator. We now prove

THEOREM 2.1. *Suppose g is continuous on $\overline{U}$. Then a necessary and sufficient condition that g be of bounded variation on each simple closed rectifiable curve in U, is that g satisfy a uniform Lipschitz condition on each compact subset of U, i.e., that for each compact subset E contained in U, there exists a constant K_E, depending only on E and g, such that $|g(z)-g(z')| \leqslant K_E|z-z'|$ for all z and z' in E.*

Proof. That the condition is sufficient is immediately clear. It is also clear, that in order to prove the converse it will suffice to show that the condition holds for each closed disk in U.

Accordingly, suppose E is a closed disk in U, for which the condition fails to hold. Then for each positive integer n there exists points z_n and z'_n in E for which

$$|g(z_n)-g(z'_n)| > n|z_n-z'_n|. \tag{2.1}$$

We note at this time that if $\{n_j\}_{j=1}^{\infty}$ is a subsequence of the sequence of positive integers, and if $s_j = z_{n_j}$ and $s'_j = z'_{n_j}$, then

$$|g(s_j)-g(s'_j)| > j|s_j-s'_j|.$$

That is, inequality (2.1) is preserved under the taking of subsequences.

Let z_0 be a cluster point of $\{z_n\}_{n=1}^{\infty}$. There exists a subsequence $\{s_j\}_{j=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ that converges to z_0. Let z'_0 be an arbitrary cluster point of $\{s'_j\}_{j=1}^{\infty}$. Then $z_0 = z'_0$; for if not, then there exists subsequences $\{t_k\}_{k=1}^{\infty}$ and $\{t'_k\}_{k=1}^{\infty}$ of $\{s_j\}_{j=1}^{\infty}$ and $\{s'_j\}_{j=1}^{\infty}$, respectively, such that

$$\lim_{k\to\infty} t_k = z_0, \quad \lim_{k\to\infty} t'_k = z'_0,$$

and

$$|g(t_k)-g(t'_k)| > k|t_k-t'_k|.$$

It is easy to see that this contradicts the continuity of g.

Thus, in view of the note already made, we can assume without loss of generality that the sequences $\{z_n\}_{n=1}^{\infty}$ and $\{z'_n\}_{n=1}^{\infty}$ each have limit z_0, for some z_0 in E. We can also assume for each $n = 1, 2, 3, \ldots$ that

$$|z_n-z_0| < \frac{1}{2n^3} \quad \text{and} \quad |z'_n-z_0| < \frac{1}{2n^3}.$$

We now observe for each $n = 1, 2, 3, \ldots$ that there exists open sets W_n and W'_n containing z_n and z'_n respectively, such that if ξ_n is in W_n and ξ'_n is in W'_n, then $|g(\xi_n)-g(\xi'_n)| > n|\xi_n-\xi'_n|$. This fact follows easily from the continuity of g. It therefore follows that we can assume for each $n = 1, 2, 3, \ldots$ that the points z_n, z_0, and z'_n are not on the same straight line.

Again by taking subsequences if necessary, we can assume that the straight line segment $\overline{z_n z'_n}$ lies inside a circular neighborhood N_n centered at z_0, whose closure $\overline{N}_n$ misses the straight line segment $\overline{z_{n-1}z'_{n-1}}$, for $n = 2, 3, \ldots$

For each $n = 1, 2, 3, \ldots$ let k_n be the positive integer such that $k_n|z_n-z'_n| \leqslant 1/n^2$ and $(k_n+1)|z_n-z'_n| > 1/n^2$. Since $|z_n-z'_n| < 1/n^3$, then $n^2|z_n-z'_n| < 1/n$, and

$$1-n^2|z_n-z'_n| > 1-\frac{1}{n}.$$

On the other hand, $1 \geqslant n^2k_n|z_n-z'_n| > 1-n^2|z_n-z'_n|$. Therefore, $1 \geqslant n^2k_n|z_n-z'_n| > 1-1/n$ and

$$\lim_{n\to\infty} n^2k_n|z_n-z'_n| = 1.$$

From this we conclude that

$$\sum_{n=1}^{\infty} k_n|z_n-z'_n| < \infty,$$

but that

$$\sum_{n=1}^{\infty} nk_n|z_n-z'_n| = \infty.$$

Since the straight line segment $\overline{z_n z'_n}$ lies in the open set $N_n-\overline{N}_{n+1}$, we can find points $z_{n,p}$, $p = 1, \ldots, k_n$, on the line through z_n perpendicular

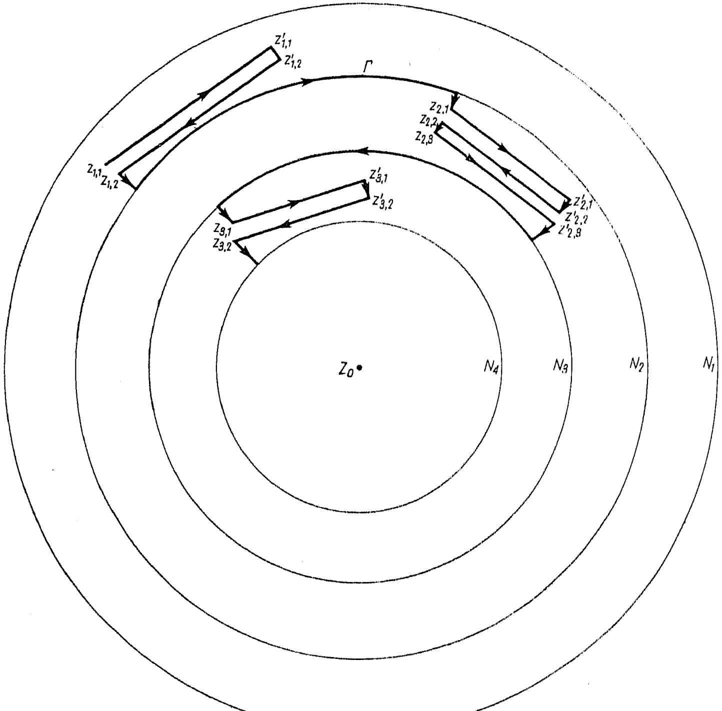

Fig. 1

to $\overline{z_n z'_n}$, and points $z'_{n,p}$ such that the line segment $\overline{z_{n,p} z'_{n,p}}$, $p = 1, \ldots, k_n$, is parallel to $z_n z'_n$, contained in $N_n - \overline{N}_{n+1}$, and such that $|g(z_{n,p}) - g(z'_{n,p})| > n|z_{n,p} - z'_{n,p}|$.

We now let Γ be the arc obtained by connecting the line segments $\overline{z'_{n,p} z_{n,p}}$ as shown in Figure 1. Γ will be rectifiable; in fact, of length less than

$$\sum_{n=1}^{\infty} k_n |z_n - z'_n| + 2\sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{2\pi}{n^3}.$$

This is seen as follows: The sum of the lengths of the line segment $\overline{z'_{n,p} z'_{n,p}}$ is equal to $\sum_{n=1}^{\infty} k_n |z_n - z'_n|$.

The sum of the lengths of the line segments connecting the k_n line segments $\overline{z_{n\,p} z'_{n,p}}$ for a fixed n, is less than $\sum_{n=1}^{\infty} 1/n^3$. The sum of the lengths of the connecting line segments that are directed toward the origin is less than $\sum_{n=1}^{\infty} 1/n^3$; and, finally, the sum of the connecting arcs that are on the boundaries of the neighborhoods $\{N_n\}_{n=1}^{\infty}$, is less than $\sum_{n=1}^{\infty} 2\pi/n^3$.

However, the variation of g over Γ is unbounded. It is at least as large as the sum of the variations of g over the line segments $\overline{z_{n,p} z'_{n,p}}$; $p = 1, \ldots, k_n$, and $n = 1, 2, 3, \ldots$, and the variation of g over each of these is at least as large as

$$|g(z_{n,p}) - g(z'_{n,p})| > n|z_{n,p} - z'_{n,p}|.$$

Thus the variation of g over Γ is at least as large as

$$\sum_{k=1}^{\infty} n k_n |z_n - z'_n| = \infty.$$

Since we have found one rectifiable arc Γ contained in E over which the variation of g is unbounded, it follows that there exists a simple closed rectifiable curve in E for which the variation of g is unbounded. Thus we have a contradiction, and the theorem is proved.

3. The algebra $L(g)$. We recall from Section 1, that the collection of functions in $A(g)$ which satisfy a uniform Lipschitz condition on each compact subset of U, will be denoted by $L(g)$. It follows from a classical Stieltjes integration formula that the functions $1, g, g^2, \ldots$ are also in $A(g)$, and from Theorem 2.1 that these functions are in $L(g)$.

LEMMA 3.1. *Suppose f is continuous on $\overline{U}$. Then $f \in A(g)$ if and only if $\int_T f dg = 0$, for each triangle T contained in U.*

Proof. Certainly if $f \in A(g)$, then $\int_T f dg = 0$ for each triangle T contained in U. We now show that the converse is true.

Let C be a simple closed rectifiable curve in U. There is a closed disk E contained in U, and containing C in its interior. By Theorem 2.1, there is a $K > 0$ such that for z and z' in E, we have $|g(z) - g(z')| \leqslant K|z - z'|$.

Let $\varepsilon > 0$ be given. There exists a partition $\{\xi_k\}_{k=0}^{n}$ of C such that if $\{t_j\}_{j=0}^{m}$ is a refinement of $\{\xi_k\}_{k=0}^{n}$, with t'_j being any point in the arc from t_{j-1} to t_j on C, then

$$\Big|\int_C f dg - \sum_{j=1}^{n} f(t'_j)[g(t_j) - g(t_{j-1})]\Big| < \frac{\varepsilon}{2}.$$

Since f is uniformly continuous on E, then there exists a $\delta > 0$ such that whenever z and z' are in E, and $|z-z'| < \delta$, then

$$|f(z)-f(z')| < \frac{\varepsilon}{2|C|K},$$

where $|C|$ is the length of curve C.

For two arbitrary points z and z' in the plane we let $\overline{zz'}$ denote the straight line segment joining z and z'. Since C is a simple closed curve, then there exists a refinement $\{t_j\}_{j=0}^{m}$ of $\{\xi_k\}_{k=0}^{n}$ such that

$$C^* = \bigcup_{j=1}^{m} \overline{t_{j-1}t_j}$$

is a simple closed polygonal curve and $|t_{j-1}-t_j| < \delta$, for $j = 1, 2, \ldots, m$. Then $\int_{C^*} fdg = 0$ since $\int_T fdg = 0$ for each triangle T contained in T, and hence

$$\Big|\int_C fdg\Big| \leqslant \Big|\int_C fdg - \sum_{j=1}^{m} f(t_j)[g(t_j)-g(t_{j-1})]\Big| + $$
$$+\Big|\sum_{j=1}^{m} f(t_j)[g(t\ )-g(t_{j-1})] - \sum_{j=1}^{m} \int_{\overline{t_{j-1}t_n}} fdg\Big|.$$

Therefore,

$$\Big|\int_C fdg\Big| < \frac{\varepsilon}{2} + \sum_{j=1}^{m}\Big|\int_{\overline{t_{j-1}t_j}} [f-f(t_j)]\,dg\Big|$$
$$\leqslant \frac{\varepsilon}{2} + \sup_{t\in\overline{t_{j-1}t_j}} \{f(t)-f(t_j)\}K\sum_{j=1}^{m}|t_j-t_{j-1}| \leqslant \varepsilon.$$

From this, the lemma now follows.

THEOREM 3.1. *The product $A(g)\cdot L(g)$ is contained in $A(g)$.*

Proof. The proof of this theorem is based on the technique used by E. Goursat to prove the Cauchy integral theorem.

Let $f\in A(g)$ and $h\in L(g)$. By Lemma 3.1, the function fh will be in $A(g)$, if $\int_T fhdg = 0$ for each triangle T contained in U. Let T_0 be one such triangle. By joining the midpoints of the sides of T_0, we obtain four congruent triangles T_{11}, T_{12}, T_{13}, and T_{14}, such that

$$\int_{T_{11}} fhdg + \int_{T_{12}} fhdg + \int_{T_{13}} fhdg + \int_{T_{14}} fhdg = \int_{T_0} fhdg.$$

For one of the triangles, which we now denote by T_1,

$$\Big|\int_{T_1} fhdg\Big| \geqslant \frac{1}{4}\Big|\int_{T_0} fhdg\Big|.$$

This provides the first step in an easy induction argument to get that there is a nested sequence of triangles $\{T_n\}_{n=0}^{\infty}$ with the following properties:

(i) $|T_n| \leqslant \frac{1}{2^n}|T_0|$, where $|T_n|$ is the length of the perimeter of T_n.

(ii) $d(T_n) \leqslant \frac{1}{2^n} d(T_0)$, where $d(T_n)$ is the length of the diameter of T_n.

(iii) $\Big|\int_{T_n} fhdg\Big| \geqslant \frac{1}{4^n}\Big|\int_{T_0} fhdg\Big|$.

There will be one point, which we denote by z_0, which is interior to each of the triangles. Since f and h are in $A(g)$, then

$$\int_{T_n} fhdg = \int_{T_n} [f-f(z_0)][h-h(z_0)]\,dg.$$

Since h and g are in $L(g)$, and the triangles are all contained in a compact subset of U, there is a constant $K < 0$ such that

$$|g(z)-g(z_0)| \leqslant K|z-z_0|, \quad \text{and} \quad |h(z)-h(z_0)| \leqslant K|z-z_0|$$

for $z\in T_n$, $n = 0, 1, 2, \ldots$

Let $\varepsilon > 0$ be given. There is a $\delta > 0$ such that if $|z-z_0| < \delta$, then $|f(z)-f(z_0)| < \varepsilon$. Let n be large enough so that T_n is contained in the δ-neighborhood of z_0. Then

$$\frac{1}{4^n}\Big|\int_{T_0} fhdg\Big| \leqslant \Big|\int_{T_n} [f-f(z_0)][h-h(z_0)]\,dg\Big| \leqslant \varepsilon\cdot K\cdot \sup_{t\in T_n}\{|t-z_0|\}\cdot V(g; T_n)$$

where $V(g; T_n)$ is the variation of g over T_n. Thus

$$\frac{1}{4^n}\Big|\int_{T_0} fhdg\Big| \leqslant \varepsilon\cdot K\cdot d(T_n)\cdot K\cdot|T_n| \leqslant \varepsilon\cdot K^2\cdot\frac{1}{4^n} d(T_0)\cdot|T_0|.$$

Hence

$$\Big|\int_{T_0} fhdg\Big| \leqslant \varepsilon\cdot K^2\cdot d(T_0)\cdot|T_0|$$

and the theorem follows.

COROLLARY 3.1. *$L(g)$ is an algebra.*

Proof. This follows immediately from Theorem 3.1, and the fact that the product of two Lipschitzian functions is again a Lipschitzian function.

Remark 3.1. In the classical case, where $g(z) = z$, then $L(g)$ is actually the same as $A(g)$. In the case where $g(z) = x = \operatorname{Re} z$, it is not difficult to show that each $f\in A(g)$ is constant on the lines parallel to the

Y-axis, and that $A(g)$ can be identified with the algebra $C([-1,+1])$ of all continuous functions on the interval $[-1,+1]$. In this case, $L(g)$ is certainly not all of $A(g)$. However, $L(g)$ is dense in $A(g)$, since $L(g)$ contains the polynomials in x, and these are dense in $C([-1,+1])$. This might lead one to suspect that $L(g)$ is always dense in $A(g)$. The following example shows, however, that this is not the case. We let $h(z)=z/\sqrt{|z|}$, and let $g(z)=h^2(z)=z^2/|z|$. It is not difficult to show that $A(g)$ consists of all functions $F(h)$, such that F is continuous on $\overline{U}$ and analytic at each point of T. Now h is not in $L(g)$, but $h^n \in L(g)$ for $n=2,3,\ldots$ The function h cannot be uniformly approximated on $\overline{U}$ by functions in $L(g)$.

4. The analytic nature of $A(g)$. We now consider the behavior of functions in $A(g)$ near points at which g is locally one-to-one. The proofs of the theorems given here would be simpler if we were to assume that g^{-1} (the inverse of g in a region where g is one-to-one) as well as g, is of bounded variation on simple closed rectifiable curves. We do not, however, make this assumption.

THEOREM 4.1. *Suppose E is a closed disk contained in U, and that the restriction of g to E, $g\mid E$, is homeomorphism. Suppose C is any simple closed rectifiable curve in E, and that z_0 is a point inside C. Then for any $f \in A(g)$,*

$$f(z_0)=\pm\frac{1}{2\pi i}\int_C \frac{f\,dg}{g-g(z_0)},$$

the sign being taken as plus or minus depending on whether $g\mid E$ is a positive or negative homeomorphism.

Proof. Let D be an arbitrary simple closed rectifiable curve in E such that z is not in D. If $g\mid E$ is a positive homeomorphism, then

$$\int_D \frac{dg}{g-g(z_0)}=\int_{g(D)} \frac{dw}{w-w_0},$$

where $w_0=g(z_0)$. The Brouwer invariance of domain theorem ([2], p. 95) gives us that $g(z_0)$ is inside $g(D)$ if and only if z_0 is inside D. Therefore,

$$\int_C \frac{dg}{g-g(z_0)}=2\pi i \text{ or } 0,$$

depending on whether or not z_0 is inside D. If $g\mid E$ is a negative homeomorphism, then

$$\int_C \frac{dg}{g-g(z_0)}=-2\pi i \text{ or } 0.$$

Let D' denote the union of D and the inside of D. If $z_0 \in E-D'$, then it follows from a polynomial approximation of Walsh ([5], p. 430), that there exists a sequence of polynomials in g, $\{P_n(g)\}_{n=1}^{\infty}$, which converges uniformly on D' to $1/(g-g(z_0))$.

By Theorem 3.1, $fg^n \in A(g)$ for $n=1,2,3,\ldots$ Therefore

$$\int_D fP_n(g)\,dg=0,$$

and hence

$$\frac{1}{2\pi i}\int_D \frac{f\,dg}{g-g(z)}=0.$$

As in the proof of the classical Cauchy integral formula, we conclude for any circle C_0 centered at z_0, and contained inside C, that

$$\frac{1}{2\pi i}\int_C \frac{f-f(z_0)}{g-g(z_0)}\,dg=\frac{1}{2\pi i}\int_{C_0} \frac{f-f(z_0)}{g-g(z_0)}\,dg.$$

The remainder of the proof is devoted to showing that

$$\frac{1}{2\pi i}\int_C \frac{f-f(z_0)}{g-g(z_0)}\,dg=0,$$

from which we get the formula

$$f(z_0)=\pm\frac{1}{2\pi i}\int_C \frac{f\,dg}{g-g(z_0)}.$$

Let

$$\theta=\frac{f-f(z_0)}{g-g(z_0)},$$

and let $\{C_n\}_{n=1}^{\infty}$ be a nested sequence of circles centered at z, contained inside C, and having radii tending to zero. Since $fg^n \in A(g)$ for $n=1,2,\ldots$ then $\theta\cdot[g^m-g^m(z_0)] \in A(g)$ for $m=1,2,\ldots$, and hence

$$\frac{1}{2\pi i}\int_C \theta\cdot g^m\,dg=g^m(z_0)\frac{1}{2\pi i}\int_{C_n}\theta\,dg.$$

Let C_n' denote the union of C_n and the inside of C_n. There exists a one-to-one mapping, h_n, taking $g(C_n')$ onto $\overline{U}$ and z_0 onto 0, that is analytic on the interior of $g(C_n')$ ([6], p. 290). Moreover, the restriction of h_n^{-1} to the unit circle is absolutely continuous ([6], p. 293).

Since $h_n(g)$ can be uniformly approximated on C_n' by polynomials in g ([5], p. 430), it follows that

$$\frac{1}{2\pi i}\int_{C_n} \theta h_n^p(h)\,dg = 0 \quad \text{for} \quad p = 1, 2, 3, \ldots$$

Let $w = h_n(g)$. The functions θ and $h_n^p(g)$ are continuous on C_n, and h_n is one-to-one on $g(C_n)$. Therefore

$$\frac{1}{2\pi i}\int_{|w|=1} \theta\Big(g^{-1}\big(h_n^{-1}(w)\big)\Big) w^p\,dh_n^{-1}(w) = 0,$$

and

$$\text{(4.1)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \theta\Big(g^{-1}\big(h_n^{-1}(e^{it})\big)\Big) e^{ipt}[h_n^{-1}(e^{it})]'\, e^{it}dt = 0.$$

Let $a_n(t) = \theta\Big(g^{-1}\big(h_n^{-1}(e^{it})\big)\Big)[h_n^{-1}(e^{it})]'e^{it}$. It follows from equation (4.1) that a_n is the almost everywhere (with respect to Lebesgue measure) radial limit of a function, say A_n, in the class H^1. In particular ([1], p. 51)

(i) $$\lim_{r\to 1} A_n(re^{it}) = a_n(t) \quad \text{a. e.},$$

(ii) $$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} |A_n(re^{it}) - a_n(t)|\,dt = 0.$$

From (i), we get that

$$\lim_{r\to 1} A_n(re^{it}) = \Big[f\Big(g^{-1}\big(h_n^{-1}(e^{it})\big)\Big) - f(z_0)\Big]\frac{[h_n^{-1}(e^{it})]'\,e^{it}}{h_n^{-1}(e^{it}) - g(z_0)} \quad \text{a. e.}$$

For each $p = 1, 2, 3, \ldots$,

$$\frac{1}{2\pi}\int_0^{2\pi} [f\Big(g^{-1}\big(h_n^{-1}(e^{it})\big)\Big) - f(z_0)]e^{ipt}dt$$

$$= \frac{1}{2\pi i p}\int_{|w|=1} f\Big(g^{-1}\big(h_n^{-1}(w)\big)\Big)\,dw^p = \frac{1}{2\pi i p}\int_{C_n} f\,dh_n^p(g) = 0.$$

It follows that $f\Big(g^{-1}\big(h_n^{-1}(e^{it})\big)\Big) - f(z)$ is the almost everywhere radial limit of an H^1 function, say F_n.

We now show that $[h_n^{-1}(e^{it})]'e^{it}$ is the almost everywhere radial limit of an H^1 function. For $p = 1, 2, 3, \ldots$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ipt}[h_n^{-1}(e^{it})]'e^{it}dt = \frac{1}{2\pi i}\int_0^{2\pi} e^{ipt}dh_n^{-1}(e^{it})$$

$$= \frac{-p}{2\pi}\int_0^{2\pi} h_n^{-1}(e^{it})e^{ipt}dt = \begin{cases} 0, & \text{if } p > 0, \\ qb_q, & \text{if } q = -p, \end{cases}$$

where

$$b_q = \frac{1}{2\pi}\int_0^{2\pi} h_n^{-1}(e^{it})e^{-iqt}dt.$$

It follows that

$$\lim_{r\to 1}\sum_{p=0}^{\infty} pb_p r^p e^{ipt} = [h_n^{-1}(e^{it})]'e^{it} \quad \text{a. e.},$$

and that

$$\sum_{p=0}^{\infty} pb_p r^p e^{ipt}$$

is an H^1 function. Moreover,

$$\sum_{p=0}^{\infty} pb_p r^p e^{ipt} = z\frac{d}{dz}\Big(\sum_{p=0}^{\infty} b_p z^p\Big) = z\frac{d}{dz}h_n^{-1}(z),$$

and so $[h_n^{-1}(e^{it})]'e^{it}$ is the almost everywhere radial limit of

$$z\frac{d}{dz}h_n^{-h}(z).$$

All H^1 functions are in the class of beschränkartige functions, denoted by N in [6] (p. 272). Moreover, a function F, analytic on U, is in the class N if and only if F is the ratio of two bounded analytic functions on U ([6], p. 277). If two functions in N have radial limits that agree on a set of positive Lebesgue measure, then they are identical ([6], p. 276). It therefore follows that

$$A_n(z) = F_n(z)\cdot\frac{z\dfrac{d}{dz}[h_n^{-1}(z)]}{h_n^{-1}(z) - h_n^{-1}(0)}.$$

We note that $A_n(0) = F\ (0)$, since

$$\lim_{z\to 0}\frac{d}{dz}[h_n^{-1}(z)]\cdot\Big[\frac{h_n^{-1}(z) - h_n^{-1}(0)}{z}\Big] = 1.$$

From (ii), it follows for $n = 1, 2, 3, \ldots$ that

$$A_n(0) = \frac{1}{2\pi}\int_0^{2\pi} a_n(t)\,dt = \int_{C_n} \theta\,dg = \int_C \theta\,dg.$$

Thus $F_n(0) = \int_C \theta\,dg$ for $n = 1, 2, 3, \ldots$

Let $\varepsilon > 0$ be given. We choose n large enough so that

$$\left|f\left(g^{-1}\left(h_n^{-1}(e^{it})\right)\right) - f(z_0)\right| < \varepsilon \quad \text{for} \quad 0 \leqslant t \leqslant 2\pi.$$

Since $F_n \in H^1$, then

$$|F_n(0)| = \left|\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} F_n(re^{it})\,dt\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\left[f\left(g^{-1}\left(h_n^{-1}(e^{it})\right)\right) - f(z_0)\right]dt\right| < \varepsilon,$$

and we conclude that $\int_C \theta\,dg = 0$.

This completes the proof of Theorem 4.1.

COROLLARY 4.1. *If g is locally one-to-one at z_0, then each function in $A(g)$ can be represented in a neighborhood of z_0 by a power series in $g - g(z_0)$.*

Proof. The power series is obtained readily by the termwise integration of the power series for the integrand in the formula

$$f(z) = \pm\frac{1}{2\pi i}\int_C \frac{f\,dg}{g - g(z)}.$$

THEOREM 4.2. *If the restriction of g to U is a light open map, then $A(g)$ is an algebra. Moreover, there exists a homeomorphism h of U onto U such that for each $f \in A(g)$, there is a function F that is analytic on U, such that $f(z) = F\left(h(z)\right)$ for $z \in U$.*

Proof. From results of Stoïlow ([3], p. 121) and Titus ([4], p. 46) we get that there is a homeomorphism h of U onto U, and a function G that is analytic on U, such that $g(z) = G\left(h(z)\right)$ for $z \in U$. Since G is locally one-to-one except at a discrete set in U, then g also has this property.

Suppose g is locally one-to-one at z_0. Let C_1 and C_2 be circles centered at z_0, such that C_2 has smaller radius then C_1, and such that g is one-to-one on C_1', the union of C_1 and the inside of C_1. By Theorem 4.1,

$$f(z) = \pm\frac{1}{2\pi i}\int_{C_1}\frac{f\,dg}{g - g(z)} \quad \text{for} \quad z \in C_2'.$$

Therefore, for z_1 and z_2 in C_2',

$$|f(z_1) - f(z_2)| = \frac{1}{2\pi}\left|\int_{C_1} f\cdot\frac{g(z_2) - g(z_1)}{[g - g(z_2)][g - g(z_1)]}\,dg\right|.$$

The denominator in the integrand is bounded for z_1 and z_2 in C_2', and hence there exists a constant K such that

$$|f(z_1) - f(z_2)| \leqslant \frac{1}{2\pi}\cdot K.\|f\|\cdot V(g; C_1)\cdot|g(z_2) - g(z_1)|,$$

where $V(g; C_1)$ is the variation of g over C_1. Since g satisfies a Lipschitz condition on C_2', then f also satisfies a Lipschitz condition on C_2'.

We now use the fact that the a conclucion similar to that in Theorem 3.1 holds if instead of using $\overline{U}$ as our basic domain we use the closed disk C_2'. Accordingly, if f and h are in $A(g)$, and C is any simple closed rectifiable curve in C_2', then $\int_C fh\,dg = 0$.

It is not difficult to show that whenever $A(g)$ is "locally an algebra" (as described in the preceeding paragraph) except at a discrete set in U, then $A(g)$ is actually an algebra.

At each point $z_0 \in U$ at which g is locally one-to-one, each $f \in A(g)$ is locally an analytic function of g, and hence also locally an analytic function of h. It then follows readily that for each $f \in A(g)$ there is a function F that is analytic on U, such that $f(z) = F\left(h(z)\right)$ for $z \in U$.

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