Concerning function algebras*

by

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1. Introduction. According to Morera’s theorem, the collection of functions which are continuous on the closure of the open unit disk $U$, and such that $\int f dz = 0$ for each $f$ in the collection and for each simple closed rectifiable curve $C$ contained in $U$, is the same as the set of functions that are continuous on $U$, and analytic at each point of $U$. This collection forms a Banach algebra under supremum norm and the usual pointwise operations.

Suppose now that instead of taking the usual line integral, that we integrate with respect to a continuous function of bounded variation. Specifically, let $g$ be continuous on $U$, and of bounded variation on each simple closed rectifiable curve $C$ contained in $U$. We denote by $\mathcal{A}(g)$, the collection of all functions which are continuous on $U$, and such that $\int f dg = 0$ for each simple closed rectifiable curve contained in $U$. The collection $\mathcal{A}(g)$ will always be a Banach space under supremum norm, and the usual pointwise operations. It is not known, however, whether or not $\mathcal{A}(g)$ is always an algebra. The study of sufficient conditions for $\mathcal{A}(g)$ to be an algebra, will be one of our main concerns in this paper.

Our first result characterizes the integrator $g$ as being a continuous function on $U$ that satisfies a uniform Lipschitz condition on each compact subset of $U$. The characterization plays an important role in the results that follow.

An important subset of $\mathcal{A}(g)$, denoted by $L(g)$, is the collection of those functions in $\mathcal{A}(g)$ which satisfy a uniform Lipschitz condition on each compact subset of $U$. It is shown in Section 3, that the product $\mathcal{A}(g) \cdot L(g)$ is contained in $\mathcal{A}(g)$. In particular, $L(g)$ is an algebra contained in $\mathcal{A}(g)$. In some cases $L(g)$ is all of $\mathcal{A}(g)$, or at least is a dense subset of

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A(g). For these cases it follows that A(g) is an algebra. An example is
given to show that L(g) need not always be dense in A(g).

Special attention is given in Section 4 to the case in which g is locally
one-to-one at a point in U. In a neighborhood in which g is one-to-one,
each function in A(g) can be represented as an analytic function of g
by a formula analogous to the Cauchy integral formula. The proof of this
theorem is somewhat involved and depends on theorems on boundary
values of analytic functions.

If the restriction of g to U is a light open map, then g is locally one-
to-one in U, except at isolated points. In this case, the collection A(g)
is an algebra, and, moreover, it can be identified with some analytic function
algebra.

It is not always the case that A(g) will have a structure resembling
that of an analytic function algebra. If we let g(z) = k + π, then A(g)
can be identified with the algebra of all continuous functions on the interval
[-1, 1].

2. Characterization of the integrator. We now prove

Theorem 2.1. Suppose g is continuous on U. Then a necessary and
sufficient condition that g be of bounded variation on each simple closed
rectifiable curve in U, is that g satisfy a uniform Lipschitz condition on
each compact subset of U, i.e., that for each compact subset E contained
in U, there exists a constant K, depending only on E and g, such that
|g(z) - g(z')| ≤ K|z - z'| for all z and z' in E.

Proof. That the condition is sufficient is immediately clear. It is
also clear, that in order to prove the converse it will suffice to show
that the condition holds for each closed disk in U.

Accordingly, suppose E is a closed disk in U, for which the condition
fails to hold. Then for each positive integer n there exist points z_n and z'_n
in E for which

|g(z_n) - g(z'_n)| > n|z_n - z'_n|.

We note at this time that if {z'_n} is a subsequence of the sequence
of positive integers, and if s' = z'_n and s' = s'_n, then

|g(z_s) - g(z'_s)| > j|z_s - z'_s|.

That is, inequality (2.1) is preserved under the taking of subsequences.

Let s be a cluster point of {z'_n} in E. There exists a subsequence {z'_n}
of {z'_n} that converges to s. Let s' be an arbitrary cluster point of
{z'_n}. Then z_n = z'_n; for if not, then there exist subsequences (b)_{n=1} and
and (c)_{n=1} of (z'_n)_{n=1} and (s'_n)_{n=1}, respectively, such that

lim_{n→∞} b_n = s', \quad \lim_{n→∞} c_n = s'_n,

and

|g(b_n) - g(c'_n)| > k|b_n - c'_n|.

It is easy to see that this contradicts the continuity of g.

Thus, in view of the note already made, we can assume without loss of
generality that the sequences {z_n} and {z'_n} each have limit z_e, for
some z_e in E. For some n = 1, 2, 3, ... we can also assume that for each
n = 1, 2, 3, ... that

|z_n - z_e| < \frac{1}{2^n^2} and |z'_n - z_e| < \frac{1}{2^n^2}.

We now observe for each n = 1, 2, 3, ... that there exists open sets W_n and W'_n containing z_n and z'_n respectively, such that if z is in W_n
and z'_n is in W'_n, then |g(z) - g(z'_n)| > n|z_n - z'_n|. This fact follows easily
from the continuity of g. It therefore follows that we can assume for each
n = 1, 2, 3, ... that the points z_n, z'_n, and z'_n are not on the same straight
line.

Again taking subsequences if necessary, we can assume that
the straight line segment z_z_n lies inside a circular neighborhood N_n
centered at z_n, whose closure N_n misses the straight line segment z_z_n,z'_n
for n = 2, 3, ...

For each n = 1, 2, 3, ... let k_n be the positive integer such that
k_n|z_n - z'_n| ≤ 1/|z'_n| and (k_n + 1)|z_n - z'_n| > 1/|z'_n|. Since |z_n - z'_n| < 1/|n|, then
n|z_n - z'_n| < 1/|n| and

1 - n|z_n - z'_n| > 1 - \frac{1}{n^2}.

On the other hand, 1 ≥ n|z_n - z'_n| > 1 - 1/n and

\lim_{n→∞} n|z_n - z'_n| = 1.

From this we conclude that

\sum_{n=1}^{∞} k_n|z_n - z'_n| < ∞,

but that

\sum_{n=1}^{∞} k_n|z_n - z'_n| = ∞.

Since the straight line segment z_n,z'_n lies in the open set N_n, N_n+1, we
find points s_n,p, p = 1, ..., k_n on the line through z_n perpendicular
The sum of the lengths of the line segments connecting the $k_n$ line segments $\overrightarrow{z_{a_n}z_n}$ for a fixed $n$, is less than $\sum_{n=1}^{\infty} 1/n^4$. The sum of the lengths of the connecting line segments that are directed toward the origin is less than $\sum_{n=1}^{\infty} 1/n^4$; and, finally, the sum of the connecting areas that are on the boundaries of the neighborhoods $\{N_n\}_{n=1}^{\infty}$, is less than $\sum_{n=1}^{\infty} 2\pi/n^5$.

However, the variation of $g$ over $\Gamma$ is unbounded. It is at least as large as the sum of the variations of $g$ over the line segments $\overrightarrow{z_{a_n}z_n}$; $p = 1, \ldots, k_n$, and $n = 1, 2, 3, \ldots$, and the variation of $g$ over each of these is at least as large as

$$\|g(z_{a_n}) - g(z_{a_n})\| > n|z_{a_n} - z_{a_n}|.$$

Thus the variation of $g$ over $\Gamma$ is at least as large as

$$\sum_{n=1}^{\infty} n|z_{a_n} - z_{a_n}| = \infty.$$

Since we have found one rectifiable arc $\Gamma$ contained in $\mathcal{C}$ over which the variation of $g$ is unbounded, it follows that there exists a simple closed rectifiable curve in $\mathcal{C}$ for which the variation of $g$ is unbounded. Thus we have a contradiction, and the theorem is proved.

3. The algebra $L(g)$. We recall from Section 1, that the collection of functions in $A(g)$ which satisfy a uniform Lipschitz condition on each compact subset of $\mathcal{C}$, will be denoted by $L(g)$. It follows from a classical Stieltjes integration formula that the functions $1, g, g^2, \ldots$ are also in $A(g)$, and from Theorem 2.1 that these functions are in $L(g)$.

**Lemma 3.1.** Suppose $f$ is continuous on $\mathcal{C}$. Then $f \in A(g)$ if and only if $\int f \, dg = 0$ for each triangle $T$ contained in $\mathcal{C}$.

**Proof.** Certainly if $f \in A(g)$, then $\int f \, dg = 0$ for each triangle $T$ contained in $\mathcal{C}$. We now show that the converse is true.

Let $O$ be a simple closed rectifiable curve in $\mathcal{C}$. There is a closed disk $B$ contained in $O$, and $\mathcal{C}$ in its interior. By Theorem 2.1, there is a $K > 0$ such that for $x$ and $x'$ in $B$, we have $|g(x) - g(x')| < K|x - x'|$.

Let $x > 0$ be given. There exists a partition $\{\xi_0, \xi_1, \ldots\}$ of $O$ such that $\{\xi_0, \xi_1, \ldots\}$ is a refinement of $\{\xi_0, \xi_1, \ldots\}$, with $t'$ being any point in the arc from $t_{i-1}$ to $t_i$ on $\mathcal{C}$, then

$$\left| \int_{\mathcal{C}} f \, dg - \sum_{i=1}^{n} f(t_i)(g(t_i) - g(t_{i-1})) \right| < \frac{x}{2}.$$
Since \( f \) is uniformly continuous on \( E \), then there exists a \( \delta > 0 \) such that whenever \( z \) and \( z' \) are in \( E \), and \( |z - z'| < \delta \), then
\[
|f(z) - f(z')| < \frac{\varepsilon}{2(|C|)}\delta.
\]
where \( |C| \) is the length of curve \( C \).

For two arbitrary points \( z \) and \( z' \) in the plane we let \( \overline{zz'} \) denote the straight line segment joining \( z \) and \( z' \). Since \( C \) is a simple closed curve, then there exists a refinement \( \{z_i\}_{i=0}^N \) of \( \overline{zz'} \) such that
\[
C^* = \bigcup_{i=1}^m \overline{z_i z_{i+1}}
\]
is a simple closed polygonal curve and \( |z_{i-1} - z_i| < \delta \), for \( j = 1, 2, \ldots, m \). Then \( \int_C f \, d\gamma = 0 \) since \( \int f \, d\gamma = 0 \) for each triangle \( T \) contained in \( T_C \), and hence
\[
\left| \int_C f \, d\gamma \right| \leq \sum_{j=1}^m \left| \int_{z_{i-1} z_i} f(t)(g(t) - g(z_{i-1})) \right| + \sum_{i=1}^m \int_{z_{i-1} z_i} \left| f(t) \right| |g(t) - g(z_{i-1})| \leq \varepsilon.
\]
Therefore,
\[
\left| \int_C f \, d\gamma \right| \leq \frac{\varepsilon}{2} + \sum_{i=1}^m \int_{z_{i-1} z_i} \left| f(f(t)) \right| |g(t) - g(z_{i-1})| \leq \frac{\varepsilon}{2} + \sup_{t \in \overline{z_{i-1} z_i}} \left| f(f(t)) \right| |g(t) - g(z_{i-1})| \leq \varepsilon.
\]

From this, the lemma now follows.

**Theorem 3.1.** The product \( A(g) \cdot L(g) \) is contained in \( A(g) \).

**Proof.** The proof of this theorem is based on the technique used by E. Goursat to prove the Cauchy integral theorem.

Let \( f \cdot A(g) \) and \( h \in L(g) \). By Lemma 3.1, the function \( fh \) will be in \( A(g) \), if \( \int f \, d\gamma = 0 \) for each triangle \( T \) contained in \( U \). Let \( T_n \) be one such triangle. By joining the midpoints of the sides of \( T_n \), we obtain four congruent triangles \( T_{11}, T_{12}, T_{13}, \) and \( T_{14} \), such that
\[
\int_{T_{11}} f \, d\gamma + \int_{T_{12}} f \, d\gamma + \int_{T_{13}} f \, d\gamma + \int_{T_{14}} f \, d\gamma = \int_C f \, d\gamma.
\]

For one of the triangles, which we now denote by \( T_1 \),
\[
\int_{T_1} f \, d\gamma \geq \frac{1}{4} \int_{T_2} f \, d\gamma.
\]

This provides the first step in an easy induction argument to get that there is a nested sequence of triangles \( \{T_n\}_{n=0}^\infty \) with the following properties:

(i) \( |T_n| \leq \frac{1}{2^n} |T_1| \), where \( |T_1| \) is the length of the perimeter of \( T_1 \).

(ii) \( d(T_n) \leq \frac{1}{2^n} d(T_1) \), where \( d(T_1) \) is the length of the diameter of \( T_1 \).

(iii) \( \int_{T_1} f \, d\gamma \geq \frac{1}{4^n} \int_{T_n} f \, d\gamma \).

There will be one point, which we denote by \( z_0 \), which is interior to each of the triangles. Since \( h \) and \( h \) are in \( A(g) \), then
\[
\int_{T_1} f \, d\gamma = \int_{T_1} (f-f(z_0))(h-h(z_0)) \, d\gamma.
\]

Since \( h \) and \( g \) are in \( L(g) \), and the triangles are all contained in a compact subset of \( U \), there is a constant \( K < 0 \) such that
\[
|g(z)-g(z_0)| \leq K |z-z_0|, \quad \text{and} \quad |h(z)-h(z_0)| \leq K |z-z_0|,
\]
for \( z \in T_n \), \( n = 0, 1, 2, \ldots \).

Let \( e > 0 \) be given. There is a \( \delta > 0 \) such that if \( |z-z_0| < \delta \), then \( |f(z)-f(z_0)| < e \). Let \( n \) be large enough so that \( T_n \) is contained in the \( \delta \)-neighborhood of \( z_0 \). Then
\[
\frac{1}{4^n} \int_{T_1} f \, d\gamma \leq \int_{T_1} (f-f(z_0))(h-h(z_0)) \, d\gamma \leq e \sup_{a \in T_n} \frac{1}{4^n} \int_{T_1} f \, d\gamma.
\]

where \( V(g; T_n) \) is the variation of \( g \) over \( T_n \). Thus
\[
\frac{1}{4^n} \int_{T_1} f \, d\gamma \leq e K \cdot d(T_n) \cdot K \cdot |T_n| \leq e K^2 \frac{1}{4^n} \int_{T_1} f \, d\gamma.
\]

Hence
\[
\left| \int_{T_1} f \, d\gamma \right| \leq e K^2 d(T_n) \cdot |T_n|.
\]

and the theorem follows.

**Corollary 3.1.** \( L(g) \) is an algebra.

**Proof.** This follows immediately from Theorem 3.1, and the fact that the product of two Lipschitzian functions is again a Lipschitzian function.

**Remark 3.1.** In the classical case, where \( g(z) = z \), then \( L(g) \) is actually the same as \( A(g) \). In the case where \( g(z) = z - z_0 \), it is not difficult to show that each \( f \cdot A(g) \) is constant on the lines parallel to the
Y-axis, and that \( A(g) \) can be identified with the algebra \( C([-1, 1]) \) of all continuous functions on the interval \([-1, 1]\). In this case, \( L(g) \) is certainly not all of \( A(g) \). However, \( L(g) \) is dense in \( A(g) \), since \( L(g) \) contains the polynomials in \( x \), and these are dense in \( C([-1, 1]) \). This might lead one to suspect that \( L(g) \) is always dense in \( A(g) \). The following example shows, however, that this is not the case. We let \( h(x) = x \sqrt{x} \), and let \( g(x) = h'(x) = \frac{1}{2x} \sqrt{x} \). It is not difficult to show that \( A(g) \) consists of all functions \( F(h) \), such that \( F \) is continuous on \( U \) and analytic at each point of \( T \). Now \( h \) is not in \( L(g) \), but \( h^k g \) in \( L(g) \), for \( k = 2, 3, \ldots \) The function \( h \) cannot be uniformly approximated on \( U \) by functions in \( L(g) \).

4. The analytic nature of \( A(g) \). We now consider the behavior of \( A(g) \) near points at which \( g \) is locally one-to-one. The proofs of the theorems given here would be simpler if we were to assume that \( g^{-1} \) (the inverse of \( g \) in a region where \( g \) is one-to-one) as well as \( g \), is of bounded variation on simple closed rectifiable curves. We do not, however, make this assumption.

**Theorem 4.1.** Suppose \( E \) is a closed disk contained in \( U \), and that the restriction of \( g \) to \( E \), \( g \mid E \), is homeomorphic. Suppose \( C \) is any simple closed rectifiable curve in \( E \), and that \( z_0 \) is a point inside \( C \). Then for any \( f \in A(g) \),

\[
f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{g(\zeta) - g(z_0)} d\zeta,
\]

the sign being taken as plus or minus depending on whether \( g \mid E \) is a positive or negative homeomorphism.

**Proof.** Let \( D \) be an arbitrary simple closed rectifiable curve in \( E \) such that \( z \) is not in \( D \). If \( g \mid E \) is a positive homeomorphism, then

\[
\int_D \frac{dg}{g - g(z_0)} = \int_D \frac{dw}{w - w_0},
\]

where \( w_0 = g(z_0) \). The Brouwer invariance of domain theorem \((2^2, \text{p. 95})\) gives us that \( g(z_0) \) is inside \( g(D) \) if and only if \( z_0 \) is inside \( D \). Therefore,

\[
\int_D \frac{dg}{g - g(z_0)} = 2\pi i \text{ or } 0,
\]

depending on whether or not \( z_0 \) is inside \( D \). If \( g \mid E \) is a negative homeomorphism, then

\[
\int_D \frac{dg}{g - g(z_0)} = -2\pi i \text{ or } 0.
\]

Let \( D' \) denote the union of \( D \) and the inside of \( D \). If \( z_0 = E - D' \), then it follows from a polynomial approximation of Walsh \((5^2, \text{p. 439})\), that there exists a sequence of polynomials in \( g \), \( (P_n(g))_{n=1}^\infty \), which converges uniformly on \( D' \) to \( 1/(g - g(z_0)) \).

By Theorem 3.1, \( f g^n A(g) \) for \( n = 1, 2, 3, \ldots \) Therefore

\[
\int_D |f P_n(g)| d\zeta = 0,
\]

and hence

\[
\frac{1}{2\pi i} \int_D \frac{f \zeta}{g - g(z_0)} d\zeta = 0.
\]

As in the proof of the classical Cauchy integral formula, we conclude for any circle \( C_1 \) centered at \( z_0 \), and contained inside \( C \), that

\[
\frac{1}{2\pi i} \int_{C_1} \frac{f - f(z_0)}{g - g(z_0)} d\zeta = \frac{1}{2\pi i} \int_{C_1} \frac{f - f(z_0)}{g - g(z_0)} d\zeta.
\]

The remainder of the proof is devoted to showing that

\[
\frac{1}{2\pi i} \int_{C_1} \frac{f - f(z_0)}{g - g(z_0)} d\zeta = 0,
\]

from which we get the formula

\[
f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{g(\zeta) - g(z_0)} d\zeta.
\]

Let

\[
\theta = \frac{f - f(z_0)}{g - g(z_0)},
\]

and let \( (C_n)_{n=1}^\infty \) be a nested sequence of circles centered at \( z_0 \), contained inside \( C \) and having radii tending to zero. Since \( f g^n A(g) \) for \( n = 1, 2, \ldots \) then \( \theta (g^n - g^n(z_0)) A(g) \) for \( m = 1, 2, \ldots \), and hence

\[
\frac{1}{2\pi i} \int_{C_1} \theta (g^n - g^n(z_0)) d\zeta = \frac{1}{2\pi i} \int_{C_1} \theta d\zeta.
\]

Let \( C'_n \) denote the union of \( C_n \) and the inside of \( C_n \). There exists a one-to-one mapping \( h_n \), taking \( C'_n \) onto \( U \) and \( z_0 \) onto 0, that is analytic on the interior of \( g(C'_n) \) \((6^2, \text{p. 290})\). Moreover, the restriction of \( h_n^{-1} \) to the unit circle is absolutely continuous \((6^2, \text{p. 293})\).
Since $h_n(g)$ can be uniformly approximated on $C_n$ by polynomials in $g$ ([5], p. 430), it follows that

$$\frac{1}{2\pi i} \int_{C_n} \theta h_n^p(g) \, dg = 0 \quad \text{for} \quad p = 1, 2, 3, \ldots$$

Let $w = h_n(g)$. The functions $\theta$ and $h_n^p(g)$ are continuous on $C_n$, and $h_n$ is one-to-one on $g(C_n)$. Therefore

$$\frac{1}{2\pi i} \int_{C_n} \theta \left( h_n^{-1}(w) \right) w^{-1} dw = 0,$$

and

$$\frac{1}{2\pi i} \int_{C_n} \theta \left( h_n^{-1}(w) \right) w^{-1} dw = 0.$$

(4.1)

Let $\alpha_n(t) = \theta \left( h_n^{-1}(e^{it}) \right) \left( h_n^{-1}(e^{it}) \right)^i e^{it}$. It follows from equation (4.1) that $\alpha_n$ is the almost everywhere (with respect to Lebesgue measure) radial limit of a function, say $A_n$, in the class $H^1$. In particular ([1], p. 54)

(i) \quad \lim_{n \to \infty} A_n(\alpha_n) = \alpha_n(0) \quad \text{a.e.,}

(ii) \quad \lim_{n \to \infty} \frac{1}{2\pi} \int \left| A_n(\alpha_n) - \alpha_n(t) \right| dt = 0.

From (i), we get that

$$\lim_{n \to \infty} A_n(\alpha_n) = \left\{ f \left( h_n^{-1}(e^{it}) \right) - f(\alpha_n) \right\} \left( h_n^{-1}(e^{it}) \right) e^{it} \quad \text{a.e.}$$

For each $p = 1, 2, 3, \ldots$,

$$\frac{1}{2\pi i} \int \left[ f \left( h_n^{-1}(e^{it}) \right) - f(\alpha_n) \right] e^{it} dt$$

$$= \frac{1}{2\pi i} \int_{C_n} f \left( h_n^{-1}(w) \right) dw = \frac{1}{2\pi i} \int_{C_n} f \left( h_n^p(g) \right) = 0.$$

It follows that $f \left( h_n^{-1}(e^{it}) \right) - f(\alpha_n)$ is the almost everywhere radial limit of an $H^1$ function, say $E_n$.

We now show that $\left( h_n^{-1}(e^{it}) \right) e^{it}$ is the almost everywhere radial limit of an $H^1$ function, say $F_n$.

limit of an $H^1$ function. For $p = 1, 2, 3, \ldots$,

$$\frac{1}{2\pi} \int \left( h_n^{-1}(e^{it}) \right) e^{it} dt = \frac{1}{2\pi} \int e^{it} dt h_n^{-1}(e^{it})$$

$$= \frac{1}{2\pi} \int \left( h_n^{-1}(e^{it}) \right) e^{it} dt = 0, \quad \text{if} \quad p > 0,$$

$$\frac{1}{2\pi} \int \left( h_n^{-1}(e^{it}) \right) e^{it} dt = \frac{1}{2\pi} \int \left( h_n^{-1}(e^{it}) \right) e^{it} dt = 0, \quad \text{if} \quad q < p,$$

where

$$h_n = \frac{1}{2\pi n} \int h_n^{-1}(e^{it}) e^{-it} dt.$$

It follows that

$$\lim_{n \to \infty} \sum_{p=1}^{\infty} \left( h_n^{-1}(e^{it}) \right) e^{it} = \left( h_n^{-1}(e^{it}) \right) e^{it} \quad \text{a.e.,}$$

and that

$$\sum_{p=1}^{\infty} p \left( h_n^{-1}(e^{it}) \right) e^{it}$$

is an $H^1$ function. Moreover,

$$\sum_{p=1}^{\infty} p \left( h_n^{-1}(e^{it}) \right) e^{it} = \frac{dz}{dx} \left( \sum_{p=1}^{\infty} b_p e^{it} \right) = \frac{dz}{dx} h_n^{-1}(x),$$

and so $\left( h_n^{-1}(e^{it}) \right) e^{it}$ is the almost everywhere radial limit of

$$\frac{dz}{dx} h_n^{-1}(x).$$

All $H^1$ functions are in the class of beschränkarte functions, denoted by $N$ in [6] (p. 272). Moreover, a function $F$, analytic on $U$, is in the class $N$ if and only if $F$ is the ratio of two bounded analytic functions on $U$ ([6], p. 277). If two functions in $N$ have radial limits that agree on a set of positive Lebesgue measure, then they are identical ([8], p. 276). It therefore follows that

$$A_n(\alpha) = F_n(\alpha) - \frac{h_n^{-1}(\alpha)}{h_n^{-1}(\alpha) - h_n^{-1}(0)}.$$

We note that $A_n(0) = F(0)$, since

$$\lim_{n \to \infty} \frac{d}{dz} \left( h_n^{-1}(e^{it}) \right) \left[ \frac{h_n^{-1}(e^{it}) - h_n^{-1}(0)}{z} \right] = 1.$$
From (ii), it follows for \( n = 1, 2, 3, \ldots \) that
\[
A_n(0) = \frac{1}{2\pi} \int_{\partial \Omega} a_n(t) dt = \int_{\partial \Omega} \hat{a}_n dg = \int_0^\infty \hat{a}_n dg.
\]
Thus \( F_n(0) = \int_0^\infty \hat{a}_n dg \) for \( n = 1, 2, 3, \ldots \).

Let \( \epsilon > 0 \) be given. We choose \( n \) large enough so that
\[
|f\left[g^{-1}(h_k^{-1}(c^{n}))\right] - f(c) < \epsilon \quad \text{for} \quad 0 < t < 2\pi.
\]

Since \( F_n \in H^2 \), then
\[
F_n(0) = \left| \lim_{t \to 0} \frac{1}{2\pi} \int_{\partial \Omega} F_n(re^{it}) du \right| = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left[g^{-1}(h_k^{-1}(c^{n}))\right] - f(c) \right| du < \epsilon,
\]
and we conclude that \( \int_0^\infty \hat{a}_n dg = 0 \).

This completes the proof of Theorem 4.1.

**Corollary 4.1.** If \( g \) is locally one-to-one at \( c_0 \), then each function in \( A(g) \) can be represented in a neighborhood of \( c_0 \) by a power series in \( g - g(c_0) \).

**Proof.** The power series is obtained readily by the termwise integration of the power series for the integrand in the formula
\[
f(c) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z) dg}{g - g(z)}.
\]

**Theorem 4.2.** If the restriction of \( g \) to \( U \) is a locally open map, then \( A(g) \) is an algebra. Moreover, there exists a homeomorphism \( h \) of \( U \) onto \( U \) such that for each \( f \in A(g) \), there is a function \( F \) that is analytic on \( U \), such that \( f(z) = F(h(z)) \) for \( z \in U \).

**Proof.** From results of Stollow ([3], p. 121) and Titus ([4], p. 46) we get that there is a homeomorphism \( h \) of \( U \) onto \( U \), and a function \( G \) that is analytic on \( U \), such that \( g(z) = G(h(z)) \) for \( z \in U \). Since \( g \) is locally one-to-one except as a discrete set in \( U \), then \( g \) also has this property.

Suppose \( g \) is locally one-to-one at \( c_0 \). Let \( C_1 \) and \( C_2 \) be circles centered at \( c_0 \), such that \( C_2 \) has smaller radius than \( C_1 \), and such that \( g \) is one-to-one on \( C_1 \), the union of \( C_1 \) and the inside of \( C_1 \). By Theorem 4.1,
\[
f(c) = \frac{1}{2\pi i} \int_{\partial C_1} \frac{f(z) dg}{g - g(z)} \quad \text{for} \quad z \in C_1.
\]

Therefore, for \( c_1 \) and \( c_2 \) in \( C_1 \),
\[
|f(c_1) - f(c_2)| = \frac{1}{2\pi i} \int_{\partial C_1} \frac{|f(z)|}{|g - g(z)|} \frac{|g(z_1) - g(z_2)|}{|g(z_1) - g(z_2)|} \frac{dg}{|g - g(z)|}.
\]

The denominator in the integrand is bounded for \( z \), and \( z \) in \( C_1 \), and hence there exists a constant \( K \) such that
\[
|f(c_1) - f(c_2)| \leq \frac{1}{2\pi i} K |V(g; C_1)| |g(z_1) - g(z_2)|,
\]
where \( V(g; C_1) \) is the variation of \( g \) over \( C_1 \). Since \( g \) satisfies a Lipshitz condition on \( C_1 \), then \( f \) also satisfies a Lipshitz condition on \( C_1 \).

We now use the fact that the conclusion similar to that in Theorem 3.1 holds if instead of using \( U \) as our basic domain we use the closed disk \( C_1' \). Accordingly, if \( f \) and \( h \) are in \( A(g) \), and \( U \) is any simple closed rectifiable curve in \( C_1' \), then \( \int_0^\infty \hat{a}_n dg = 0 \).

It is not difficult to show that whenever \( A(g) \) is “locally an algebra” (as described in the preceding paragraph) except at a discrete set in \( U \), then \( A(g) \) is actually an algebra.

At each point \( z \in U \), \( g \) is locally one-to-one, each \( f \in A(g) \) is locally an analytic function of \( g \), and hence also locally an analytic function of \( h \). It then follows readily that for each \( f \in A(g) \), there is a function \( F \) that is analytic on \( U \), such that \( f(z) = F(h(z)) \) for \( z \in U \).

**Bibliography**


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