but A. Pełczyński (see [8], p. 368) has remarked that every subspace of E with an unconditional basis is reflexive.

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On sequences of continuous functions and convolution

by

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1. In the study of Mikusiński operators the question arises "given a sequence of continuous functions g_n on the half-line $t \geqslant 0$ is there a single non-zero continuous g such that, for each n, g is of the form

$$g(t) = \int_0^t g_n(t-u) f_n(u)\,du, \qquad t \geqslant 0, \tag{1}$$

where f_n is a continuous function?" For an affirmative answer it is obviously necessary that there exist some interval $[0, T]$, $T > 0$, such that none of the g_n vanish identically on $[0, T]$. If this condition is satisfied the answer given by Theorem 3 below is "yes, there is always such a function g".

In what follows we will utilize the following notation. The functions involved are complex values functions on the half-line $t \geqslant 0$; juxtaposition of functions denotes convolution so that equation (1) will be written $g = g_n f_n$. C is the vector space of continuous functions, and L is the vector space of locally integrable functions. For g in C or in L we will use the semi-norm

$$\|g\|_T = \int_0^T |g|(t)\,dt,$$

and a sequence g_n is convergent in L to g if $\|g_n - g\|_T \to 0$ for every $T > 0$. The fundamental inequality for this semi-norm (in addition to the triangle inequality) is that, for any two functions g and f in L, $\|gf\|_T \leqslant \|g\|_T \|f\|_T$. The set C_0 (or L_0) is the set of all g in C (or L) such that $\|g\|_T > 0$ for all $T > 0$; that is, it consists of those functions which vanish on no neighborhood of the origin. In particular, a function g in C_0 is not the zero function. The symbol h will be used for that function in C which is such that $h(t) = 1$ for all $t \geqslant 0$.

The basic principle in what follows is a theorem of Ċ. Foiaş which says

THEOREM 1. *For g in L_0, $T>0$, $\varepsilon>0$, and f in L there is a k in L such that $\|f-gk\|_T<\varepsilon$.*

For a proof of the above theorem the reader is referred to [1] or [2].

COROLLARY. *For g in L_0, $T>0$, and $\varepsilon>0$, there is a k in L such that $s=kg$ has the properties*

(i) $$\|s\|_T\leqslant 1,$$

(ii) $$\|hs-h\|_T<\varepsilon.$$

Proof. Take f in L with the properties that $\|f\|_T=a<1$ and $\|hf-h\|_T<\varepsilon/2$. By Theorem 1 there is a k in L such that $\|f-gk\|_T<\operatorname{Min}[1-a,\varepsilon/2]$. Then $s=kg$ satisfies (i) and (ii).

We will denote the convolution product $\prod_1^N s_n$ by S_N, and will use the notation

$$S_{N,M}=\prod_N^M s_n,\qquad H_N=h\prod_1^N s_n,\qquad H_{N,M}=h\prod_N^M s_n.$$

The next theorem can be interpreted to mean that if $s_n\to 1$ (the identity in the field of Mikusiński operators) in a sufficiently tractable manner then the infinite product $\prod_1^\infty s_n$ is convergent.

THEOREM 2. *Take s_n in L, $\|s_n\|_n\leqslant 1$, and suppose that $\|hs_n-h\|_n\leqslant\varepsilon_n$ where $\sum_{n\geqslant 1}\varepsilon_n<\infty$. Then H_M is convergent in L and $H=\operatorname{Lim}_M H_M$ is zero if and only if $s_n=0$ for some n. For each n we have $H=s_np_n$ where p_n is in L.*

Proof. We note that, since $\|s_n\|_T\leqslant 1$ for $n>T$, $\|S_{N,M+P}\|_T\leqslant\|S_{NM}\|_T$ if $M>T$. The convergence of H_M follows from the inequalities

$$\|H_M-H_{M+P}\|_T=\|S_M(h-H_{M+1,M+P})\|_T\leqslant\|S_N\|_T\|h-H_{M+1,M+P}\|_T$$
$$\leqslant\|S_N\|_T\Big(\sum_M\varepsilon_n\Big),$$

where $N>T$ is fixed. The same inequalities hold for $H_{R,M}$ with R fixed and $H=\operatorname{Lim}_M H_M=\operatorname{Lim}_M S_RH_{R+1,M}=S_R(\operatorname{Lim}_M H_{R+1,M})$ which proves the last statement in the theorem.

It only remains to show that H is zero if and only if some s_n is zero. Take $N>T$ and $\sum_{n\geqslant N}\varepsilon_n<T/2$; then

$$\|H_{N,M}\|_T\geqslant\|h\|_T-\sum_N^M\varepsilon_n>T/2.$$

Thus

$$\operatorname{Lim}_M\|H_{N,M}\|_T=\|\operatorname{Lim}_M H_{N,M}\|_T>0.$$

Since $H=S_N(\operatorname{Lim}_M H_{N+1,M})$ and the latter limit is non-zero, H is zero if and only if $S_N=0$, i. e. if and only if some factor $s_n=0$ for $n\leqslant N$.

THEOREM 3. *Let g_n be a sequence in C. A necessary and sufficient condition that there exists a non-zero g in C such that each g_n factors g, $g=g_nf_n$, with f_n in C, is that there is an initial interval $[0,T]$ such that for no n does g_n vanish on $[0,T]$.*

Proof. The condition stated in the theorem is clearly necessary; we will show that it is also sufficient. Since each g_n is a function in C_0 shifted to the right no more than T units we can just as well suppose that all g_n are in C_0. With this assumption we can find for each n, according to the Corollary to Theorem 1, a k_n in L such that the continuous function $s_n=g_nk_n$ has semi-norms $\|s_n\|_n\leqslant 1$ and $\|hs_n-h\|_n<1/n^2$. By Theorem 2 the product $H=h\prod_1^\infty s_n$ is convergent, and the function $h=H$ is the desired function.

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