On w^*-sequential convergence, type P^* bases, and reflexivity *

by

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1. If X is a Banach space and S a subspace of X^*, let $K_X(S)$ denote the w^*-sequential closure of S in X^*. McWilliams ([6], Thm. 1) has recently shown that if $f \in K_X(S)$ and

$$\varphi(f) = \inf\{\sup_n \|f_n\| : \{f_n\} \subset S \text{ and } w^*\text{-}\lim_n f_n = f\}, \tag{1.1}$$

then $K_X(S)$ is closed in the norm topology of X^* if and only if there is a number $C \geqslant 1$ such that $\varphi(f) \leqslant C\|f\|$ for all $f \in K_X(S)$. In accordance with this, for a subspace S of X^* let

$$C_S = \inf\{C : \varphi(f) \leqslant C\|f\| \text{ for all } f \in K_X(S)\}, \tag{1.2}$$

$$Q_X = \sup\{C_S : S \text{ is a subspace of } X^*\}. \tag{1.3}$$

It is clear that $Q_X = \infty$ if and only if either

(1.4) there exists a sequence $\{S_N\}$ of subspaces of X^* such that $C_{S_N} \to \infty$ as $N \to \infty$,

or

(1.5) there is a subspace S in X^* such that $C_S = \infty$.

If X is reflexive, then $K_X(S) = S$ for every closed subspace S of X^*, and hence trivially $Q_X = 1$. Similarly, since w-sequential and w^*-sequential convergence coincide in $(m)^*$ ([4], Theorem 9, p. 168), where (m) is the (non-reflexive) space of bounded sequences, it follows that $Q_{(m)} = 1$.

In Section 2 it is proved that if X has a type P^* basis ([8], p. 354) then $Q_X = \infty$. It then follows that for X to be reflexive it is necessary and sufficient that $Q_Y < \infty$ for every closed subspace Y of X. Further, it is shown that if X has an unconditional basis and $Q_X < \infty$, then X is reflexive.

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In Section 3 it is shown that (1.5) is satisfied for (l^1) and for (c_0). For a quasi-reflexive space [2], (1.5) can never be satisfied, but (1.4) is satisfied for the quasi-reflexive space of James ([5], p. 523). Further, it is shown for a quasi-reflexive space X that $K_{X^*}(J_X(X)) = X^{**}$, where J_X is the canonical mapping from X into X^{**}, and that if S is a subspace of X^*, then $K_X(K_X(S)) = K_X(S)$.

Finally, in Section 4 some unsolved problems are mentioned.

2. If $\{x_n\}$ is a basis for a Banach space X, then $\{x_n\}$ is said to be *of type P^** if

$$\sup_n \|x_n\| < \infty \quad \text{and} \quad \sup_n \Big\| \sum_{i=1}^{n} h_i \Big\| < \infty,$$

where $\{h_i\}$ is the sequence of functionals in X^* biorthogonal to $\{x_i\}$.

THEOREM 1. *If X is a Banach space with a basis $\{x_i\}$ of type P^*, then $Q_X = \infty$.*

Proof. We may assume that $\sup_i \|x_i\| \leqslant 1$. Let $\{h_i\}$ be the sequence in X^* biorthogonal to $\{x_i\}$ and let

$$T = \sup_n \Big\| \sum_{i=1}^{n} h_i \Big\|.$$

From [8] (Prop. 3, p. 356) there is a $g_0 \in X^*$ such that $g_0(x_n) = 1$ for each n. Let N be a positive number greater than 1. Define $\{f^{Nj}\}_{j=1}^{\infty}$ as follows: for each j,

$$f^{Nj} = N\|g_0\| g_0 - (N\|g_0\| - 1) \sum_{i=1}^{j} h_i.$$

Then $\|f^{Nj}\| \leqslant N\|g_0\|^2 + (N\|g_0\| - 1)T$ for each j. On the other hand, since $\|x_n\| \leqslant 1$ for each n, $\|f^{Nj}\| \geqslant |f^{Nj}(x_{j+1})| = N\|g_0\|$. It is clear that

$$\lim_{j\to\infty} f^{Nj}(x_k) = g_0(x_k) = 1 \quad \text{for each } k.$$

Thus, since $\{f^{Nj}\}$ is bounded in norm, $w^*\text{-}\lim_j f^{Nj} = g_0$. Suppose

$$g = \sum_{i=1}^{m} a_i f^{Ni},$$

where $a_1, \ldots, a_m$ are scalars. Then

$$\|g\| \geqslant |g(x_{m+1})| = N\|g_0\| \Big| \sum_{i=1}^{m} a_i \Big|. \tag{2.1}$$

Let S_N be the (not necessarily closed) subspace of X^* spanned by $\{f^{Nj}\}$. Suppose $\{g^n\}$ is a sequence in S_N converging to g_0 in the w^*-topology. For each n,

$$g^n = \sum_{i=1}^{m_n} a_i^{(n)} f^{Ni} \quad \text{and} \quad g^n(x_1) = \sum_{i=1}^{m_n} a_i^{(n)}.$$

Since $\lim g^n(x_1) = g_0(x_1) = 1$, it follows that for each $\varepsilon > 0$ there is an $M > 0$ such that for $n > M$,

$$\Big| \sum_{i=1}^{m_n} a_i^{(n)} - 1 \Big| < \varepsilon. \tag{2.2}$$

Thus from (2.1) and (2.2),

$$\liminf_n \|g^n\| \geqslant N\|g_0\|. \tag{2.3}$$

Now $g_0 \in K(S_N)$ and, by (2.3), $\varphi(g_0) \geqslant N\|g_0\|$. Thus $C_{S_N} \geqslant N$, and so $Q_X = +\infty$.

Remark 1. *For every $N > 1$ the subspace S_N constructed in the proof of Theorem 1 has the property that $K_X(S_N) = X^*$, and hence C_{S_N} is finite.*

Proof. Let f be a non-zero element of X^*. For each positive integer n let

$$d_k^{(n)} = \begin{cases} \dfrac{f(x_{k+1}) - f(x_k)}{N\|g_0\| - 1} & \text{for } 1 \leqslant k < n, \\[2ex] \dfrac{N\|g_0\| f(x_1) - f(x_n)}{N\|g_0\| - 1} & \text{for } k = n, \end{cases}$$

and let

$$p^n = \sum_{k=1}^{n} d_k^{(n)} f^{Nk}.$$

We note that

$$\sum_{k=1}^{n} d_k^{(n)} = f(x_1) \quad \text{and} \quad p^n(x_j) = \begin{cases} N\|g_0\| f(x_1) & \text{for } j > n, \\ f(x_j) & \text{for } 1 \leqslant j \leqslant n. \end{cases}$$

If $x \in X$, then $x = \sum_{i=1}^{\infty} a_i x_i$ for some scalar sequence $\{a_i\}$; since $g_0(x_i) = 1$ for each i, the series $\sum_{i=1}^{\infty} a_i$ converges. If $\varepsilon > 0$ is given, then there is an $M > 0$ such that for $n > M$,

$$\Big| \sum_{i=n+1}^{\infty} a_i \Big| < \frac{\varepsilon}{2N\|g_0\|\|f\|}, \tag{2.4}$$

and

$$\left|\sum_{i=n+1}^{\infty} a_i f(x_i)\right| < \frac{\varepsilon}{2}. \tag{2.5}$$

Thus for $n > M$,

$$|p^n(x)-f(x)| = \left|N\|g_0\|f(x_1)\sum_{i=n+1}^{\infty} a_i - \sum_{i=n+1}^{\infty} a_i f(x_i)\right| < \varepsilon$$

by (2.4) and (2.5); i. e., $w^*\text{-}\lim_n p^n = f$, and thus $f \in K_X(S_N)$.

Corollary 1. *A Banach space X is reflexive if and only if $Q_Y < \infty$ for every norm-closed subspace Y of X.*

Proof. A closed subspace Y of a reflexive space X is reflexive, and so $Q_Y = 1$.

On the other hand, if X is not reflexive, then there is a non-shrinking basic sequence $\{z_n\}$ in X([7], Thm. 1, p. 372) and hence a basic sequence $\{y_n\}$ of type P^* ([8], Thm. 1, p. 358). If $Y = [y_n]$ is the closed linear span of $\{y_n\}$, then $Q_Y = \infty$ by Theorem 1.

It is easy to verify that the unit vector basis of (l^1) and the basis $\{z_n\}$ of (c_0), where

$$z_n = (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots, 0), \tag{2.6}$$

are of type P^*; thus $Q_{(c_0)} = Q_{(l^1)} = \infty$. Letting $X = (m)$ and $Y = (c_0)$, we see that it is possible for Y to be a closed subspace of X and for Q_Y to be infinite while Q_X is finite. This cannot happen in the presence of a continuous projection from X onto Y.

Theorem 2. *Let X be a Banach space, Y a closed subspace of X and T a continuous projection from X onto Y. Then $Q_Y \leqslant \|T\| Q_X$.*

Proof. Since the range of T is all of Y, T^{*-1}, where T^* is the adjoint of T, exists and it is easy to verify that

$$\|T^{*-1}f\| \leqslant \|f\| \leqslant \|T\|\|T^{*-1}f\|, \tag{2.7}$$

for any f in the range of T^*.

Suppose W is a subspace of Y^* and $Y' \in K_Y(W)$. If $S = T^*(W)$ and $f = T^*Y'$, then $f \in K_X(S)$. Suppose that $\{f_n\}$ is a sequence in S such that $w^*\text{-}\lim_n f_n = f$. Let $Y'_n = T^{*-1}f_n$ for each n. Then $w^*\text{-}\lim_n Y'_n = Y'$, $\{Y'_n\} \subset W$, and from (2.7) $\|Y'_n\| \leqslant \|f_n\|$ for each n. Thus $\varphi(Y') \leqslant \sup_n \|f_n\|$, and so $\varphi(Y') \leqslant \varphi(f) \leqslant C_S\|f\| \leqslant C_S\|T\|\|Y'\|$. Thus $C_W \leqslant \|T\| C_S$, and hence $Q_Y \leqslant \|T\| Q_X$.

The next theorem is proved in a similar manner.

Theorem 3. *If T is an isomorphism (i. e., linear homeomorphism) from a Banach space X onto a Banach space Y, then*

$$(\|T^{-1}\|\|T\|)^{-1}Q_X \leqslant Q_Y \leqslant (\|T^{-1}\|\|T\|)Q_X. \tag{2.8}$$

Theorem 4. *If X is a Banach space with an unconditional basis $\{x_i\}$ and if $Q_X < \infty$, then X is reflexive.*

Proof. Suppose there is a subspace of X isomorphic to (c_0). Then, by [1] (C. 6, p. 157), there is a subspace Y of X isomorphic to (c_0) such that there exists a continuous projection from X onto Y. It follows from Theorems 2 and 3 that $Q_X = \infty$, contradicting the hypothesis.

Suppose X contains a subspace isomorphic to (l^1). Then $\{x_i\}$ is non-shrinking ([3], Thm. 3, p. 76), and so ([8], Prop. 5, p. 367) the sequence $\{f_i\} \subset X^*$ biorthogonal to $\{x_i\}$ is a non-boundedly-complete basis for $[f_i]$. Now by [3] (Thm. 2, p. 74), the space $[f_i]$ and hence also X^* contain a subspace isomorphic to (c_0). Thus ([1], Thm. 4, p. 155) there is a subspace Y of X isomorphic to (l^1) and a continuous projection from X onto Y. By Theorems 2 and 3, $Q_X = \infty$, contradicting the hypothesis. Thus X has no subspace isomorphic to (c_0) or (l^1), so by [5] (Thm. 2, p. 521) X is reflexive.

3. If X is a space with a type P^* basis, then Theorem 1 provides a method for constructing a sequence $\{S_N\}$ of subspaces of X^* satisfying (1.4). An example in [6] and the following example show that (1.5) is satisfied in (l^1) and (c_0) respectively.

Example. Let $X = (c_0)$ so that $X^* = (l^1)$. The sequence $\{z_i\}$ defined by (2.6) is a basis of type P^* for (c_0) and the biorthogonal functionals $\{h_i\}$ in (l^1) associated with $\{z_i\}$ are given by $h_i = (\underbrace{0, \ldots, 0}_{i-2}, 1, -1, 0, 0, \ldots)$.

In the notation of Theorem 1, the functionals $f^{nj} = \{f_p^{nj}\}$ are defined by

$$f_p^{nj} = \begin{cases} 1 & \text{if} \quad p = 1, \\ n-1 & \text{if} \quad p = j+1, \qquad n \geqslant 2. \\ 0 & \text{otherwise}, \end{cases}$$

Let $\{M_k\}$ be the collection of disjoint sets of positive integers defined as follows: for each k, $M_k = \{n\colon n = 2^{k-1}(2p-1),\ p = 1, 2, 3, \ldots\}$. For each pair of positive integers n, j with $2 \leqslant n$, $1 \leqslant j$, define the element H^{nj} of (l^1) by $H^{nj} = (H_m^{nj})$ where

$$H_m^{nj} = \begin{cases} 0 & \text{if} \quad m \notin M_{n-1}, \\ f_p^{nj} & \text{if} \quad m = 2^{n-2}(2p-1). \end{cases}$$

Let $S = [H^{nj}]$. If $\{f^i\}$ is the unit vector basis of (l^1), then $w^*\text{-}\lim_j H^{qj} = f^{2^{q-1}}$, so $f^{2^{q-1}} \in K_X(S)$, for every positive integer $q \geqslant 2$. If $\{g^n\} \subset S$ and

$w^*\text{-}\lim_n g^n = f^{2^{q-2}}$ then an argument similar to that of Theorem 1 shows that $\liminf_n \|g^n\| \geqslant q$. It follows that $C_S \geqslant q$ for arbitrary $q \geqslant 2$, i. e., $C_S = +\infty$.

We now show that (1.5) cannot be satisfied in quasi-reflexive spaces.

THEOREM 5. *If X is quasi-reflexive and S is a subspace of X^{**}, then $K_{X^*}(S)$ is norm-closed in X^{**}, and $K_{X^*}(K_{X^*}(S)) = K_{X^*}(S)$.*

Proof. Since $K_{X^*}(S) = K_{X^*}(\bar{S})$, it may be assumed that S is norm-closed. Let $S_1 = S \cap J_X(X)$, where J_X is the canonical mapping of X into X^{**}. Then S is the direct sum of S_1 and a finite-dimensional subspace S_2 of X^{**}. The projection P of S onto S_1 along S_2 is norm-continuous. Thus if $F \in K_{X^*}(S)$, so that F is the w^*-limit of a norm-bounded sequence $\{F_n\}$ in S, then the sequence $\{F_n - PF_n\}$ is a bounded sequence in S_2. Since S_2 is finite-dimensional, there is a subsequence $\{F_{n_i} - PF_{n_i}\}$ which converges in norm to some $G \in S_2$. Thus $\{PF_{n_i}\}$ converges in the w^*-topology of X^{**} to $F-G$, so that $F-G \in K_{X^*}(S_1)$. Since $F = (F-G)+G$, and since $K_{X^*}(S_1)$ and S_2 are contained in $K_{X^*}(S)$, it follows that $K_{X^*}(S)$ is the sum, not necessarily direct, of $K_{X^*}(S_1)$ and S_2. Since S_1 is a norm-closed subspace of $J_X(X)$, it is clear that $K_{X^*}(S_1) \cap J_X(X) = S_1$. Thus $K_{X^*}(S_1)$ is the direct sum of S_1 and a finite-dimensional subspace S_3 of X^{**}, so that $K_{X^*}(S_1)$ and hence also $K_{X^*}(S)$ are norm-closed in X^{**} ([3], p. 14).

Now $K_{X^*}(S)$ is the sum of S_1, S_2, and S_3. Since S_2 and S_3 are finite-dimensional, it follows that $K_{X^*}(S)$ is the direct sum of S_1 and some finite-dimensional subspace S_4 of X^{**}. Hence $K_{X^*}(K_{X^*}(S))$ is the sum of $K(S_1)$ and S_4, but this sum is equal to $K_{X^*}(S)$.

THEOREM 6. *Let X be quasi-reflexive, Y a closed subpsace of X, and S a subspace of Y^*. Then $K_Y(S)$ is norm-closed in Y^*, and $K_Y(K_Y(S)) = K_Y(S)$.*

Proof. It has been shown in [2], pp. 908-909, that Y must be quasi-reflexive and that there must exist a topological isomorphism T from Z^* onto Y for some quasi-reflexive space Z. Now $T^*(S)$ is a subspace of Z^{**} and hence $K_{Z^*}(T^*(S))$ is norm-closed in Z^{**} by Theorem 5. It is clear that $K_Y(S) = (T^*)^{-1}[K_{Z^*}(T^*(S))]$, so that $K_Y(S)$ is norm-closed in Y^*. Further, $K_Y(K_Y(S)) = (T^*)^{-1}[K_{Z^*}(K_{Z^*}(T^*(S)))] = (T^*)^{-1}[K_{Z^*}(T^*(S))] = K_Y(S)$.

It is easy to show that the basis $\{x_n\}$, where $x_n = (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$, of the quasi-reflexive space E of R. C. James ([5], p. 523), is of type P^*. Thus by Theorem 1, (1.4) is satisfied but Theorem 6 shows that (1.5) cannot be satisfied for E.

The following theorem improves the result of Civin and Yood ([2], p. 909) to the effect that if X is quasi-reflexive, then X is reflexive if and only if it is weakly complete.

THEOREM 7. *If X is quasi-reflexive, then $K_{X^*}(J_X(X)) = X^{**}$.*

Proof. The result is trivial if X is separable, for then X^* is also separable. In the general case, there is a topological isomorphism T from Y^* onto X for some quasi-reflexive space Y; here Y and Y^* are of the same deficiency n as X itself. Now Y has a reflexive subspace Z such that Y/Z is separable ([2], p. 910). Then $(Y/Z)^*$ is isometrically isomorphic with the annihilator A of Z in Y^* (Day [3], p. 25). Since Z is reflexive, it follows that Y/Z, A, and TA are quasi-reflexive with deficiency n in their respective second conjugates ([2], pp. 908-909).

If i is the identity mapping from $B = TA$ into X, then i^{**} is an isometric isomorphism from B^{**} into X^{**}, and it is easily verified that $i^{**}(K_{B^*}(J_B B)) \subseteq K_{X^*}(J_X X)$. Now $K_{B^*}(J_B B)$ is equal to B^{**}, which is the direct sum of $J_B B$ and an m-dimensional subspace S of B^{**}; hence $i^{**}S \subseteq K_{X^*}(J_X X)$. It may be verified directly that $(i^{**}S) \cap J_X X = (0)$; indeed, if $i^{**}F = J_X x$, where $F \in S$ and $x \in X$, then $x \in B$, but then $J_B x = F$, so that $x = 0$. Since $K_{X^*}(J_X X)$ contains the direct sum of $J_X X$ and the n-dimensional subspace $i^{**}S$, it must be that $K_{X^*}(J_X X) = X^{**}$.

4. We raise the following questions concerning possible improvements of the results in this paper:

(**P1**) If X is separable and $Q_X < \infty$, must X be reflexive?

More specifically,

(**P2**) If X has a basis and $Q_X < \infty$, must X be reflexive?

Singer ([8], p. 368) has posed the following question:

(**P3**) If X is a non-reflexive space with a basis, must X have a basis of type P^*?

An affirmative answer to (P3) would, by virtue of Theorem 1, answer (P2) affirmatively.

(**P4**) Are (P2) and (P3) equivalent?

(**P5**) If $Q_X < \infty$, then must every subspace of X with an unconditional basis be reflexive?

We remark that the method of proof of Theorem 4 is ineffectual in trying to answer (P5). Consider $X = C[0,1]$ and Y a subspace of $C[0,1]$ such that Y is isometrically isomorphic to l^1. Now $C[0,1]^*$ is weakly complete and hence can contain no subspace isomorphic to (c_0). Thus by [1], Thm. 4, p. 155, there is no continuous projection from $C[0,1]$ onto any subspace isomorphic to (l^1).

It is interesting to note that the converse of (P5) is false. Again consider the quasi-reflexive space E of James. We have shown that $Q_E = \infty$, but A. Pełczyński (see [8], p. 368) has remarked that every subspace of E with an unconditional basis is reflexive.

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On sequences of continuous functions and convolution

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1. In the study of Mikusiński operators the question arises "given a sequence of continuous functions g_n on the half-line $t \geqslant 0$ is there a single non-zero continuous g such that, for each n, g is of the form

$$g(t) = \int_0^t g_n(t-u) f_n(u)\,du, \qquad t \geqslant 0, \tag{1}$$

where f_n is a continuous function?" For an affirmative answer it is obviously necessary that there exist some interval $[0, T]$, $T > 0$, such that none of the g_n vanish identically on $[0, T]$. If this condition is satisfied the answer given by Theorem 3 below is "yes, there is always such a function g".

In what follows we will utilize the following notation. The functions involved are complex values functions on the half-line $t \geqslant 0$; juxtaposition of functions denotes convolution so that equation (1) will be written $g = g_n f_n$. C is the vector space of continuous functions, and L is the vector space of locally integrable functions. For g in C or in L we will use the semi-norm

$$\|g\|_T = \int_0^T |g|(t)\,dt,$$

and a sequence g_n is convergent in L to g if $\|g_n - g\|_T \to 0$ for every $T > 0$. The fundamental inequality for this semi-norm (in addition to the triangle inequality) is that, for any two functions g and f in L, $\|gf\|_T \leqslant \|g\|_T \|f\|_T$. The set C_0 (or L_0) is the set of all g in C (or L) such that $\|g\|_T > 0$ for all $T > 0$; that is, it consists of those functions which vanish on no neighborhood of the origin. In particular, a function g in C_0 is not the zero function. The symbol h will be used for that function in C which is such that $h(t) = 1$ for all $t \geqslant 0$.

The basic principle in what follows is a theorem of Ċ. Foiaş which says