On the theory of \((\mathcal{F})\)-sequences

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**Introduction.** It very often happens that considering an \((\mathcal{F})\)-space we are virtually confronted with an inverse sequence of pseudonormed spaces which yields the \((\mathcal{F})\)-space as its projective limit.

However, it may happen that together with restricting our attention to the inverse limit only, we are losing some of important properties of the spaces from the initial sequence.

This paper suggests a method of handling an inverse sequence as a whole. The object introduced for such purposes we call an \((\mathcal{F})\)-sequence. The concept of \((\mathcal{F})\)-sequence, announced in [5] comes as a consequence of a careful analysis of results and methods of [2]-[4]. Applications to some essential points considered in [2]-[4] will appear separately.

Though the bare notion of \((\mathcal{F})\) and pre-(\((\mathcal{F})\))-sequences gives scarce intuition as to its most important applications, it is still a very natural thing to consider these notions and their elegant mathematical form should appeal even without any important applications at hand.

**Terminology and notation.** We denote by \((a_n)\) the set of elements of a sequence \(x_1, x_2, \ldots\) of elements of some \(X\) which justifies writing the inclusion \((a_n) \subseteq X\).

An operation is said to be linear iff it is additive and homogeneous and we do not require any kind of continuity. This differs from the standpoint of [1].

*Pseudonorms* will always be understood as subadditive non-negative and positive-homogeneous functionals vanishing in zero. As usual a pseudonorm may assume the value zero on non-zero element.

Suppose \(X\) and \(Y\) are subsets of the same set \(Z\) and \(Y\) is provided with some topology \(\tau\). We say that \(X\) is of the second category in \((Y, \tau)\) iff \(X \cap Y\) is of the second category in \((Y, \tau)\).

Consider two linear topological spaces \((X_i, \tau_i)\), \(i = 1, 2\). We say that \((X_2, \tau_2)\) is coarser than \((X_1, \tau_1)\) and we write \((X_1, \tau_1) \leq (X_2, \tau_2)\) iff \(X_1\) is a subspace of \(X_2\) and the identical injection of \((X_2, \tau_2)\) into \((X_1, \tau_1)\) is continuous.
A sequence $\mathfrak{V} = (\{V_n, ||\cdot||_n\})$ of linear spaces each provided with a pseudonorm is said to be a pre-$\mathcal{F}$-sequence iff $(V_n, ||\cdot||_n) \preccurlyeq (V_{n+1}, ||\cdot||_{n+1})$, for $n = 1, 2, \ldots$ and $x_n \in V_n$ vanishes whenever $||x_n||_n = 0$ for every $n$.

If the pseudonorms of a pre-$\mathcal{F}$-sequence $\mathfrak{V}$ are not given directly, we shall write $||\cdot||_n$ for the pseudonorm of the $n$-th space of the sequence $\mathfrak{V}$.

With any pre-$\mathcal{F}$-sequence $\mathfrak{V} = (\{V_n, ||\cdot||_n\})$ we associate the following three notions:

(i) The linear space

$$\mathfrak{V} = V_1,$$

(ii) The linear locally convex metrizable space

$$\mathfrak{V} = \bigcap_{n=1}^{\infty} V_n,$$

where $\tau$ denotes the topology of $\bigcap_{n=1}^{\infty} V_n$ induced by pseudonorms

$$\{||\cdot||_n; n = 1, 2, \ldots\}.$$

(iii) The metric-function $\rho_n$, defined and translation-invariant on $\mathfrak{V}$ considered as an abelian group with the linear space addition as the group operation, given by fixing the distance $\rho_n(x) = \rho_n(x - 0)$ of $x \in V_n$ from 0 as follows:

$$\rho_n(x) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(x),$$

where

$$\rho_n(x) = \begin{cases} ||x||_1 (1 + ||x||_2) & \text{for } x \in V_2, \\ 1 & \text{for } x \in \mathfrak{V} - V_n. \end{cases}$$

Since $\rho_n$ are all additive, $\rho_n$ must be subadditive as well. The metric-function $\rho_n$ and the topological group $\mathfrak{V}$, where

$$\mathfrak{V}^* = (\mathfrak{V}, ||\cdot||_\rho),$$

are said to be assigned to $\mathfrak{V}$.

A pre-$\mathcal{F}$-sequence $\mathfrak{V}$ is said to be an $\mathcal{F}$-sequence iff $\mathfrak{V}$ is complete.

It is easy to see that $\bigcap_{n=1}^{\infty} V_n$ is a closed subgroup of $\mathfrak{V}$.

Examples. I. Consider the $k$-dimensional Euclidean space $\mathbb{R}^k$ and denote by $\mathcal{P}$ the space of all complex valued functions $f$ defined on $\mathbb{R}^k$ with continuous derivatives $D^p f$ for every $p = (p_1, \ldots, p_k)$ with $\sum p_i < \infty$. For any compact $K \subseteq \mathbb{R}^k$ we write $\mathcal{P}^s(K) = \{f \in \mathcal{P}; \text{ support } f \subseteq K\}$.

With any descending sequence $\{K_n\}$ of compact subsets of $\mathbb{R}^k$ such that each $K_n$ coincides with the closure of its interior, we associate the $\mathcal{F}$-sequences

$$\mathfrak{V}_n = (\mathcal{P}^s(K_n), ||\cdot||_{\rho_n}) \quad \text{and} \quad \mathfrak{V} = (\bigcap_{n=1}^{\infty} \mathcal{P}^s(K_n), ||\cdot||_{\rho}),$$

where $\rho_n$ is an increasing sequence of natural numbers.

In the first case the spaces $(\mathcal{P}^s(K_n), ||\cdot||_{\rho_n})$ are all complete and then $\mathfrak{V}$ is automatically complete, and in the second case none of $(\mathcal{P}^s(K_n), ||\cdot||_{\rho_n})$ is complete while $\mathfrak{V}$ is still complete. Moreover, if $\bigcap_{n=1}^{\infty} K_n$ is discrete, then $\mathfrak{V}$ is trivial, i.e., represents $\mathcal{F}$-space that consists of only point zero.

II. Consider a $k$-dimensional complex linear space $\mathbb{C}^k$. For $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ we write $|z| = \sum |z_i|$ and $\text{Im} z = (\text{Im} z_1, \ldots, \text{Im} z_k) \in \mathbb{R}^k$.

Take any entire function $f$ defined on $\mathbb{C}^k$. We put

$$\|f\|_p = \sup_{z \in \mathbb{C}^k} (1 + |z|)^p |f(z)| e^{||z||_\rho}, \quad a > 0,$$

and

$$\mathcal{F}_a = \{f \text{-entire on } \mathbb{C}^k; ||f\|_p < \infty\}.$$

For any increasing $(m_n)$ and decreasing $(a_n)$, $a_n > 0$, tending to some $a > 0$, we introduce an $\mathcal{F}$-sequence

$$\mathfrak{V} = (\{\mathcal{F}_a, ||\cdot||_\rho\}).$$

It is clear, that for $K_n$ from Example I equal to $(x \in \mathbb{R}^k; |x| < a_n)$, the Fourier transform $F$ maps $\mathfrak{V}$, where $\mathfrak{V}$ is defined in Example I, into $\mathfrak{V}$. It turns out that $F$ is closed in the sense which will be introduced in this paper, every $\mathcal{F}_a$ contains some $\mathcal{P}^s(K)$, and, vice versa, every $\mathcal{P}^s(K)$ contains some $\mathcal{F}_a$. This can serve as an illuminating illustration to Theorems 3 and 4 yielding in the same time a more general formulation of the Paley-Wiener theorem.

III. Consider a sequence $\mathfrak{V} = (\{Z_n, ||\cdot||_n\})$ of linear spaces with pseudonorms such that $(Z_n, ||\cdot||_n) \supseteq (Z_{n+1}, ||\cdot||_{n+1})$ for $n = 1, 2, \ldots$, and the linear space $X = \bigcup_{n=1}^{\infty} Z_n$ and a space $X'$ of linear functionals over $X$, $X'$ closed with respect to the pointwise addition, multiplication by scalars and convergence of sequences of functionals.

We define the polar $\mathfrak{V}'$ of $\mathfrak{V}$ in $X'$ setting $\mathfrak{V}' = (\{Z_n', ||\cdot||_n\})$, where

$$||z'||_n = \sup_{x \in Z_n} |z'(x)| \text{ for } x \in Z_n,$$

and $Z'_n = \{z \in X': ||z||_n < \infty\}.$
It is clear that \( \mathcal{Z} \) is an \((\mathcal{F})\)-sequence and that \( \bigcap_{n=1}^{\infty} \mathcal{Z}_n \) consists of all functional continuous on the inductive limit of \( 3 \). It is clear, moreover, that the spaces \( (\mathcal{Z}_n, \|\cdot\|) \) need not be complete.

IV. Take a topological space \( E \) and let \((R_n)\) be a sequence of open subsets of \( E \), \( R_n \subset R_{n-1} \) for \( n = 1, 2, \ldots \), such that \( E = \bigcup_{n=1}^{\infty} R_n \). For any \( U \subset E \) denote by \( C(U) \) the linear space of all continuous scalar-valued functions defined on \( E \) and bounded on \( U \). For \( f \in C(U) \) let \( \|f\|_U = \sup \{|f(x)| : x \in U\} \).

The sequence \( \mathcal{C} = \{\|C(R_n), \|\|\|\}_n \} \) is an \((\mathcal{F})\)-sequence. Here \( \bigcap_{n=1}^{\infty} C(R_n) \) consists of all continuous functions on \( E \) which are bounded on every \( R_n \). The spaces that constitute \( \mathcal{C} \) are generally not complete.

V. Consider two \((\mathcal{F})\)-spaces \((X_i, \tau_i), i = 1, 2\), where the topologies \( \tau_i \) are given by pointwise non-decreasing sequences of pseudonorms \( \{\|\cdot\|_n\} \) for \( i = 1, 2 \) respectively.

Let \( E \) be the space of linear mappings of \( X_1 \) into \( X_1 \) closed with respect to the pointwise addition, multiplication by scalars and convergence. We put

\[
\|A_{kn}\| = \sup \{||Ax|| : \|x\|_n \leq 1\}, \quad L_{kn} = \{A \in E : \|A_{kn}\| \leq 1\},
\]

and further we set

\[
\|f\|_{X, Y} = \sup \{|f(x, y)| : \|x\|_1, \|y\|_2 \leq 1\}
\]

and further we set

\[
B_{kn} = \{g \in E : \|g\|_{X, Y} \leq 1\}.
\]

Exactly like in the previous example to any increasing sequence \( \{\|\cdot\|_n\} \) there corresponds the \((\mathcal{F})\)-sequence

\[
\mathcal{C} = \{\|B_{kn}, \|\|\|\}_n \}.
\]

Setting \( X = X_1 \) and \( Y = \text{the adjoint to } (X_1, \tau_1) \), where \( (X_1, \tau_1) \) are from Example V, we find that assigning to \( A \in E \) the functional \( \varphi \in E \), where \( \varphi(x, y) = \varphi(x) \cdot \sigma(y) \), we obtain a mapping of \( \mathcal{C} \) into \( \mathcal{C} \). This, together with Example II, provides the reader with some intuitive background as to what kind of mappings are subject to the discussion presented in this paper.

However, we do not propose to draw any conclusions from Theorems 3 and 4 of this paper as yet, waiting for another paper which together with further development of the theory will bring better opportunities for a careful study of the examples.

**Proposition 1.** Consider a pre-\((\mathcal{F})\)-sequence \( \mathcal{B} = \{\{V_n, \|\cdot\|_n\}\} \).

A sequence \( \{x_n\} \subset \mathcal{B} \) tends to zero in \( \mathcal{B} \) iff to each \( p \) there correspond \( m_p \) such that \( x_n \in V_p \) for \( n \geq m_p \) and \( \lim_{n \to \infty} \|x_n\|_p = 0 \); \( \{x_n\} \subset \mathcal{B} \) tends to \( x \) in \( \mathcal{B} \) iff \( \{x_n - x\} \subset \mathcal{B} \) tends to zero in \( \mathcal{B} \); \( \{x_n\} \subset \mathcal{B} \) satisfies the Cauchy condition in \( \mathcal{B} \) iff to every \( p \) there correspond \( m_p \) such that \( x_n - x_m \in V_p \) for \( m, n \geq m_p \) and \( \lim_{n \to \infty} \|x_n - x_m\|_p = 0 \).

B. Denote by \( \mathcal{C} \) the operation of closure in \( \mathcal{B} \). A point \( x \in \mathcal{B} \) belongs to the closure \( \mathcal{C} \) of some \( G \subset \mathcal{B} \) for every \( n \) the set \( G \cap (x - V_n) \) is non-void and

\[
\inf \{|x - u| : u \in G \cap (x - V_n)\} = 0.
\]

C. If \( \{x_n\} \subset \mathcal{B} \) satisfies the Cauchy condition in \( \mathcal{B} \), then for every scalar \( t \) we have \( \{tx_n\} \subset \mathcal{B} \) satisfying the Cauchy condition in \( \mathcal{B} \). If, in particular, the sequence \( \{x_n\} \subset \mathcal{B} \) tends to some \( x \in \mathcal{B} \), then \( \{tx_n\} \subset \mathcal{B} \) tends to \( tx \) in \( \mathcal{B} \).

D. Take \( x \in \mathcal{B} \) and a sequence of scalars \( \{t_n\} \subset \mathcal{B} \) tending to some scalar \( t \) different from countably many \( t_n \). If \( \{t_nx_n\} \subset \mathcal{B} \) tends to \( tx \) in \( \mathcal{B} \), then \( x \in \mathcal{B} \).

**Proof.** Convergence of \( \{x_n\} \) to zero in \( \mathcal{B} \) is equivalent to \( \varphi(x_n) \rightarrow x_n \rightarrow 0 \) tending to zero for every \( p \) separately which again is equivalent to the condition given in A. The next follows from the fact that \( \mathcal{B} \) is a topological group. Similarly, the Cauchy condition for \( \{x_n\} \subset \mathcal{B} \) amounts to \( \varphi(x_n - x_m) \rightarrow x - x \rightarrow 0 \) tending to zero for every \( p \) separately which again is equivalent to the condition given in A. An element \( x \) belongs to \( \mathcal{B} \) iff \( \varphi(x - g) \rightarrow x - g \rightarrow 0 \) for every \( n \). Hence, for every \( n \) the set \( \mathcal{G} \cap (x - V_n) \) must be non-void and \( \varphi(x - g) \rightarrow x - g \rightarrow 0 \) which proves B. The last two assertions are simple consequences of the first one.

**Proposition 2.** To every pre-\((\mathcal{F})\)-sequence \( \mathcal{B} = \{\{V_n, \|\cdot\|_n\}\} \) there corresponds an \((\mathcal{F})\)-sequence \( \mathcal{B}^* = \{\{V_n^*, \|\cdot\|_n^*\}\} \) such that \( V_n^* \subset V_n \) for \( n = 1, 2, \ldots \), where \( \mathcal{C} \) denotes the closure in \( \mathcal{B}^* \) and \( \|\cdot\|_n^* \) coincides with \( \|\cdot\|_n \) restricted to \( V_n^* \) respectively for \( n = 1, 2, \ldots \). The \((\mathcal{F})\)-sequence \( \mathcal{B}^* \) is called the completion of \( \mathcal{B} \).
Proof. Let $(V^{-}, e^{-})$ denote the completion of the abelian topological group $(\mathcal{B})$. Then, $(\mathcal{B})$ is a dense subgroup of $(V^{-}, e^{-})$ and $\mathcal{O}$ is the restriction of $\mathcal{O}$ to $(\mathcal{B})$. From Proposition 1 it follows that the multiplication by scalars in $(\mathcal{B})$ can be extended over $V^{-}$ in such a way that $V^{-}$ becomes a linear space. Indeed, if $(a_n) \subset (\mathcal{B})$ tends to some $e \in V^{-}$ in the sense of $e^{-}$, then for any scalar $t$ the sequence $(ta_n)$ in $\mathcal{O}$ tends to some $e \cdot t e^{-}$.

Setting $s_1 = t$ we obtain a non-ambiguous definition of multiplication by scalars in $V^{-}$ which makes a linear space out of the group $V^{-}$. Since all $\mathcal{O}$ are continuous with respect to $\mathcal{O}$ and then $e^{-}$ as well, we can extend them over $V^{-}$ to $\mathcal{O}$. Since $e^{-}$ is the extension of $\mathcal{O}$ over $V^{-}$, we shall have

$$e^{-}(x) = \sum_{n=0}^{\infty} 2^{-n} \mathcal{O}_n(x)$$

for $e \in V^{-}$.

Setting $V_{p} = \{xe^{-} : \mathcal{O}_n(x) < 1\}$ we find that $V^{-} = V_{1}$, $V_n \supset V_{n+1}$, and, moreover, for every $p$ the pseudonorm $\|\cdot\|_p$ extends from $V_p$ over $V^{-}$ to $\|\cdot\|_p$. Indeed, if $(a_n) \subset (\mathcal{B})$, $\mathcal{O}_n(a_n) \to 0$ and $\mathcal{O}_n(x) < 1$, then $\mathcal{O}_n(x_n) < 1$ for $n \geq n_0$ and $(a_n : n \geq n_0) \subset V_p$. Then $\|x_n - a_n\|_p \to 0$ and setting $\|x\|_p = \lim_\to_\to (\|x_n\|_p)$ we find that the definition of $\|\cdot\|_p$ does not depend on the choice of $(a_n)$ and produces the extension of $\|\cdot\|_p$ to a continuous pseudonorm on $V^{-}$ such that $\mathcal{O}(x) = \|x\|_p (1 + \|x\|_p)$ for $x \in V^{-}$. It is clear that $\mathcal{B}^{-} = (V_n, \|\cdot\|_n)$ is an $(\mathcal{F})$-sequence with $(\mathcal{B})^{-} = (V^{-}, e^{-})$, satisfying all the requirements of the Proposition. This finishes the proof of Proposition 2.

Consider a pre-$(\mathcal{F})$-sequence $\mathcal{B} = (\{V_n, \|\cdot\|_n\})$. With any natural $p$ we associate the $p$-th shift $p\mathcal{B}$ of $\mathcal{B}$ defined as

$$p\mathcal{B} = (\{V_n, \|\cdot\|_n : n = p+1, p+2, \ldots\})$$

i. e. $p\mathcal{B}$ is produced from $\mathcal{B}$ by dropping first $p$ elements of the sequence $\mathcal{B}$. Using the notion of $p$-th shift we can express the space that appears in the $p$-th element of $\mathcal{B}$ as $p\mathcal{B}$. Hence, we can write the identity $\mathcal{B} = (\{\{p\mathcal{B}, \|\cdot\|_n\})$. Here comes a proposition showing that for every $\mathcal{B}$ we have the decomposition $(\mathcal{B}) = \bigcap_{p} \cap \{\{n\mathcal{B}, \|\cdot\|_n\}) into disjoint open and closed subsets of $\mathcal{B}$.

Proposition 3. Consider a pre-$(\mathcal{F})$-sequence $\mathcal{B}$. For any natural $n$ the identical injection of $[n\mathcal{B}]$ into $(\mathcal{B})$ is bicontinuous and $[n\mathcal{B}]$ is open and closed in $(\mathcal{B})$.

Proof. The bicontinuity of the identical injection of $[n\mathcal{B}]$ into $(\mathcal{B})$ follows from the fact that the convergence of $\mathcal{O}_n(x_n)$ to zero amounts to the convergence to zero of $\mathcal{O}_n(x_n)$ separately for every $p$ taken from any fixed countable set of natural numbers. Hence, for $(x_n) \subset [n\mathcal{B}]$ the condition $\lim_\to_\to (\mathcal{O}_n(x_n) = 0$ is equivalent to the condition $\lim_\to_\to (\mathcal{O}_n(x_n) = 0$.

Every $(\mathcal{B})$ is closed in $(\mathcal{B})$. Indeed, if $(x_n)$ tends to some $x \in (\mathcal{B})$, then $x = \mathcal{E}(x_n)$ for sufficiently great $n$ and then $x \subset (\mathcal{B})$ as well. Similarly, if $(x_n) \subset (\mathcal{B})$, then $x = \mathcal{E}(x_n)$ for sufficiently great $n$ and this time, since every $x_n$ does not belong to $(\mathcal{B})$, $x$ does not belong to $(\mathcal{B})$ either. This finishes the proof of Proposition 3.

It is worth noticing that the fact of bicontinuity of the identical imbedding of $(\mathcal{B})$ in $(\mathcal{B})$ generalizes to the bicontinuity of such imbedding of any $(\mathcal{B})$ obtained by dropping of any fixed number of elements of $\mathcal{B}$ in such a way that the remaining is still an infinite sequence. This is which was practically proved in the above given proof.

Consider two pre-$(\mathcal{F})$-sequences $\mathcal{B}_1, \mathcal{B}_2$, and set $\mathcal{B}_1 > \mathcal{B}_2$, if there exists a number $p$ such that $[p\mathcal{B}_1]$ is a subspace of $[p\mathcal{B}_2]$ and the identical injection of $[p\mathcal{B}_2]$ into $[p\mathcal{B}_1]$ is continuous; $\mathcal{B}_1$ is said to be equivalent to $\mathcal{B}_2$, if $\mathcal{B}_1 > \mathcal{B}_2$ and $\mathcal{B}_2 > \mathcal{B}_1$. Obviously $\mathcal{B}_1 > \mathcal{B}_2$ implies $\mathcal{B}_2 > \mathcal{B}_1$ and every pre-$(\mathcal{F})$-sequence $\mathcal{B}$ is equivalent to any subsequence of itself.

For example, the $(\mathcal{F})$-sequences $\mathcal{B}$ and $\mathcal{B}_2$, introduced in Example I, are equivalent by virtue of the well-known Sobolev lemma, assuring that the uniform convergence of derivatives up to a given order follows from $L$-convergence of derivatives up to some higher order.

Consider a pre-$(\mathcal{F})$-sequence $\mathcal{B} = (\{V_n, \|\cdot\|_n\}) and a linear $(\mathcal{B})$-closed subspace $L$ of $(\mathcal{B})$. We define the quotient pre-$(\mathcal{F})$-sequence (of $(\mathcal{B})$)

$$\mathcal{B}/L = (\{V_n/L, \|\cdot\|_n\})$$

as follows. We set $[\mathcal{B}]/L \equiv [\mathcal{B}]/L and we denote by $V_n/L$ the subspace of $[\mathcal{B}]/L$ with representatives taken from $V_n$. We define in the following the pseudonorms

$$\|x/L\|_n = \inf \{\|x - k\|_n : k \in L \cap (x + V_n)\}$$

for $x/L \subset V_n/L$. Notice that $L \cap (x + V_n)$ is non-void for $x/L \subset V_n/L$. If $x/L \subset V_n/L and \|x/L\|_n = 0$ for $n = 1, 2, \ldots$, then $x/L = 0$, i.e. $x/L = 0$.

Indeed, if $\|x/L\|_n \neq 0$, there exists $k \in L \cap (x + V_n)$ such that $\|x - k\|_n < 1/n$, where $(M_n) \subset M_n > 0$, are chosen
in such a way that $M_n |\eta|_n \leq M_{n+1} |\eta|_{n+1}$ for $x \in V_n$, $n = 1, 2, \ldots$. Hence $(k_n)$ tends to $x$ in $(\mathcal{S})$ and, since $L$ is closed in $(\mathcal{S})$, we have $x \in L$.

In view of the fact that the relations

$$(V_n/L_1, \mathcal{S}) = (V_{n+1}/L_1, \mathcal{S}), \quad n = 1, 2, \ldots,$$

hold as simple consequences of the corresponding relations in $\mathcal{B}$ itself, we conclude that $(\mathcal{S})$ is a pre-$(\mathcal{F})$-sequence.

Looking at $(\mathcal{S})$ from the topological group viewpoint we can always construct the factor-group $(\mathcal{S})/L$ which is a topological group in the case of closed $L$ and, moreover, the group $(\mathcal{S})/L$ is complete whenever the original group $(\mathcal{S})$ is complete. In view of the definition of $(\mathcal{S})/L$ we know that $(\mathcal{S})/L$ and $(\mathcal{S})/L$ are identical algebraically.

**Proposition 4.** Consider a pre-$(\mathcal{F})$-sequence $\mathcal{B}$ and a linear $(\mathcal{S})$-closed subspace $L \subset \mathcal{B}$. The topological groups $(\mathcal{S})/L$ and $\mathcal{B}/L$ are topologically the same.

**Proof.** Let $\mathcal{B} = \{ (V_n, \| \cdot \|_n) \}$ and take $x_n/L \in \mathcal{B}/L$, where $L$ is a $(\mathcal{S})$-closed subspace of $(\mathcal{S})$. The sequence $(x_n/L)$ tends to $0$ in $(\mathcal{S})/L$ iff there exists $k_n \in L$ such that $\rho_n(x_n - k_n)$ tends to zero, i.e., $x_n - k_n \in V_p$ for $n \geq n_0$, and $\lim_{n \to \infty} \rho_n(x_n - k_n) = 0$ for $p = 1, 2, \ldots$, where $(m_n)$ is a properly chosen sequence. This can be equivalently expressed by saying that for every $p$ we have $x_n/L \in V_p/L$ for $n \geq n_0$ and $\lim_{n \to \infty} \rho_n(x_n) = 0$ which means exactly that $(x_n/L)$ tends to zero in $(\mathcal{S})/L$. This finishes the proof of Proposition 4.

To establish a relation between topologies of the elements of a given pre-$(\mathcal{F})$-sequence $\mathcal{B}$ and the topology of the assigned group $(\mathcal{S})$ we prove the following statement:

**Proposition 5.** Let $\mathcal{B}$ be a pre-$(\mathcal{F})$-sequence and define for any positive $r$ and natural $n$ the following subsets of $\mathcal{B}$:

$$K_\mathcal{B}(r) = \{ x \in \mathcal{B}; \rho(x) < r \}, \quad K_{\mathcal{B},n}(r) = \{ x \in \mathcal{B}; \rho(x) < r \}. $$

A. To every natural $p$ and $\varepsilon > 0$ there corresponds $\eta > 0$ such that

$$K_\mathcal{B}(\eta) \subset K_{\mathcal{B},n}(\varepsilon).$$

B. To every $\varepsilon > 0$ there corresponds a natural $p$ and $\eta > 0$ such that

$$K_\mathcal{B}(\varepsilon) \supset K_{\mathcal{B},n}(\eta).$$

**Proof.** If $x \in V_p \subset \mathcal{B}$, then

$$\rho_\mathcal{B}(x) = \rho_\mathcal{B}(\varepsilon) = 1$$

for $n \geq p$ and

$$g(x) \equiv g(x) \geq \sum_{n \geq p+1} 2^{-n} \phi_n(x) = \sum_{n \geq p+1} 2^{-n} = 2^{-n}.$$
which amounts to the boundedness of \( ||\cdot || \) on \( K_B(x) \). Hence \( x \) and \( \beta \) are equivalent.

The implication \( z \rightarrow y \) follows directly from Proposition 1 and then to complete the proof it is sufficient to show the implication \( y \rightarrow z \). We show it proving its contraposition. If \( ||\cdot || \) is not bounded on any \( K_B(x) \), then from A of Proposition 5 it follows that \( ||\cdot || \) is not bounded on any \( K_{B,n}(x) \) and then to each \( n \) there correspond \( x_n \in K_{B,n}(1/n) \) such that \( ||x_n|| \geq 1 \). Hence, by virtue of A of Proposition 1, \( (x_n) \) tends to zero in \( [B] \) and \( ||x_n|| \geq 1 \) for every \( n \) which contradicts \( y \).

This concludes the proof of Proposition 6.

From now on we shall start using the notion of topological category. Applied in the case of the assigned group this notion admits some extra properties which are explained in the following proposition:

**Proposition 7.** Take a pre-\((\mathcal{F})\)-sequence \( \mathcal{C} \). If \( Y \subset [B] \) is of the second category in \( [B] \), then \( Y \) is of the second category in \( [B] \) as well.

**Proof.** Suppose \( Y \) is of the second category in \( [B] \) and take a decomposition \( Y = \bigcup_{n=1}^{\infty} Y_n \). There must be then a \( Y_{n_0} \) which closure contains a ball \( K_\varepsilon \) taken in \( [B] \). Since in view of Proposition 1, \( [B] \) is open in \( [B] \), the ball \( K_\varepsilon \) is open in \( [B] \) as well and \( Y_{n_0} \) is not nowhere dense in \( [B] \) which concludes the proof of Proposition 7.

Now comes the well-known result which is often called the Banach and Steinhaus Theorem:

**Theorem 1.** Consider a pre-\((\mathcal{F})\)-sequence with the second category assigned group \( [B] \). Every \( [B] \)-lower semi-continuous pseudonorm defined on \( [B] \) is \( (\mathcal{F}) \)-continuous.

**Proof.** In view of the Proposition 6 it is sufficient to show that every \( [B] \)-lower semi-continuous pseudonorm \( ||\cdot || \) is bounded on at least one open subset of \( [B] \).

We have

\[
[B] = \bigcup_{n=1}^{\infty} \{ x \in [B] : ||x|| \leq n \}
\]

and since \( ||\cdot || \) is lower semi-continuous, every \( \{ x \in [B] : ||x|| \leq n \} \) is closed in \( [B] \). Furthermore, \( [B] \) is of the second category and then at least one of \( \{ x \in [B] : ||x|| \leq n \} \) is not nowhere dense in \( [B] \). Hence there is \( n_0 \) and an open subset \( U \) of \( [B] \) such that \( ||x|| \leq n_0 \) for \( x \in U \) and the Theorem follows.

In order to prove some other properties of pre-\((\mathcal{F})\)-sequences we shall need certain new notions.

Consider two pre-\((\mathcal{F})\)-sequences \( \mathcal{C}_i \) \( (i = 1, 2) \) and a linear mapping \( T \) of \( [B]_1 \) into \( [B]_2 \).

The mapping \( T \) is said to be nearly-open iff the following condition holds:

1. To every \( \varepsilon > 0 \) there correspond \( \eta > 0 \) such that

\[
\text{Cl}_0 \{ K_{B,n}(r) \} \subset K_{B,n}(\eta),
\]

where \( \text{Cl}_0 \) denotes the operation of closure in \( [B]_1 \) and \( K_{B,n}(r) \) \( (i = 1, 2) \) are defined according to Proposition 5.

The mapping \( T \) is said to be open iff the following condition holds:

2. To every \( \varepsilon > 0 \) there correspond \( \eta > 0 \) such that

\[
K_{B,n}(\varepsilon) \supset K_{B,n}(\eta),
\]

where \( K_{B,n}(r) \) are defined according to Proposition 5.

The mapping \( T \) is said to be complete-closed iff the following condition holds:

3. If \( (a_n) \subset [B] \) satisfies the Cauchy condition in \( [B] \) and the sequence \( \{ T_{a_n} \} \) tends in \( [B] \) to some \( y \in [B] \), then there exists \( \varepsilon \in [B] \) such that \( \{ a_n \} \subset \varepsilon \) tends to \( \varepsilon \) and \( y = T_x \).

The mapping \( T \) is said to be closed iff the following condition holds:

4. If \( (a_n) \subset [B] \) tends in \( [B] \) to some \( \varepsilon \in [B] \) and \( \{ T_{a_n} \} \) tends in \( [B] \) to some \( y \in [B] \), then \( y = T_x \).

The coromomop analysis will explain the notion of nearly open and open mappings entirely in terms of elements of participating pre-\((\mathcal{F})\)-sequences.

**Proposition 8.** Consider pre-\((\mathcal{F})\)-sequences \( \mathcal{C}_i \) \( (i = 1, 2) \) and a linear mapping of \( [B]_1 \) into \( [B]_2 \).

A. The mapping \( T \) is open if and only if the following condition holds:

5. To every \( n \) and \( \varepsilon > 0 \) there correspond \( m \) and \( \eta > 0 \) such that

\[
K_{B,n}(\varepsilon) \supset K_{B,m}(\eta),
\]

where \( K_{B,n}(r) \) \( (i = 1, 2) \) are defined according to Proposition 5.

**B. Define**

\[
S_{\mathcal{C}_i}(y) = \{ x \in [B]_1 : ||Tx + y - x|| \leq \eta \} \quad \text{and} \quad K_{B,n}(r) = K_{B,n}(\eta).
\]

The mapping \( T \) is nearly open if and only if the following condition holds:

6. To every \( n \) and every \( \varepsilon > 0 \) there correspond \( m \) and \( \eta > 0 \) such that for every \( y \in K_{B,n}(r) \) and every \( p \) we have

a. \( S_{\mathcal{C}_i}(y) \) is non-empty,

b. \( ||Tx - y||_{[B]} = \inf \{ ||Tx - y||_{[B]} : x \in S_{\mathcal{C}_i}(y) \} = 0 \).

**Proof.** Let us put briefly \( K_{B,n}(r) = K(r) \) and \( K_{B,n}(r) = K_{B,n}(r) \) for \( i = 1, 2 \).
Ad A. Suppose that \( 2 \) holds and fix any \( \varepsilon > 0 \) and natural \( n \). Then, using A of Proposition 5 we find \( \varepsilon > 0 \) such that \( h_{\alpha}(\varepsilon) \geq K_{\alpha}(\varepsilon) \) and then, applying 2, we find \( \varepsilon > 0 \) such that \( TK_{\alpha}(\varepsilon) \geq K_{\alpha}(\varepsilon) \). Applying again Proposition 5, but now part B, we find \( \eta > 0 \) and \( m \) such that \( K_{\alpha}(\eta) \geq K_{\alpha}(\eta) \) and we finally get \( TK_{\alpha}(\varepsilon) \geq K_{\alpha}(\eta) \). This proves the implication 2 \( \Rightarrow \) 5.

Let conversely 5 hold. Applying B of Proposition 5 we find that to any fixed \( \varepsilon > 0 \) there correspond \( \varepsilon > 0 \) and natural \( n \) such that \( K_{\alpha}(\varepsilon) \geq K_{\alpha}(\varepsilon) \) and then from 5 we find \( \eta > 0 \) and \( m \) such that \( TK_{\alpha}(\varepsilon) \geq K_{\alpha}(\eta) \) and consequently \( TK_{\alpha}(\varepsilon) \geq K_{\alpha}(\eta) \) which shows that 5 implies 2 and concludes the proof of the first part of Proposition 8.

Ad B. Write \( S_{\alpha}(\varepsilon) = \{ x \in [\mathcal{C}] : T x \varepsilon + \| p \mathcal{C} \| \cap K_{\alpha}(\varepsilon) \} \). From Proposition 5 it follows that 6 can be replaced by the following equivalent conditions:

1. To every \( \varepsilon > 0 \) there correspond \( \eta > 0 \) such that for every \( y \in K_{\alpha}(\eta) \) and every \( p \) we have
   a. \( S_{\alpha}(\varepsilon) \) is non-void,
   b. \( \inf \| T x \varepsilon + \| p \mathcal{C} \| \cap K_{\alpha}(\varepsilon) \| = 0 \).

Indeed, suppose that 6 is satisfied and take arbitrary \( \varepsilon > 0 \). From B of Proposition 5 it follows that there are \( \varepsilon > 0 \) and \( n \) such that \( K_{\alpha}(\varepsilon) \geq K_{\alpha}(\varepsilon) \). From 6 it follows that there are \( \eta > 0 \) and \( m \) such that for \( y \in K_{\alpha}(\eta) \) and every \( p \) we have \( S_{\alpha}(\varepsilon) \) non-void and \( \inf \| T x \varepsilon + \| p \mathcal{C} \| \cap K_{\alpha}(\varepsilon) \| = 0 \). From Proposition 5 we find \( \eta > 0 \) such that \( K_{\alpha}(\eta) \geq K_{\alpha}(\eta) \) and consequently \( TK_{\alpha}(\varepsilon) \geq K_{\alpha}(\eta) \) which implies the previous fact that \( \eta \) and \( \varepsilon \) hold for the \( \varepsilon \) and \( \eta \). Hence 6 implies 7.

Suppose, conversely, that 7 holds and take arbitrary \( \varepsilon > 0 \) and natural \( n \). From B of Proposition 5 we find \( \varepsilon > 0 \) such that \( K_{\alpha}(\varepsilon) \geq K_{\alpha}(\varepsilon) \). From 7 it follows that there is \( \eta > 0 \) such that for every \( y \in K_{\alpha}(\eta) \) and every \( p \) we have \( S_{\alpha}(\varepsilon) \) non-void and \( \inf \| T x \varepsilon + \| p \mathcal{C} \| \cap K_{\alpha}(\varepsilon) \| = 0 \). Applying the above-given inclusion we find that for \( y \in K_{\alpha}(\eta) \) and every \( p \) we have \( S_{\alpha}(\varepsilon) \) non-void and \( \inf \| T x \varepsilon + \| p \mathcal{C} \| \cap K_{\alpha}(\varepsilon) \| = 0 \). Using B of Proposition 5 we find \( \eta > 0 \) and natural \( m \) such that \( K_{\alpha}(\varepsilon) \geq K_{\alpha}(\varepsilon) \) and we conclude that \( \eta \) and \( \varepsilon \) hold for the \( \varepsilon \) and \( \eta \). This establishes the implication 7 \( \Rightarrow \) 6 and then also the equivalence of 6 and 7.

To finish the proof of B of Proposition 8 it is sufficient to notice that \( T \mathcal{C}(\varepsilon) = (y + p \mathcal{C}) \cap K_{\alpha}(\varepsilon) \) and then, by virtue of B of Proposition 1, 7 amounts exactly to 1, i.e. states that \( T \) is nearly open.

This way Proposition 8 has been proved.

It should be mentioned that Proposition 8 has been introduced only for the explanatory purposes, i.e. to show how great a simplification it is to use the concept of the assigned group instead of translating everything on the language of sequences of pseudonormed spaces. Though we expect to use Proposition 8 in some other occasions, it will not be used in any proof given in this paper thanks to the application of this efficient tool which is the assigned group.

**Proposition 9.** Consider a pre-(\( F \))-sequence \( \mathcal{Q} \). For every \( \epsilon > 0 \) and every natural \( n \) we have

\[
| p \mathcal{Q} | \leq \sum_{m=1}^{\infty} | x \in [\mathcal{Q}] : \varepsilon(x/m) < \epsilon + 2\epsilon^p |.
\]

**Proof.** If \( x \in [\mathcal{Q}] \), then \( \lim q_{\mathcal{Q}}(x/m) = 0 \) for \( i = 1, 2, \ldots, p \) and for any given \( \epsilon > 0 \) we can find \( m \) such that

\[
\sum_{i=1}^{p} 2^{-i} q_{\mathcal{Q}}(x/m) < \epsilon.
\]

Then

\[
q_{\mathcal{Q}}(x/m) < \epsilon + \sum_{i=p+1}^{\infty} 2^{-i} \epsilon = \epsilon + 2\epsilon^p
\]

which proves our assertion.

**Proposition 10** (Banach [1]). Consider pre-(\( F \))-sequences \( \mathcal{Q}_i \) \((i = 1, 2)\) and a linear mapping \( T \) of \([\mathcal{Q}_i] \) into \([\mathcal{Q}_i] \). If to every \( p \) there correspond \( h_{\alpha} \) such that \( T[p \mathcal{Q}_i] \) is of the second category in \([h_{\alpha} \mathcal{Q}_i] \), then \( T \) is nearly open.

**Proof.** From Proposition 7 it follows that we can put all \( h_0 \) equal to 1. Let \( q_{\alpha} \) for \( i = 1, 2 \). Take any \( \epsilon > 0 \) and let \( p \) be such that \( \epsilon/4 > 2\epsilon^p \). From Proposition 9 it follows that

\[
| p \mathcal{Q}_i | \leq \sum_{m=1}^{\infty} | x \in [\mathcal{Q}_i] : \varepsilon(x/m) < \epsilon/2 |.
\]

Setting \( H_{\alpha} = T[p \mathcal{Q}_i] : \varepsilon_{\alpha}(x/m) < \epsilon/2 \) we obtain

\[
| p \mathcal{Q}_i | \leq \sum_{m=1}^{\infty} H_{\alpha}.
\]

Since \( T[p \mathcal{Q}_i] \) is of the second category in \([\mathcal{Q}_i] \), there exists \( m_0 \) such that \( H_{\alpha} \) is nowhere dense in \([\mathcal{Q}_i] \). Hence, there are \( q_{\alpha} \in [\mathcal{Q}_i] \) and \( r > 0 \) such that \( H_{\alpha} = (y \in [\mathcal{Q}_i] : q_{\alpha}(y-y_\alpha) < r) \), \( G^{\circ} \) the closure of \( G \) in \([\mathcal{Q}_i] \).

From now on we repeat the proof given in [1]. It follows that \( T[p \mathcal{Q}_i] \subset (y \in [\mathcal{Q}_i] : q_{\alpha}(y-y_\alpha) < r) \). Indeed, \( q_{\alpha}(y-y_\alpha) < r/c_{\mathcal{Q}_i} \) implies \( q_{\alpha}(y-y_\alpha) < r/c_{\mathcal{Q}_i} \). Hence,

\[
q_{\alpha}(y-y_\alpha) < r/c_{\mathcal{Q}_i} < \epsilon/m_0 \leq \epsilon/m_0
\]
and then \( m, y \in H_{m, y} \) or, finally, \( y \in H_{m} \). There exists \( y_1 \in H_1 \) with \( \varepsilon (y_0 - m_0 - y_1) < \eta / 2m_0 \). Then

\[
(y \in \mathfrak{B}_1 : \varepsilon (y - y_0 / m_0) < \eta / 2m_0) \Rightarrow (y \in \mathfrak{B}_1 : \varepsilon (y - y_1) < \eta)
\]

and

\[
H_1 \supseteq (y \in \mathfrak{B}_1 : \varepsilon (y - y_1) < \eta).
\]

Take \( y \in \mathfrak{B}_1 \) with \( \varepsilon (y) < \eta \). We have \( \eta > \varepsilon (y) > \varepsilon (y_1) \) and then \( y_1 - y \in H_{1}^{-} \). Hence \( y - y_1 - y \in H_{1}^{-} \), further \( y \varepsilon (y_1 - H_{1}^{-}) = (y - y_1 - H_{1}^{-}) \) and we conclude that

\[
(y \in \mathfrak{B}_1 : \varepsilon (y) < \eta) \subseteq (y_1 - H_{1}^{-})^{-}.
\]

Moreover, if \( y = y_1 - u \varepsilon (y_1 - H_{1}^{-}) \), then there are \( n_1, \ v \in \mathfrak{B}_1 \) with \( \varepsilon (n_1), \ v \varepsilon (y_1) < \varepsilon / 2 \) such that \( y_1 = T n_1 \) and \( u = T v \). Setting \( x = y_1 - v \) we have \( y = T n_1 \), where \( \varepsilon (n_1) \subseteq \varepsilon (x_1) + \varepsilon (v) < \varepsilon \) and we find that

\[
(y \in \mathfrak{B}_1 : \varepsilon (y) < \eta) \cap \mathfrak{B}_1 (\varepsilon (x_1) + \varepsilon (v) < \varepsilon) \]

and then the Proposition is proved.

Here comes a statement which explains the connection between complete-closed and closed mappings.

**Proposition 11.** Consider \( \mathfrak{B}_i \) (\( i = 1, 2 \)) and a mapping \( T \) of \( \mathfrak{B}_1 \) into \( \mathfrak{B}_2 \). If \( \mathfrak{B}_1 \) is complete, then \( T \) is complete-closed whenever it is closed.

**Proof.** This is a triviality.

The next proposition, which is originally due to Banach, explains the connection between nearly open and open mappings and together with Proposition 10 appears to be the essential part of the theorems known as open-mapping and closed-graph theorems.

**Proposition 12 (Banach [1]).** Let \( \mathfrak{B}_i \) (\( i = 1, 2 \)) be two \( \mathfrak{B} \)-sequences. Every complete-closed nearly open linear mapping of \( \mathfrak{B}_1 \) into \( \mathfrak{B}_2 \) is open.

**Proof.** The proof is the almost exact repetition of that given by Banach in [1] but because of some necessary rearrangements of the Banach's proof we should like to repeat it here in full.

Let \( \mathfrak{B}_i \mathfrak{R}_i K_i (r) \mathfrak{R}_i K_i (r) \) for \( i = 1, 2 \), take \( \varepsilon > 0 \) and let \( \eta_\varepsilon > 0 \), be such that \( \lim \eta_\varepsilon = 0 \) and

\[
\mathfrak{C}_0 T (\varepsilon \mathfrak{B}_1 : \varepsilon (\varepsilon < 2^{\alpha}) \Rightarrow (y \in \mathfrak{B}_1 : \varepsilon (y) < \eta_\varepsilon).
\]

Put \( \eta \mathfrak{R}_\varepsilon \eta_\varepsilon \) and let \( y \varepsilon K_i (\eta) \). We can always find \( x \varepsilon K_i (2^{\alpha}) \) such that \( \varepsilon (y - x) < \eta_\varepsilon \). Suppose we have found \( x_1, \ldots, x_{n-1} \) such that

\[
\varepsilon (y - T (x_1, \ldots, x_{n-1})) < \eta_\varepsilon \quad \text{and} \quad x_n \varepsilon K_i (2^{\alpha}) \text{ for } i = 1, \ldots, n-1.
\]

If such is the case, then applying (**) we find \( x_n \) from \( K_i (2^{\alpha}) \) such that

\[
\varepsilon (y - T (x_1, \ldots, x_{n-1}) - x_n) < \eta_\varepsilon \quad \text{and} \quad x_n \varepsilon K_i (2^{\alpha})
\]

if \( i = 1, \ldots, n-1 \), and it follows that we can define by induction a sequence \( (x_n) \) satisfying (**) for each \( n \). Setting \( x_n = x_1 + \ldots + x_n \) we find that \( (x_n) \) satisfies the Cauchy condition in \( \mathfrak{B}_i \) while \( (T x_n) \) tends to \( y \) in \( \mathfrak{B}_1 \). Since \( T \) is complete-closed, there must be the limit \( x \) of \( (x_n) \) in \( \mathfrak{B}_1 \) with \( y = T x \). We have

\[
\varepsilon (x) \leq \sum_{n=1}^{\infty} \varepsilon (x_n) < \varepsilon \sum_{n=1}^{\infty} 2^{-n} = \varepsilon
\]

and then \( y = K_i (\varepsilon) \) which completes the proof of Proposition 12.

We are provided now with all the necessary informations to express the main results, i.e. the open mapping and closed graph theorems.

**Theorem 2 (The Open Mapping Theorem I).** Consider \( \mathfrak{B} \)-sequences \( \mathfrak{B}_i \) (\( i = 1, 2 \)) and a complete-closed mapping \( T \) of \( \mathfrak{B}_1 \) into \( \mathfrak{B}_2 \). If to every \( p \) there correspond \( k_p \) such that \( T[p \mathfrak{B}_1] \) is of the second category in \( \mathfrak{B}_2 \), then \( T \) is open.

**Proof.** It follows after successive application of Propositions 10 and 12.

**Theorem 3 (The Open Mapping Theorem II).** Consider \( \mathfrak{B} \)-sequences \( \mathfrak{B}_i \) (\( i = 1, 2 \)) and a complete mapping \( T \) of \( \mathfrak{B}_1 \) into \( \mathfrak{B}_2 \). If to every \( p \) there correspond \( k_p \) such that \( T[p \mathfrak{B}_1] \) is of the second category in \( \mathfrak{B}_2 \), then \( T \) is open.

**Proof.** This follows from Theorem 2 after application of Proposition 11 and the Baier theorem on categories.

**Corollary.** Let \( \mathfrak{B}_1 \) (\( i = 1, 2 \)) be two \( \mathfrak{B} \)-sequences, \( \mathfrak{B}_2 \) with \( \mathfrak{B}_1 \), such that to every \( p \) there correspond \( k_p \) such that \( |\mathfrak{B}_1| = |\mathfrak{B}_2| \). Then \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) are equivalent.

**Proof.** Since \( \mathfrak{B}_2 \) is finer than \( \mathfrak{B}_1 \), there exists \( p_0 \) such that \( |\mathfrak{B}_2| < \mathfrak{B}_1 \) and the identical injection of \( \mathfrak{B}_2 \) into \( \mathfrak{B}_1 \) is continuous. Denote this injection by \( T \). Of course \( T \) is closed and \( T[p \mathfrak{B}_2] = |\mathfrak{B}_2| \) for \( p > p_0 \). This means that the assumptions of Theorem 3 are satisfied and \( T \) must be open as a mapping of \( \mathfrak{B}_1 \) into \( \mathfrak{B}_2 \). Therefore we have \( |\mathfrak{B}_1| = \mathfrak{B}_1 \) for some \( q_0 \), the identical injection \( T^{-1} \) of \( |\mathfrak{B}_2| \) into \( \mathfrak{B}_1 \) is continuous and the Corollary follows.

**Theorem 4 (The Closed Graph Theorem; cf. [2]).** Let \( \mathfrak{B}_i \) (\( i = 1, 2 \)) be two \( \mathfrak{B} \)-sequences. Every complete closed mapping of \( \mathfrak{B}_1 \) into \( \mathfrak{B}_2 \) such that every \( p \mathfrak{B}_1 \) contains \( T[p \mathfrak{B}_2] \) is continuous.
Proof. Let \( L = (\sigma, \mathcal{V}; T, \nu = 0) \). The mapping \( \mathcal{T} \) can be factorized to a one-to-one closed mapping \( \mathcal{F} \) of \( \mathcal{V} / L \) into \( \mathcal{V} \). Here \( L \) is closed in \( \mathcal{V} \) and then \( \mathcal{V} / L \) is an \( \mathcal{F} \)-sequence. Denote by \( Y \) the image of \( \mathcal{T} \) in \( \mathcal{V} \) and let \( \mathcal{Z} = \{ Y \cap \alpha \mathcal{V}, \| \cdot \|_{\alpha} \} \). The mapping \( \mathcal{F}^{-1} \) is closed, maps \( \mathcal{Z} \) onto \( \mathcal{V} / L \) and, moreover, \( \mathcal{F}^{-1} \| \cdot \|_{\alpha} \rightarrow \| \cdot \|_{\mathcal{V} / L} \).

Applying Theorem 3 we find that \( \mathcal{F}^{-1} \) is open, consequently \( \mathcal{V} \) is continuous and then \( T \) must be continuous as well which finishes the proof of Theorem 4.

References


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