Another characterization of Hilbert spaces

by

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1. Introduction. Let X be a real Banach space of dimension not less than two. We denote the norm by $\|\cdot\|$, the (solid) unit sphere by Σ, its boundary by $\partial\Sigma$. We set $\operatorname{sgn} x = \|x\|^{-1}x$ for each $x \in X \setminus \{0\}$. In [3], Clarkson introduced the following concept, in order to make a detailed analysis of the triangle inequality in uniformly convex spaces: for any $x, y \in X\setminus\{0\}$, we define the *angular distance* (Clarkson's "angle") $\alpha[x, y] = \|\operatorname{sgn} y - \operatorname{sgn} x\|$. In [6], Massera and Schäffer established the following property of the angular distance, which turns up in many geometrical questions; we include a proof for the sake of completeness.

LEMMA 1. *For any* $x, y \in X\setminus\{0\}$, *we have*

$$\|y-x\| \geqslant \tfrac{1}{2}\alpha[x, y]\max\{\|x\|, \|y\|\}. \tag{1}$$

Proof. Because of the symmetry, it is sufficient to consider the case $\|x\| \leqslant \|y\|$. Set $\alpha = \alpha[x, y]$. If $\|x\| \leqslant (1-\frac{1}{2}\alpha)\|y\|$, (1) follows from $\|y\| = \|y-x+x\| \leqslant \|y-x\| + (1-\frac{1}{2}\alpha)\|y\|$; if $(1-\frac{1}{2}\alpha)\|y\| \leqslant \|x\| \leqslant \|y\|$, it follows from $\alpha\|y\| = \big\|y - \|y\|\operatorname{sgn} x\big\| = \big\|y - x - (\|y\| - \|x\|)\operatorname{sgn} x\big\| \leqslant \|y-x\| + \frac{1}{2}\alpha\|y\|$.

It is easy to see that the coefficient $\frac{1}{2}$ in (1) cannot be replaced by any larger number, by taking $y = -\varepsilon x$ with a small positive ε. However, it is fairly obvious that when X is a Hilbert space the coefficient can be chosen as close to 1 as one pleases, provided $\alpha[x, y]$ is sufficiently small. The main purpose of this paper is to show that this property characterizes Hilbert spaces whenever the dimension is not less than three (but not when it is two).

The main tool will be the characterization of Hilbert spaces by the "symmetry of orthogonality", which (at least for spaces with strictly convex unit spheres) goes back to Blaschke's characterization of the ellipsoids in three-dimensional affine space as the convex bodies with plane "shadow boundaries" ([1]; see [2], (16.14), (17.25)), and was first stated in its general form by James ([4], Theorem 1), although the equivalent geometric fact is given by Kakutani ([5], Theorem 5).

We shall also show that this characterization may be rephrased as follows: if the dimension of X is not less than three, then X is a Hilbert space if and only if the length of any rectifiable curve that contains no interior point of Σ is not increased by radial projection onto $\partial\Sigma$.

2. Inferior limits. For convenience we introduce the notation

$$\psi(x, y) = \frac{\|y-x\|}{\alpha[x, y]\max\{\|x\|, \|y\|\}}$$

for any $x, y \in X \setminus \{0\}$, $\alpha[x, y] \neq 0$. This function is obviously symmetric and positively homogeneous; if $\|x\| = \|y\|$, then $\psi(x, y) = 1$. Lemma 1 states that $\psi(x, y) \geqslant \frac{1}{2}$ for all x, y for which ψ is defined. These remarks show that

$$(2) \qquad \tfrac{1}{2} \leqslant \mu = \liminf_{\alpha[x,y]\to+0} \psi(x, y) \leqslant 1.$$

The condition we are interested in is $\mu = 1$.

For any unit vectors u, v such that $u \pm v \neq 0$ we have $\alpha[u, u+\lambda v] = \|u+\lambda v\|^{-1}\big\|(1-\|u+\lambda v\|)u+\lambda v\big\| \to 0$ as $\lambda \to +0$, whence

$$\liminf_{\lambda\to+0} \lambda/\alpha[u, u+\lambda v] = \liminf_{\lambda\to+0} \psi(u, u+\lambda v) \geqslant \mu.$$

Therefore $\mu = 1$ implies

$$\liminf_{\lambda\to+0} \lambda/\alpha[u, u+\lambda v] \geqslant 1$$

for all $u, v \in \partial\Sigma$ such that $u \pm v \neq 0$. It will turn out in the end that this apparently weaker, highly "local" condition is actually equivalent to $\mu = 1$.

Two incidental remarks: it will appear in the proof of Theorem 1 (implication (b) $\to$ (c)) that in any case the *limit* $\lim \lambda/\alpha[u, u+\lambda v]$ exists; and the reader is invited to verify that if some plane section of Σ is a parallelogram, and we choose u, v as adjacent vertices of this, then that limit is precisely $\frac{1}{2}$; in such a space the lower bound in (2) is therefore attained.

3. Orthogonality. If $x, y \in X$, we say that x is *orthogonal* to y, and write $x \perp y$, if $\|x\| \leqslant \|x+\lambda y\|$ for all real λ; we say that *orthogonality is symmetric* if $x \perp y$ implies $y \perp x$. Since orthogonality is separately homogeneous in each member, it is sufficient to consider unit vectors. If x, y are non-collinear unit vectors and Y is the two-dimensional subspace spanned by them, $x \perp y$ may be interpreted geometrically as the fact that there exists at x a supporting line of the unit disk $\Sigma \cap Y$ of Y with the direction of y. We therefore state for reference:

LEMMA 2. *Orthogonality is symmetric in X if and only if the unit disk $\Sigma_Y = \Sigma \cap Y$ of every two-dimensional subspace Y has the following property:*

(Π) *If $u, v \in \partial\Sigma_Y$ and v has the direction of a supporting line of Σ_Y at u, then u has the direction of a supporting line of Σ_Y at v.*

We require a slightly sharper condition:

LEMMA 3. *Orthogonality is symmetric in X if and only if the unit disk $\Sigma_Y = \Sigma \cap Y$ of every two-dimensional subspace Y has the following property:*

(Π_0) *If $u, v \in \partial\Sigma_Y$ and v has the direction of an extreme supporting line of Σ_Y at u, then u has the direction of a supporting line of Σ_Y at v.*

Proof. (Π) obviously implies (Π_0). Assume that (Π_0) holds, and let $u, v \in \partial\Sigma_Y$ be given, with v in the direction of a supporting line of Σ_Y at u. Let w_1, w_2 be those unit vectors of the extreme supporting lines of Σ_Y at u that lie in the same half-plane as v with respect to the line $0u$. By (Π_0), w_1, w_2 both lie on the unique supporting line parallel to $0u$ in that half-plane, and therefore the arc w_1w_2 of $\partial\Sigma_Y$ that contains v (and which might of course consist in a single point) lies entirely in this supporting line. Therefore (Π) holds, and (Π), (Π_0) are equivalent. The conclusion follows from Lemma 2.

4. The main result.

THEOREM 1. *Let X be a real Banach space of dimension not less than two. The following conditions are equivalent:*

(a) $\quad \mu = \liminf\limits_{\alpha[x,y]\to+0} \dfrac{\|y-x\|}{\alpha[x, y]\max\{\|x\|, \|y\|\}} = 1;$

(b) $\quad \liminf\limits_{\lambda\to+0} \lambda/\alpha[u, u+\lambda v] \geqslant 1$ *for all* $u, v \in \partial\Sigma$, $u \pm v \neq 0$;

(c) $\quad$ *orthogonality is symmetric.*

The preceding conditions are satisfied if, and when $\dim X \geqslant 3$ *only if, X is a Hilbert space.*

Proof. We prove (X a Hilbert space) $\to$ (a) $\to$ (b) $\to$ (c). The implication (c) $\to$ (X a Hilbert space) when $\dim X \geqslant 3$ is the characterization proved by James ([4], Theorem 1) that was mentioned in Section 1. The proof of (c) $\to$ (a) for $\dim X = 2$ is given in the next section.

A Hilbert space satisfies (a). Let $(\cdot,\cdot)$ denote the inner product. Since ψ is symmetric, we may assume $\|x\| \leqslant \|y\|$. Set $\alpha = \alpha[x, y]$, $\gamma = (\operatorname{sgn} x, \operatorname{sgn} y)$. Then $\alpha^2 = 2(1-\gamma)$ and $\|y-x\|^2 - (1-\frac{1}{4}\alpha^2)\alpha^2\|y\|^2 = \|x\|^2 + \|y\|^2 - 2\gamma\|x\|\|y\| - (1-\gamma^2)\|y\|^2 = (\|x\|-\gamma\|y\|)^2 \geqslant 0$. Therefore $\psi(x, y) \geqslant (1-\frac{1}{4}\alpha^2)^{1/2}$. Together with (2) this implies (a).

(a) *implies* (b). Trivial (see Section 2).

(b) *implies* (c). Let Y be a two-dimensional subspace of X and $\Sigma_Y = \Sigma \cap Y$ its unit sphere. For any $u \in \partial\Sigma_Y$ and any $y \in Y$ we denote by $\sigma(u, y)$ the distance between y and the line $0u$, i. e., $\sigma(u, y) = \inf\{\|y-\lambda u\|: \lambda \text{ real}\}$. For fixed u, $\sigma(u, \cdot)$ is a seminorm; for fixed y, $\sigma(\cdot, y)$ is continuous on $\partial\Sigma_Y$.

Assume that $u, v \in \partial\Sigma_Y$, and that v has the direction of an extreme supporting line of Σ_Y at u; for definiteness, we suppose it has the sense in which the supporting line is tangent to Σ_Y. There exists a unit vector w in the same half-plane with respect to $0u$, and such that $\sigma(u, w) = 1$. On account of the definition of v, $v = \lim_{\lambda\to+0} \operatorname{sgn}\big(\operatorname{sgn}(u+\lambda w) - u\big)$. Therefore, by (b) applied to u, w,

$$1 = \|v\| \geqslant \sigma(u, v) = \lim_{\lambda\to+0} \sigma(u, \|u+\lambda w\|^{-1}\lambda w)/\alpha[u, u+\lambda w]$$
$$= \lim_{\lambda\to+0} \lambda\sigma(u, w)/\alpha[u, u+\lambda w] \geqslant 1.$$

Therefore $\sigma(u, v) = \sigma(u, -v) = 1$, and $v, -v$ lie on supporting lines of Σ_Y parallel to $0u$. Therefore Σ_Y satisfies (Π_0), and Lemma 3 implies that (c) holds.

5. The two-dimensional case.

End of the proof of Theorem 1. (c) *implies* (a) *when* $\dim X = 2$. We may identify X with the plane Y in Section 3, so that $\Sigma_Y = \Sigma$. We use the function σ defined in the preceding section.

There exist sequences (x_n), (y_n) in X such that $\lim_{n\to\infty} \alpha[x_n, y_n] = +0$ and $\lim_{n\to\infty} \psi(x_n, y_n) = \mu$. On account of the homogeneity of ψ we may assume that $x_n \in \partial\Sigma$ for all n; and since $\partial\Sigma$ is compact we may assume, taking a subsequence if necessary, that $\lim_{n\to\infty} x_n = u \in \partial\Sigma$ exists. We claim that $\lim_{n\to\infty} y_n = u$. If this were not the case we might assume, taking a subsequence if necessary, that $\|y_n - x_n\| \geqslant \eta > 0$ for all n. But then we find, using (2),

$$1 \geqslant \mu = \lim_{n\to\infty} \psi(x_n, y_n) \geqslant \lim_{n\to\infty} \frac{\eta}{(1+\eta)\,\alpha[x_n, y_n]} = \infty,$$

which is absurd; our claim is thus established. We have

$$\lim_{n\to\infty} \|y_n\| = 1, \tag{3}$$

$$\lim_{n\to\infty} \|y_n - x_n\|/\alpha[x_n, y_n] = \mu. \tag{4}$$

We may assume, taking a subsequence if necessary, that $\lim_{n\to\infty} \operatorname{sgn}(\operatorname{sgn} y_n - x_n) = v \in \partial\Sigma$ exists; since (x_n), $(\operatorname{sgn} y_n)$ converge to u, v has the direction of a supporting line of Σ at u. (c) implies, by Lemma 2, that v is on a supporting line of Σ parallel to $0u$, so that $\sigma(u, v) = 1$. Further,

$$|\sigma(x_n, v) - \sigma\big(x_n, \operatorname{sgn}(\operatorname{sgn} y_n - x_n)\big)| \leqslant \|v - \operatorname{sgn}(\operatorname{sgn} y_n - x_n)\| \to 0 \quad \text{as} \quad n\to\infty, \tag{5}$$

$$\|y_n\|\sigma(x_n, \operatorname{sgn} y_n - x_n) = \sigma(x_n, y_n) = \sigma(x_n, y_n - x_n) \leqslant \|y_n - x_n\|. \tag{6}$$

Using the continuity of $\sigma(\cdot, v)$ on $\partial\Sigma$, and formulas (5), (3), (6), (4), (2), in that order, we obtain

$$1 = \sigma(u, v) = \lim_{n\to\infty} \sigma(x_n, v) = \lim_{n\to\infty} \sigma\big(x_n, \operatorname{sgn}(\operatorname{sgn} y_n - x_n)\big)$$
$$= \lim_{n\to\infty} \sigma(x_n, \operatorname{sgn} y_n - x_n)/\alpha[x_n, y_n] \leqslant \lim_{n\to\infty} \|y_n - x_n\|/\alpha[x_n, y_n] = \mu \leqslant 1.$$

Therefore $\mu = 1$, and (a) is verified.

Radon [7] was the first to point out that the unit disk of a Banach plane that satisfies (Π) need not be an ellipse, even if it is strictly convex (when (Π), (Π_0) are identical and may be interpreted in terms of "conjugate diameters"); all planes with such strictly convex unit disks are described in [2], p. 104. An example with a unit disk that is not strictly convex is a plane with an affinely regular hexagon as the unit disk.

6. Rectifiable curves. We return to a general Banach space X. A curve Γ in $X\setminus\{0\}$ is given (parametrically) by a continuous function $f: [\alpha, \beta] \to X\setminus\{0\}$, where the domain is a compact interval of real numbers. We set $d(\Gamma) = \min\{\|f(\tau)\|: \tau \in [\alpha, \beta]\}$, the distance between the origin and the curve. If Γ is rectifiable, $l(\Gamma)$ denotes its length. If σ is a positive real number, $\sigma\Gamma$ denotes the curve represented by σf; then $d(\sigma\Gamma) = \sigma d(\Gamma)$, $l(\sigma\Gamma) = \sigma l(\Gamma)$. We denote by $\operatorname{sgn} f$ the function given by $\operatorname{sgn} f(\tau) = \operatorname{sgn}\big(f(\tau)\big)$ and by $\operatorname{sgn}\Gamma$ the corresponding curve; we say that $\operatorname{sgn}\Gamma$ is obtained from Γ by *radial projection onto* $\partial\Sigma$.

LEMMA 4. *If Γ is a rectifiable curve in $X\setminus\{0\}$, then $\operatorname{sgn}\Gamma$ is rectifiable, and $l(\operatorname{sgn}\Gamma) \leqslant l(\Gamma)/\mu d(\Gamma)$.*

Proof. Let Γ be described by $f: [\alpha, \beta] \to X\setminus\{0\}$. For a given number ϱ, $0 < \varrho < 1$, let $\varepsilon > 0$ be so small that $\psi(x, y) \geqslant \varrho\mu$ provided $\alpha[x, y] \leqslant \varepsilon$. Since $\operatorname{sgn} f$ is continuous, there exists $\delta > 0$ such that $\alpha[f(\tau'), f(\tau'')] \leqslant \varepsilon$ for all $\tau', \tau'' \in [\alpha, \beta]$ with $|\tau'' - \tau'| \leqslant \delta$.

Consider any partition $\alpha = \tau_0 < \tau_1 < \ldots < \tau_{k-1} < \tau_k = \beta$ of $[\alpha, \beta]$ with $\tau_{i+1} - \tau_i \leqslant \delta$, $i = 0, \ldots, k-1$. Set $f_i = f(\tau_i)$. Then $\alpha[f_i, f_{i+1}] \leqslant \varepsilon$, and

$$l(\Gamma) \geqslant \sum_0^{k-1} \|f_{i+1} - f_i\| = \sum_0^{k-1} \psi(f_i, f_{i+1})\,\alpha[f_i, f_{i+1}]\max\{\|f_i\|, \|f_{i+1}\|\}$$
$$\geqslant \varrho\mu d(\Gamma) \sum_0^{k-1} \|\operatorname{sgn} f_{i+1} - \operatorname{sgn} f_i\|.$$

Since this is true for any sufficiently fine partition, $\operatorname{sgn}\Gamma$ is rectifiable, and $l(\Gamma) \geqslant \varrho\mu d(\Gamma)\, l(\operatorname{sgn}\Gamma)$; since ϱ was arbitrarily close to 1, the conclusion follows.

THEOREM 2. *Let X be a real Banach space of dimension not less than two. Conditions* (a), (b), (c) *of Theorem* 1 *are equivalent to each of the following conditions*:

(d) *for every rectifiable curve Γ in $X\setminus\{0\}$, $l(\operatorname{sgn}\Gamma)\leqslant l(\Gamma)/d(\Gamma)$*;

(e) *for every rectifiable curve Γ in X that contains no interior point of the unit sphere* (i. e., *with $d(\Gamma)\geqslant 1$*), *$l(\operatorname{sgn}\Gamma)\leqslant l(\Gamma)$.*

If the dimension of X is not less than three, X satisfies these equivalent conditions if and only if X is a Hilbert space.

Proof. (d) obviously implies (e); conversely, if Γ is any curve, and we set $\sigma=d(\Gamma)$, then $d(\delta^{-1}\Gamma)\geqslant 1$ and (e) implies $l(\operatorname{sgn}\Gamma)=l(\operatorname{sgn}\sigma^{-1}\Gamma)\leqslant l(\sigma^{-1}\Gamma)=l(\Gamma)/d(\Gamma)$, so that (e) implies (d). Now (a) implies (d) by Lemma 4. The conclusion will follow from Theorem 1 if we prove that (d) implies (b).

Let $u, v\in\partial\Sigma$, $u\pm v\neq 0$, be given. For each λ, $0<\lambda<1$, we consider the curve Γ_λ given by $f_\lambda(\tau)=u+\tau v$, $\tau\in[0,\lambda]$ (a line segment). Now $d(\Gamma_\lambda)\geqslant 1-\lambda$, $l(\Gamma_\lambda)=\lambda$, $l(\operatorname{sgn}\Gamma_\lambda)\geqslant\|\operatorname{sgn}f(\lambda)-\operatorname{sgn}f(0)\|=a[u, u+\lambda v]$. By (d),

$$\lambda/a[u, u+\lambda v]\geqslant l(\Gamma_\lambda)/l(\operatorname{sgn}\Gamma_\lambda)\geqslant d(\Gamma_\lambda)\geqslant 1-\lambda,$$

and (b) follows on taking the inferior limit as $\lambda\to+0$.

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Reçu par la Rédaction le 13. 5. 1964

Summability in $l(p_1, p_2, \ldots)$ spaces*

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A Banach space E will be said to have the BS-*property* provided every bounded sequence in E admits a subsequence z_a whose sequence of arithmetic means

$$z_1,\ \tfrac{1}{2}(z_1+z_2),\ \tfrac{1}{3}(z_1+z_2+z_3),\ \ldots$$

is norm-convergent to a point of E. This property was established by Banach and Saks [1] for the spaces L_p and l_p $(1<p<\infty)$, and by Kakutani [2] for all uniformly convex Banach spaces. Nishiura and Waterman [6] recently showed that the BS-property does not imply uniform convexifiability, that it does imply reflexivity, and that reflexivity is equivalent to a different summability property. In his review of [6], Sakai [7] asked for an example of a reflexive Banach space which lacks the BS-property. The purpose of this note is to supply such an example by means of the $l(p_1, p_2, \ldots)$ spaces of Nakano [5]. (I am indebted to Mr. K. Sundaresan for calling my attention to these spaces in a different connection.)

Let P denote the set of all sequences in $]1,\infty[$ and let s denote the linear space of all sequences of real numbers. For $p=(p_1, p_2, \ldots)\in P$ and $x=(x_1, x_2, \ldots)\in s$, let

$$\mu_p(x)=\sum_{i=1}^{\infty}|x_i|^{p_i}/p_i.$$

Let $l(p)$ denote the set of all points $x\in s$ such that $\|x\|_p<\infty$, where

$$\|x\|_p=\inf\left\{\lambda>0:\ \mu_p\left(\frac{1}{\lambda}x\right)\leqslant 1\right\}.$$

Then $l(p)$ is a linear subspace of s and $\|\ \|_p$ is a norm for $l(p)$. It follows from results of Nakano (or by direct reasoning analogous to that for the classical l_p spaces) that the spaces $l(p)$ are all reflexive Banach spaces (for $p\in P$), and that $l(p)$ is uniformly convex if and only if $1<\inf\{p_i\}$

* Research supported in part by the National Science Foundation, U. S. A. (NSF-GP-378).