On linear processes of approximation (II)

by

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§ 1. As in the first part of our work, we are concerned with the so-called *method of test-functions*. To reproduce the main features of the problem, let us be given a sequence of linear operators $B_n(f)$ which transform a Banach space X of functions into itself. It is required to know whether, resp. how precisely, the sequence $B_n(f)$ approximate all the functions of X. There are situations in which this can be decided by studing the order of approximation in a certain subclass of X. We then call this class a *test-class* for the approximation problem in question. As to the well-known results in this direction, we refer to chapter 1 of our paper [3]. We supply two additional remarks.

(a) The first test-condition, using a finite class of test-functions, seems to have been discovered by H. Bohman (see [1], chapter 3).

(b) For X-spaces of infinite dimension it is not possible to have a finite test-class, unless presupposing for B_n's something more than the mere boundedness of their norms. For let us consider a finite subset $f_1, f_2, \ldots, f_r$ of X. We consider the linear subspace X_r spanned by f_j's. Let f_{r+1} be an element of X outside X_r. Denoting by X_{r+1} the linear subspace of X spanned by X_r and by f_{r+1}, we have for every $f \in X_{r+1}$ the unique decomposition $f = \varphi + \alpha f_{r+1}$, $\varphi \in X_r$, α a number. Then we define $B(f) = \varphi$ for $f \in X_{r+1}$ and by the Hahn-Banach principle extend it to the whole space X. Putting $B_n = B$, we have $B_n(f_j) = f_j$, $j = 1, 2, \ldots, r$, but $B_n(f_{r+1}) = 0 \nrightarrow f_{r+1}$.

§ 2. Let X be either the space $C_{2\pi}$ or one of $L^p[-\pi, \pi]$, $p \geqslant 1$, with the usual norms. The elements of $L_p[-\pi, \pi]$ are considered 2π-periodic. We note that in each case we have the following inequalities which hold for every continuous $\varphi(u)$, $f \in X$ and arbitrary α, β (δ_0 fixed)(1):

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(1) We will also make use of the observation, which is trivial for $X = C_{2\pi}$ resp. $L^p[-\pi, \pi]$, that $|f(t)| < g(t)$ implies $\|f\| < \|g\|$. In fact, we used this property of X also in [3], but we failed to mention it.

(2.1) $$\left\|\int_a^\beta |f(u+t)-f(t)|\varphi(u)\,du\right\| \leqslant \int_a^\beta \|f(u+t)-f(t)\|\,|\varphi(u)|\,du,$$

(2.2) $$\left\|\int_a^\beta |f(u+t+\delta_0)-f(u+t)|\varphi(u)\,du\right\| \leqslant \|f(t+\delta_0)-f(t)\|\int_a^\beta |\varphi(u)|\,du.$$

We make a convention, once for all throughout this paper, that whenever a function of more variables than one occurs under the sign of norm, it should be understood that the norm is taken according to the variable t and all the other variables are parameters.

Let $\{B_n(f;t)\}$ be a sequence of linear transformations of X which transform the subspace $C_{2\pi}$ into itself. The norms of B_n's are supposed to be bounded:

(2.3) $$\|B_n\| \overset{\text{def}}{=} \sup_{f\in X} \frac{\|B_n(f;t)\|}{\|f\|} = O(1).$$

As in the previous paper, it will be understood that constants involved with O-estimates are numerical after having fixed X and $\{B_n\}$.

First of all we are quoting the main result of [3], in fact, in a slightly refined form.

THEOREM 1. *Let $\{\lambda_n\}$ be a monotonously increasing sequence of positive integers and suppose, in addition to* (2.3),

(2.4) $$B_n(1;t) = 1,$$

(2.5) $$B_n(e^{i\tau};t) = e^{it}+O\left(\frac{1}{\lambda_n}\right),$$

(2.6) $$\sum_{k=1}^{\lambda_n}\left|B_n\left(e^{ik\tau}\sin^2\frac{\tau-t}{2};t\right)\right| = O\left(\frac{1}{\lambda_n}\right).$$

Then for every $f\in X$

(2.7) $$\|B_n(f(\tau);t)-f(t)\| = O(1)\left\{\int_{-\infty}^{\infty}\left\|f\left(t+\frac{y}{\lambda_n}\right)-f(t)\right\|\min(1,y^{-2})\,dy\right\}.$$

The proof of this theorem runs, with small modifications, along the lines of [3]. As indicated in [3], the above theorem is appropriate e. g. for processes which approximate as the Fejér means but it is not appropriate for processes of Jackson type. In the present paper we will provide a theorem applicable in the latter case. We use the concept of the modulus of continuity

(2.8) $$\omega(\delta;f;X) \overset{\text{def}}{=} \sup_{|h|\leqslant\delta}\|f(t+h)-f(t)\|.$$

We note that, as a consequence of $\omega(\vartheta\delta;f;X)\leqslant 2\vartheta\omega(\delta;f;X)$, $\vartheta\geqslant 1$, we have

(2.9) $$\omega(|y|;f;X) \leqslant 2|y|\nu\omega\left(\frac{1}{\nu};f;X\right) \quad \text{for} \quad |y|\geqslant\frac{1}{\nu}>0.$$

Then we assert the following

THEOREM 2. *Let $\{\lambda_n\}$ be a monotonously increasing sequence of positive integers and suppose, in addition to* (2.3),

(2.10) $$B_n(1;t)-1 = 0,$$

(2.11) $$B_n(e^{i\tau};t)-e^{it} = O\left(\frac{1}{\lambda_n\log\lambda_n}\right),$$

(2.12) $$B_n\left(e^{i\tau}\sin^2\frac{\tau-t}{2};t\right) = O\left(\frac{1}{\lambda_n^2}\right),$$

(2.13) $$\sum_{k=1}^{\lambda_n}\left|B_n\left(e^{i(k+\frac{1}{2})\tau}\sin^3\frac{\tau-t}{2};t\right)\right| = O\left(\frac{1}{\lambda_n^2}\right).$$

Then for every $f\in X$

(2.14) $$\|B_n(f(\tau);t)-f(t)\| = O\left(\omega\left(\frac{1}{\lambda_n};f;X\right)\right).$$

We will prove this theorem in § 3. The lemmas needed in the course of this proof have been postponed to § 4. In the last § 5 we make remarks and give some examples.

§ 3. Let μ be the greatest integer not exceeding $\lambda_n/2$. We will employ the de la Vallée-Poussin means

(3.1) $$v_\mu = v_\mu(f;t) \overset{\text{def}}{=} 2\sigma_{2\mu}(f;t)-\sigma_\mu(f;t);$$

$\sigma_\nu(f;t)$ stand for the familiar Fejér-sums

(3.2) $$\sigma_\nu = \sigma_\nu(f;t) = \sum_{|k|\leqslant\nu}\left(1-\frac{|k|}{\nu}\right)a_k e^{ikt},$$

where a_k are the Fourier coefficients of $f(t)$:

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy.$$

We need the estimate

(3.3) $$\|v_\mu(f;t)-f(t)\| = O\left(\omega\left(\frac{1}{\mu};f;X\right)\right);$$

for $X = C_{2\pi}$ it is a combination of Jackson's approximation theorem and a well known theorem of de la Vallée-Poussin; as to $X = L^p[-\pi,\pi]$, the extension does not present difficulties.

Splitting

$$B_n(f;t)-f(t) = \{v_\mu(f;t)-f(t)\}+B_n(f-v_\mu;t)+\{B_n(v_\mu;t)-v_\mu(f;t)\}$$
$$= \{v_\mu(f;t)-f(t)\}+B_n(f-v_\mu;t)+2\{B_n(\sigma_{2\mu};t)-\sigma_{2\mu}(f;t)\}-$$
$$-\{B_n(\sigma_\mu;t)-\sigma_\mu(f;t)\},$$

we see that the first two terms are estimated satisfactorily by (3.3) and (2.3). All that remains to be proved is

$$\|B_n(\sigma_\nu;t)-\sigma_\nu(f;t)\| = O\left(\omega\left(\frac{1}{\nu};f;X\right)\right) \quad \text{for} \quad \nu=\mu,\ \nu=2\mu. \tag{3.4}$$

Setting

$$\eta_n(k;t) \overset{\text{def}}{=} B_n(e^{ik(\tau-t)};t)-1, \tag{3.5}$$

we get by the linearity of B_n

$$\Delta_{n,\nu}(f;t) \overset{\text{def}}{=} B_n(\sigma_\nu;t)-\sigma_\nu(f;t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y+t)\sum_{|k|\leqslant\nu}\left(1-\frac{|k|}{\nu}\right)\eta_\mu(k;t)e^{-iky}dy.$$

Putting $f(\tau)\equiv f(t)$ in this formula, we have, using (2.10) and subtracting (without any loss of generality f can be supposed real)

$$\Delta_{n,\nu}(f;t) = \frac{1}{\pi}\operatorname{Re}\int_{-\pi}^{\pi}\{f(y+t)-f(t)\}\Lambda_{n,\nu}(t,y)\,dy, \tag{3.6}$$

where

$$\Lambda_{n,\nu}(t,y) \overset{\text{def}}{=} \sum_{k=1}^{\nu}\left(1-\frac{k}{\nu}\right)\eta_n(k;t)e^{-iky}. \tag{3.7}$$

Our proof will be finished if we show

$$J \overset{\text{def}}{=} \left\|\int_{-\pi}^{\pi}\{f(y+t)-f(t)\}\Lambda_{n,\nu}(t,y)\,dy\right\| = O\left(\omega\left(\frac{1}{\nu};f;X\right)\right). \tag{3.8}$$

In fact, we have

$$J \leqslant \left\|\int_{-2\pi/(\nu+1)}^{2\pi/(\nu+1)}\right\| + \left\|\int_{2\pi/(\nu+1)}^{\pi}\right\| + \left\|\int_{-\pi}^{-2\pi/(\nu+1)}\right\| \equiv J_1+J_2+J_3.$$

By formula (4.1) of lemma 1 (§ 4), we have $\Lambda_{n,\nu}(t,y) = O(\nu)$, so that by (2.1)

$$J_1 = O(\nu)\int_{-2\pi/(\nu+1)}^{2\pi/(\nu+1)}\|f(y+t)-f(t)\|\,dy = O\left(\omega\left(\frac{1}{\nu};f;X\right)\right).$$

By lemma 3 (§ 4), (4.15),

$$J_2 \leqslant \frac{|\eta_n(\nu-1;t)|}{\nu}\left\|\int_{2\pi/(\nu+1)}^{\pi}\{f(y+t)-f(t)\}\frac{e^{-iy(\nu+1)}}{(1-e^{-iy})^2}\,dy\right\| + \tag{3.9}$$
$$+O\left(\frac{1}{\nu\log\nu}\right)\left\|\int_{2\pi/(\nu+1)}^{\pi}\frac{|f(y+t)-f(t)|}{y^2}\,dy\right\| + O\left(\frac{1}{\nu^2}\right)\left\|\int_{2\pi/(\nu+1)}^{\pi}\frac{|f(y+t)-f(t)|}{y^3}\,dy\right\|.$$

Using lemma 4 and formula (4.1) of lemma 1 (§ 4), the first term of (3.9) is $O(\omega(1/\nu;f;X))$.

As to the second term of (3.8), (2.1) and (2.9) yield

$$\frac{1}{\nu\log\nu}\left\|\int_{2\pi/(\nu+1)}^{\pi}\frac{|f(y+t)-f(t)|}{y^2}\,dy\right\| \leqslant \frac{1}{\nu\log\nu}\int_{2\pi/(\nu+1)}^{\pi}\frac{\|f(y+t)-f(t)\|}{y^2}\,dy$$
$$\leqslant \frac{1}{\nu\log\nu}\int_{2\pi/(\nu+1)}^{\pi}\frac{\omega(y;f;X)}{y^2}\,dy \leqslant \frac{2}{\log\nu}\,\omega\left(\frac{1}{\nu};f;X\right)\int_{2\pi/(\nu+1)}^{\pi}\frac{dy}{y} = O\left(\omega\left(\frac{1}{\nu};f;X\right)\right),$$

and a similar argument gives for the third term

$$\frac{1}{\nu^2}\left\|\int_{2\pi/(\nu+1)}^{\pi}\frac{|f(y+t)-f(t)|}{y^3}\,dy\right\| = O\left(\omega\left(\frac{1}{\nu};f;X\right)\right).$$

J_3 can be handled as (3.9), so that (3.8) follows.

§ 4. We will use the following notation: for the sequence $\eta_n(k;t)$, $k=0,1,2,\ldots$, we define

$$\Delta\eta_n(k;t) = \eta_n(k;t)-\eta_n(k+1;t), \qquad \Delta^2\eta_n(k;t) = \Delta\eta_n(k;t)-\Delta\eta_n(k+1;t),$$
$$\Delta^3\eta_n(k;t) = \Delta^2\eta_n(k;t)-\Delta^2\eta_n(k+1;t).$$

LEMMA 1. *As a consequence of* (2.10)-(2.13), *we have*

$$\eta_n(k;t) = O(1), \qquad k=0,1,\ldots,\lambda_n, \tag{4.1}$$

$$\eta_n(0;t) = 0, \qquad \eta_n(1;t) = O\left(\frac{1}{\lambda_n\log\lambda_n}\right), \tag{4.2}$$

$$\Delta\eta_n(k;t) = O\left(\frac{1}{\lambda_n}\right), \qquad k=0,1,\ldots,\lambda_n, \tag{4.3}$$

$$\Delta^2\eta_n(k;t) = O\left(\frac{1}{\lambda_n^2}\right), \qquad k=0,1,\ldots,\lambda_n. \tag{4.4}$$

Proof. Using (3.5), we have

$$\Delta\eta_n(k;t) = 2iB_n\left(e^{i(k+\frac{1}{2})(\tau-t)}\sin\frac{\tau-t}{2};t\right),$$

$$\Delta^2\eta_n(k;t) = -4iB_n\left(e^{i(k+1)(\tau-t)}\sin^2\frac{\tau-t}{2};t\right),$$

$$\Delta^3\eta_n(k;t) = -8iB_n\left(e^{i(k+3/2)(\tau-t)}\sin^3\frac{\tau-t}{2};t\right).$$

Inserting these expressions into (2.12) and (2.13), we get

$$\Delta^2\eta_n(0;t) = O\left(\frac{1}{\lambda_n^2}\right), \tag{4.5}$$

$$\sum_{k=0}^{\lambda_n-1}|\Delta^3\eta_n(k;t)| = O\left(\frac{1}{\lambda_n^2}\right). \tag{4.6}$$

By (4.5) and (4.6)

$$\Delta^2\eta_n(k;t) = \Delta^2\eta_n(0;t)-\sum_{j=0}^{k-1}\Delta^3\eta_n(j;t) = O\left(\frac{1}{\lambda_n^2}\right),\quad k=0,1,\ldots,\lambda_n, \tag{4.7}$$

so that

$$\sum_{k=0}^{\lambda_n}|\Delta^2\eta_n(k;t)| = O\left(\frac{1}{\lambda_n}\right). \tag{4.8}$$

Next, by (2.10) and (2.11),

$$\Delta\eta_n(0;t) = O\left(\frac{1}{\lambda_n\log\lambda_n}\right), \tag{4.9}$$

whence by (4.8)

$$\Delta\eta_n(k;t) = \Delta\eta_n(0;t)-\sum_{j=0}^{k-1}\Delta^2\eta_n(j;t) = O\left(\frac{1}{\lambda_n}\right),\quad k=0,1,\ldots,\lambda_n. \tag{4.10}$$

Finally, by $\eta_n(0;t)=0$ and (4.10),

$$\eta_n(k;t) = \eta_n(0;t)-\sum_{j=0}^{k-1}\Delta\eta_n(j;t) = O\left(\frac{k}{\lambda_n}\right) = O(1),\quad k=0,1,\ldots,\lambda_n.$$

Apart from the second inequality of (4.2), we have proved all the assertions of our lemma. This remaining inequality follows directly from (2.11). The lemma has been proved.

Let c_k, d_k, $k=0,1,\ldots,\nu$, be arbitrary numbers and set $d_{\nu+1}=d_{\nu+2}=d_{\nu+3}=0$. Write

$$C_k = c_0+c_1+c_2+\ldots+c_k,\qquad k=0,1,2,\ldots,\nu,$$
$$C_k^{(1)} = C_0+C_1+C_2+\ldots+C_k,\qquad k=0,1,2,\ldots,\nu,$$
$$C_k^{(2)} = C_0^{(1)}+C_1^{(1)}+C_2^{(1)}+\ldots+C_k^{(1)},\qquad k=0,1,2,\ldots,\nu,$$

further

$$\Delta d_k = d_k-d_{k+1},\quad \Delta^2 d_k = \Delta d_k-\Delta d_{k+1},\quad \Delta^3 d_k = \Delta^2 d_k-\Delta^2 d_{k+1},$$
$$k=0,1,\ldots,\nu.$$

Lemma 2. *We have*

$$\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)c_kd_k = \sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)C_k^{(2)}\Delta^3d_k+\frac{3}{\nu}\sum_{k=0}^{\nu}C_k^{(2)}\Delta^2d_{k+1}. \tag{4.11}$$

Proof. As we have proved in [3] (formula (4.1)) (²)

$$\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)c_kd_k = \sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)C_k^{(1)}\Delta^2d_k+\frac{2}{\nu}\sum_{k=0}^{\nu}C_k^{(1)}\Delta d_{k+1}. \tag{4.12}$$

Substituting $C_k^{(1)} = C_k^{(2)}-C_{k-1}^{(2)}$ into (4.12), we come readily to (4.11).

Remark. Setting $c_0=1$, $c_1=c_2=\ldots=c_\nu=0$, we have from (4.11)

$$\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)\binom{k+2}{2}\Delta^3d_k+\frac{3}{\nu}\sum_{k=0}^{\nu}\binom{k+2}{2}\Delta^2d_{k+1} = d_0, \tag{4.13}$$

and setting $c_0=1$, $c_1=-1$, $c_2=c_3=\ldots=c_\nu=0$, we get similarly

$$\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)(k+1)\Delta^3d_k+\frac{3}{\nu}\sum_{k=0}^{\nu}(k+1)\Delta^2d_{k+1} = d_0-\left(1-\frac{1}{\nu}\right)d_1. \tag{4.14}$$

Lemma 3. *We have for* $0<|y|\leqslant\pi$, $\nu\leqslant\lambda_n$,

$$\Lambda_{n,\nu}(t,y) = \frac{\eta_n(\nu-1;t)}{\nu}\frac{e^{-iy(\nu+1)}}{(1-e^{-iy})^2}+O\left(\frac{1}{(\nu\log\nu)y^2}\right)+O\left(\frac{1}{\nu^2|y|^3}\right). \tag{4.15}$$

Proof. We insert in (4.11) $c_0=0$, $c_k=e^{-iky}$, $k\geqslant 1$, further $d_k=\eta_n(k;t)$, $k=0,1,\ldots,\nu$; using the summation formula

$$C_k^{(2)} = \binom{k+2}{2}\frac{e^{-iy}}{1-e^{-iy}}-(k+1)\frac{e^{-iy}}{(1-e^{-iy})^2}+\frac{e^{-2iy}}{(1-e^{-iy})^3}(1-e^{-i(k+1)y}),$$

(²) Actually, owing to a misprint in the place quoted, the last summation was made to run from $k=1$ to $k=\nu$.

we come to (see (3.7))

$$\Lambda_{n,\nu}(t,y)=\left\{\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)\binom{k+2}{2}\Delta^3 d_k+\frac{3}{\nu}\sum_{k=0}^{\nu}\binom{k+2}{2}\Delta^2 d_{k+1}\right\}\frac{e^{-iy}}{1-e^{-iy}}-$$

$$-\left\{\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)(k+1)\Delta^3 d_k+\frac{3}{\nu}\sum_{k=0}^{\nu}(k+1)\Delta^2 d_{k+1}\right\}\frac{e^{-iy}}{(1-e^{-iy})^2}+$$

$$+\left\{\sum_{k=0}^{\nu-3}\left(1-\frac{k}{\nu}\right)|\Delta^3 d_k|+\frac{3}{\nu}\sum_{k=0}^{\nu-3}|\Delta^2 d_{k+1}|\right\}O(|y|^{-3})+$$

$$+\frac{2}{\nu}\frac{e^{-2iy}}{(1-e^{-iy})^3}(1-e^{-i(\nu-1)y})\Delta^3 d_{\nu-2}+\frac{1}{\nu}\frac{e^{-2iy}}{(1-e^{-iy})^3}(1-e^{-i\nu y})\Delta^3 d_{\nu-1}+$$

$$+\frac{3}{\nu}\frac{e^{-2iy}}{(1-e^{-iy})^3}(1-e^{-i(\nu-1)t})\Delta^2 d_{\nu-1}+\frac{3}{\nu}\frac{e^{-2iy}}{(1-e^{-iy})^3}(1-e^{-i\nu y})\Delta^2 d_\nu.$$

Making use of (4.13), (4.14) and bearing in mind that $d_{\nu+1}=d_{\nu+2}=d_{\nu+3}=0$, we obtain

$$\Lambda_{n,\nu}(t,y)=\frac{\eta_n(0;t)}{1-e^{-iy}}-\left\{\eta_n(0;t)-\left(1-\frac{1}{\nu}\right)\eta_n(1;t)\right\}\frac{e^{-iy}}{(1-e^{-iy})^2}+$$

$$+\frac{\eta_n(\nu-1;t)}{\nu}\frac{e^{-2iy}}{(1-e^{-iy})^3}(e^{-iy(\nu-1)}-e^{-iy\nu})+\frac{2}{\nu}\frac{e^{-2iy}}{(1-e^{-iy})^3}(1-e^{-iy(\nu-1)})\times$$

$$\times\Delta\eta_n(\nu-2;t)+\left\{\sum_{k=0}^{\nu-3}|\Delta^3\eta_n(k;t)|+\frac{3}{\nu}\sum_{k=0}^{\nu-3}|\Delta^2\eta_n(k+1;t)|\right\}O(|y|^{-3}).$$

Using lemma 1 and formula (4.6), we conclude the proof of our lemma.

LEMMA 4. *We have*

$$\left\|\int_{2\pi/(\nu+1)}^{\pi}\{f(y+t)-f(t)\}\frac{e^{-iy(\nu+1)}}{(1-e^{-iy})^2}\,dy\right\|=O\left(\nu\omega\left(\frac{1}{\nu};f;X\right)\right)\tag{4.16}$$

and

$$\left\|\int_{-\pi}^{-2\pi/(\nu+1)}\{f(y+t)-f(t)\}\frac{e^{-iy(\nu+1)}}{(1-e^{-iy})^2}\,dy\right\|=O\left(\nu\omega\left(\frac{1}{\nu};f;X\right)\right).\tag{4.17}$$

Proof. The integral inside the norm sign in (4.16) will be called $r_\nu(f;t)$. We substitute in this integral $y-\pi/(\nu+1)$ for y. Then

$$r_\nu(f;t)=-\int_{3\pi/(\nu+1)}^{\pi+\pi/(\nu+1)}\frac{f\left(y+t-\frac{\pi}{\nu+1}\right)-f(y+t)}{(1-e^{-i(y-\pi/(\nu+1))})^2}e^{-iy(\nu+1)}dy-$$

$$-\int_{3\pi/(\nu+1)}^{\pi+\pi/(\nu+1)}\frac{f(y+t)-f(t)}{(1-e^{-i(y-\pi/(\nu+1))})^2}e^{-iy(\nu+1)}dy.$$

Hence, by addition

$$2r_\nu(f;t)=-\int_{3\pi/(\nu+1)}^{\pi+\pi/(\nu+1)}\frac{f\left(y+t-\frac{\pi}{\nu+1}\right)-f(y+t)}{(1-e^{-i(y-\pi)/(\nu+1)})^2}e^{-iy(\nu+1)}dy+$$

$$+\int_{2\pi/(\nu+1)}^{\pi}\{f(y+t)-f(t)\}\left\{\frac{1}{(1-e^{-iy})^2}-\frac{1}{(1-e^{-i(y-\pi)/(\nu+1)})^2}\right\}e^{-iy(\nu+1)}dy+$$

$$+\int_{2\pi/(\nu+1)}^{3\pi/(\nu+1)}\frac{f(y+t)-f(t)}{(1-e^{-i(y-\pi)/(\nu+1)})^2}e^{-iy(\nu+1)}dy-\int_{\pi}^{\pi+\pi/(\nu+1)}\frac{f(y+t)-f(t)}{(1-e^{-i(y-\pi)/(\nu+1)})^2}e^{-iy(\nu+1)}dy$$

$$\equiv K_1+K_2+K_3+K_4.$$

Hence

$$2\|r_\nu(f;t)\|\leqslant\sum_{j=1}^{4}\|K_j\|.\tag{4.18}$$

Applying (2.2),

$$\|K_1\|=O\left(\omega\left(\frac{\pi}{\nu+1};f;X\right)\right)\int_{3\pi/(\nu+1)}^{\infty}\frac{dy}{y^2}=O\left(\nu\omega\left(\frac{1}{\nu};f;X\right)\right).\tag{4.19}$$

As to K_2, for $2\pi/(\nu+1)\leqslant y\leqslant\pi$,

$$\frac{1}{(1-e^{-iy})^2}-\frac{1}{(1-e^{-i(y-\pi)/(\nu+1)})^2}=O(y^{-4})\{(1-e^{-i(y-\pi)(\nu+1)})^2-(1-e^{-iy})^2\}$$

$$=O\left(\frac{1}{\nu y^3}\right).$$

By this, (2.1) and (2.9),

$$\|K_2\|=O\left(\int_{2\pi/(\nu+1)}^{\pi}\frac{\omega(y;f;X)}{y^3\nu}\,dy\right)=O\left(\nu\omega\left(\frac{1}{\nu};f;X\right)\right).\tag{4.20}$$

Next,

$$\|K_3\| = O\left(\omega\left(\frac{1}{\nu}; f; X\right)\right) \int_{2\pi/(\nu+1)}^{\infty} \frac{dy}{y^2} = O\left(\nu\omega\left(\frac{1}{\nu}; f; X\right)\right). \tag{4.21}$$

As to the last integral K_4, we observe that it can be estimated without pain by $O(\nu^{-1}\|f\|)$. However, with a view to making the final result as simple as possible, we shall use the "Hille-Klein inequality" (see J. Czipszer [2]).

For $\varphi \in L^1[-\pi, \pi]$ *with* $\int_{-\pi}^{\pi} \varphi(y)\,dy = 0$ *and* $h > 0$, *we have*

$$\int_a^{a+h} |\varphi(y)|\,dy \leqslant 12\omega(h; \varphi; L^1). \tag{4.22}$$

We write K_4 in the form

$$K_4 = -\int_{\pi}^{\pi+\pi/(\nu+1)} \frac{f(y+t)-f(y-\pi+t)}{(1-e^{-i(y-\pi)/(\nu+1)})^2} e^{-iy(\nu+1)}dy - \int_{\pi}^{\pi+\pi/(\nu+1)} \frac{f(y-\pi+t)-f(t)}{(1-e^{-i(y-\pi)/(\nu+1)})^2} e^{-iy(\nu+1)}dy$$

$$= O\left(\int_{\pi}^{\pi+\pi/(\nu+1)} |f(y+t)-f(y-\pi+t)|\,dy\right) + O\left(\int_0^{\pi/(\nu+1)} |f(y+t)-f(t)|\,dy\right).$$

To estimate the first term, we use (4.22) with $\varphi(y) = f(y+t)-f(y-\pi+t)$. Hence

$$K_4 = O\left(\omega\left(\frac{1}{\nu}; f; L^1\right)\right) + O\left(\int_0^{\pi/(\nu+1)} |f(y+t)-f(t)|\,dy\right). \tag{4.23}$$

We note that

$$\omega\left(\frac{1}{\nu}; f; L^1\right) = O\left(\omega\left(\frac{1}{\nu}; f; X\right)\right)$$

for all of our spaces X, whence by (4.23) and (2.1)

$$\|K_4\| = O\left(\omega\left(\frac{1}{\nu}; f; X\right)\right). \tag{4.24}$$

We get the desired (4.16) by (4.18), (4.19), (4.20), (4.21) and (4.24). (4.17) follows along the same lines.

§ 5. We first make some additional remarks.

a) Note that $B_n(f; t) = v_n(f; t)$ (see (3.1)) with $\lambda_n = n-2$ serve as an example of a process satisfying all the conditions of Theorem 2.

b) The following example shows that condition (2.3) cannot be deduced from the remaining conditions of Theorem 2. We set

$$B_n(f; t) = v_n(f; t) + \omega_n \int_{-\pi}^{\pi} f(\tau)\cos n\tau\,d\tau$$

with an $\omega_n \to \infty$ and observe that (2.10)-(2.13) are satisfied if we put $\lambda_n = n-3$. Nevertheless, $\|B_n\| \geqslant \omega_n - 4 \to \infty$ together with ω_n, so that the conclusion of our theorem cannot be satisfied. This example shows that condition (2.3) cannot be replaced by $\|B_n\| = O(\omega_n)$ for some $\omega_n \to \infty$. We note also that, as in this example all expressions (2.10)-(2.13) vanish identically, nothing better can be obtained in this respect on replacing the O-bounds concerned by stronger ones. The same applies to Theorem 1.

c) Condition (2.11) certainly cannot be weakened to

$$B_n(e^{i\tau}; t) - e^{it} = O\left(\frac{\omega(\lambda_n)}{\lambda_n \log \lambda_n}\right)$$

for some $\omega(x) \to \infty$ with $x \to \infty$, at least in case of $X = C_{2\pi}$. This is shown by the following example. We limit ourselves to $\omega(x) = O(\log x)$, for otherwise we could consider $\min(\omega(x), \log x)$ for $\omega(x)$. Let us put then

$$B_n(f; t) = \left(1 - \frac{\omega(n)}{\log n}\right) f(t) + \frac{\omega(n)}{\log n}\sigma_{n+2}(f; t), \tag{5.1}$$

σ_{n+2} being the Fejér means (3.2). Putting $\lambda_n = n$, these B_n's satisfy all the conditions of Theorem 2, except (2.11). Supposing (2.14) true, we could deduce for every $f(t) \in C_{2\pi}$ satisfying

$$\max_t |f(t+h) - f(t)| \leqslant |h|, \tag{5.2}$$

the inequality

$$\max_t |B_n(f; t) - f(t)| = O\left(\frac{1}{n}\right). \tag{5.3}$$

Inserting (5.1) into (5.3), we get

$$\max_t |\sigma_{n+2}(f; t) - f(t)| = O\left(\frac{\log n}{n\omega(n)}\right). \tag{5.4}$$

Denoting the class of functions satisfying (5.2) by D, we have by a well-known theorem of approximation theory (see [4])

$$\lim_{n\to\infty}\frac{n}{\log n}\sup_{f\epsilon D}\max_{t}|f(t)-\sigma_n(f;t)|>0,$$

which evidently contradicts (5.4). We note that also in this example all expressions (2.10)-(2.13) except (2.11) vanish identically.

d) In case of $X=C_{2\pi}$, considering a function which satisfies

$$f(t+h)-f(t)=O(|h|)$$

at a fixed value of t, it is not allowed to conclude from the assumptions of Theorem 2

$$B_n(f;t)-f(t)=O\left(\frac{1}{\lambda_n}\right)$$

at the same value of t. A counter-example is furnished by

$$B_n(f;t)=v_n(f;t)\quad\text{and}\quad\lambda_n=n-2.$$

e) As an application we consider summation methods of Fourier series with triangular coefficient matrices. Let

$$B_n(f;t)=\frac{a_0}{2}+\sum_{k=1}^{n}(1-\mu_k^{(n)})(a_k\cos kt+b_k\sin kt),\tag{5.5}$$

where a_k, b_k are the Fourier coefficients of f and $\mu_{n+1}^{(n)}=1$. Then putting $\lambda_n=n-3$ and assuming $\mu_0^{(n)}=0$, our conditions (2.10)-(2.13) reduce to

$$\mu_1^{(n)}=O\left(\frac{1}{n\log n}\right),\tag{5.6}$$

$$\Delta^2\mu_0^{(n)}=O\left(\frac{1}{n^2}\right),\tag{5.7}$$

$$\sum_{k=0}^{n-4}|\Delta^3\mu_k^{(n)}|=O\left(\frac{1}{n^2}\right).\tag{5.8}$$

We have to remember, however, that these conditions do not imply the boundedness of $\{\|B_n\|\}$. Supposing in addition that

$$\text{either }\Delta^2\mu_k^{(n)}\geqslant 0\text{ or }\Delta^2\mu_k^{(n)}\leqslant 0,\quad k=0,1,\ldots,n-1,\tag{5.9}$$

and

$$\sum_{k=0}^{n}\frac{1-\mu_k^{(n)}}{n-k+1}=O(1),\tag{5.10}$$

we get (by a theorem of S. M. Nikolskij [5])

$$\|B_n\|=O(1).$$

Thus for the B_n's defined in (5.5), the conditions (5.6)-(5.10) are sufficient for

$$B_n(f;t)-f(t)=O\left(\omega\left(\frac{1}{n};f;C\right)\right).$$

f) An important particular case of (5.5) is obtained by setting

$$\mu_k^{(n)}=\varphi\left(\frac{k}{n+1}\right),$$

where $\varphi(x)$ is a function defined in $[0,1]$.

In order that (5.6)-(5.10) be satisfied it is sufficient that $\varphi''(x)$ exist everywhere, be of constant sign and of bounded variation throughout $0\leqslant x\leqslant 1$, moreover that $\varphi(0)=\varphi'(0)=0$, $\varphi(1)=1$.

Putting e. g. $\varphi(x)=1-\cos\frac{\pi}{2}x$, we get substantially the Rogosinski means

$$\frac{1}{2}\left\{S_n\left(t+\frac{\pi}{2n+2}\right)+S_n\left(t-\frac{\pi}{2n+2}\right)\right\}=R_n(f;t)$$

of the Fourier series of $f(t)$. In this case our theorem yields the well-known result

$$R_n(f;t)=f(t)+O\left(\omega\left(\frac{1}{n};f;C\right)\right).$$

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