Isometries of certain Banach algebras

by

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0. Introduction. Let B denote a complex Banach space. By an *isometry* of B we will mean a map φ of B to B which is linear, norm-preserving, and surjective. The purpose of this article is to describe the isometries of two well-known function spaces which, under the norms considered, are not only Banach spaces but Banach algebras:

(1) the algebra $C^{(1)}([0,1])$ (henceforth denoted by $C^{(1)}$) of complex functions continuously differentiable on $[0,1]$, with norm given by

$$\|f\| = \max_{x\in[0,1]}(|f(x)|+|f'(x)|) \quad \text{for} \quad f\in C^{(1)}$$

and

(2) the algebra $AC([0,1])$ (to be denoted by AC) of absolutely continuous complex functions on $[0,1]$, with norm

$$\|f\| = \max_{x\in[0,1]}|f(x)|+\int_0^1 |f'(x)|\,dx \quad \text{for} \quad f\in AC.$$

It is shown that any isometry of $C^{(1)}$ or of AC is induced by a point map of the interval $[0,1]$ onto itself.

1. The algebra $C^{(1)}([0,1])$. We prove the following proposition:

PROPOSITION. *Given $x\in[0,1]$, $\theta\in[-\pi,\pi]$, then there exists $h\in C^{(1)}$ such that*

$$|h(x)|+|h'(x)| > |h(y)|+|h'(y)|$$

for $y\in[0,1]$, $y\neq x$, with $|h(x)| = h(x) > 0$, and $|h'(x)| = e^{i\theta}h'(x) > 0$.

Proof. Let f be the real non-negative continuous function on $[0,1]$ which has the value 1 at x, has slope 1 on $(0,x)$ (if this set is non-void) and has slope -1 on $(x,1)$ (if this latter set is non-void). Next let $g\in C^{(1)}$ be given by

$$g(y) = \int_0^y f(s)\,ds - \int_0^x f(s)\,ds.$$

Then $|g(x)|+|g'(x)| > |g(y)|+|g'(y)|$ for $y \in [0,1]$, $y \neq x$, and $g(x)=0$, from which it follows readily that the function $h \in C^{(1)}$, defined by

$$h(y) = e^{-i\Theta} g(y)+1,$$

has the desired properties.

If X is any compact Hausdorff space, we will denote by $C(X)$ the Banach algebra of continuous complex functions defined on X with norm $\|\cdot\|_\infty$ determined by

$$\|g\|_\infty = \sup_{x \in X} |g(x)| \quad \text{for} \quad g \in C(X).$$

Now let W denote the compact space $[0,1]\times[-\pi,\pi]$. Given $f \in C^{(1)}$, we define $\tilde{f} \in C(W)$ by

$$\tilde{f}(x,\theta) = f(x)+e^{i\Theta} f'(x), \quad (x,\theta) \in W.$$

The following, which we state as a lemma, is then obvious:

LEMMA 1.1. *The mapping $f \to \tilde{f}$ establishes a linear and norm-preserving correspondence between $C^{(1)}$ and the closed subspace S of $C(W)$, $S = [\tilde{f}: f \in C^{(1)}]$.*

We recall that a linear functional f^* contained in the unit ball U^* of $C^{(1)*}$, the dual space of $C^{(1)}$, is called an *extreme point* of U^* if it is not the midpoint of a segment lying in U^*. Clearly f^* is extreme in U^* if and only if $e^{i\eta} f^*$ is extreme for all $\eta \in [-\pi,\pi]$.

Next given $(x,\theta) \in W$, we define a continuous linear functional $L_{(x,\Theta)}$ on $C^{(1)}$ by

$$L_{(x,\Theta)}(f) = f(x)+e^{i\Theta} f'(x), \quad f \in C^{(1)},$$

and prove the following

LEMMA 1.2. *An element f^* of $C^{(1)*}$ is an extreme point of the unit ball U^* of $C^{(1)*}$ if and only if f^* is of the form $e^{i\eta} L_{(x,\theta)}$ for some $\eta \in [-\pi,\pi]$, $(x,\theta) \in W$.*

Proof. It is well known ([1], p. 441) that each extreme point $\tilde{f}^*$ of the unit ball of the dual space of S is of the form

(i) $$\tilde{f}^*(\tilde{f}) = e^{i\eta} \tilde{f}(x,\theta), \quad \tilde{f} \in S,$$

where η is a fixed element of $[-\pi,\pi]$ and (x,θ) is fixed in W. Thus, by virtue of the correspondence established between $C^{(1)}$ and S in Lemma 1.1, each extreme point of U^* is of the form specified.

The converse depends upon a result of K. de Leeuw ([3], p. 61) which sacrificing some generality, we will state as follows. Suppose that X is any compact Hausdorff space, that A is a closed linear subspace of $C(X)$, and that x belongs to X. If there exists an $f \in A$ with $f(x) = \|f\|_\infty$ and

$$|f(y)| \leqslant \|f\|_\infty, \quad y \in X, y \neq x,$$

with equality holding only for those $y \in X$ that satisfy

$$g(y) = g(x) \quad \text{for all} \quad g \in A,$$

then the functional $h \to h(x)$, $h \in A$, is an extreme point of the unit ball of A^*. Employing this result, one obtains as an immediate consequence of the proposition that each functional of the form (i) is extreme in the unit ball of S^*. Hence, again applying Lemma 1.1, each $e^{i\eta} L_{(x,\theta)}$ is an extreme point of U^*.

We now suppose that φ is an isometry of $C^{(1)}$. The adjoint φ^* is then an isometry of $C^{(1)*}$, and thus carries the set of extreme points of U^* onto itself.

LEMMA 1.3. *The image by φ of the constant function 1 of $C^{(1)}$ is a constant function $e^{i\lambda}$, $\lambda \in [-\pi,\pi]$.*

Proof. For each extreme point $e^{i\eta} L_{(x,\Theta)}$ of U^*, $|e^{i\eta} L_{(x,\Theta)}(1)| = 1$. Thus for each extreme point, $|e^{i\eta} L_{(x,\Theta)}(\varphi(1))| = |e^{i\eta} \varphi^* L_{(x,\Theta)}(1)| = 1$. From this it follows that $|(\varphi(1))'|$ can assume only the values zero and one. Hence it is identically zero.

For $x \in [0,1]$, $\theta \in [-\pi,\pi]$, we denote by $e^{i\lambda} L_{(y_{x,\Theta}, v_{x,\Theta})}$ the functional $\varphi^* L_{(x,\Theta)}$. Note that λ is fixed in $[-\pi,\pi]$, independent of x and θ, i. e. $\varphi^* L_{(x,\Theta)}(1) = L_{(x,\Theta)}(\varphi(1)) = e^{i\lambda}$.

LEMMA 1.4. *If $x \in [0,1]$, then for all $\Theta \in [-\pi,\pi]$, $y_{x,\Theta} = y_{x,0}$.*

Proof. For fixed x in $[0,1]$, we consider the map of $[-\pi,\pi]$ to $[0,1]$ given by

$$\Theta \to y_{x,\Theta}.$$

The fact that this mapping is continuous is easily verified. (One may, for example, employ the proposition.) Hence the image of $[-\pi,\pi]$ in $[0,1]$ is a connected subset of $[0,1]$. It is, in fact, a singleton. For otherwise we could find g in $C^{(1)}$, such that $g \equiv 0$ on a subinterval $I \subseteq [y_{x,\Theta}: \Theta \in [-\pi,\pi]]$ while for some $y_{x,\Theta_0} \notin I$, $|g'(y_{x,\Theta_0})| > |g(y_{x,\Theta_0})| > 0$. Hence for an infinity of Θ with $y_{x,\Theta} \in I$,

$$L_{(x,\Theta)}(\varphi(g)) = \varphi^* L_{(x,\Theta)}(g) = 0,$$

while

$$L_{(x,\Theta_0)}(\varphi(g)) = \varphi^* L_{(x,\Theta_0)}(g) \neq 0,$$

which is absurd. Thus, for all Θ, $y_{x,0} = y_{x,\Theta}$ as claimed.

Finally, we define a point map τ of $[0,1]$ to $[0,1]$ by

$$\tau(x) = y_{x,0}.$$

Consideration of $(\varphi^{-1})^*$ shows that τ is onto, and, applying Lemma 1.4, one-one.

THEOREM 1.5. *Let φ be an isometry of $C^{(1)}$. Then, for $f\in C^{(1)}$,*

$$\big(\varphi(f)\big)(x) = e^{i\lambda} f\big(\tau(x)\big),$$

with $e^{i\lambda} = \varphi(1)$. Moreover, τ is one of the two functions F, $1-F$, where F is the identity mapping of $[0,1]$ onto itself: $F(x) = x$ for $x\in[0,1]$.

Proof. Given $x\in[0,1]$, consider the function g of the proposition, constructed so that $g(x) = 0$, $g'(x)$ is positive real and greater than $|g(y)|+|g'(y)|$ for all $y\in[0,1]$, $y\neq x$. For all Θ in $[-\pi,\pi]$ we have

$$\|g\| = e^{-i\Theta}L_{(x,\Theta)}(g) = e^{-i\Theta}\varphi^* L_{(x,\Theta)}(\varphi^{-1}(g)) = e^{i(\lambda-\Theta)}L_{(\tau(x),\psi_{x,\Theta})}(\varphi^{-1}(g)),$$

which clearly implies that $\big(\varphi^{-1}(g)\big)\big(\tau(x)\big) = 0$, and that $\psi_{x,\Theta} = \psi_{x,0}+\Theta$.

Now given any element $f\in C^{(1)}$ with $f(x) = 0$, then for all $\Theta\in[-\pi,\pi]$,

$$f'(x) = e^{-i\Theta}L_{(x,\Theta)}(f) = e^{-i\Theta}\varphi^* L_{(x,\Theta)}(\varphi^{-1}(f)) = e^{i(\lambda-\Theta)}L_{(\tau(x),\psi_{x,0}+\Theta)}(\varphi^{-1}(f))$$

so that $\big(\varphi^{-1}(f)\big)\big(\tau(x)\big) = 0$.

For arbitrary $f\in C^{(1)}$, define $g(y)$ by

$$g(y) = f(y)-f(x), \qquad y\in[0,1].$$

Then $g(x) = 0$, so that

$$0 = \big(\varphi^{-1}(g)\big)\big(\tau(x)\big) = \big(\varphi^{-1}(f)\big)\big(\tau(x)\big) - f(x)\big(\varphi^{-1}(1)\big)\big(\tau(x)\big)$$
$$= \big(\varphi^{-1}(f)\big)\big(\tau(x)\big) - e^{-i\lambda}f(x).$$

Thus, replacing f by $\varphi(f)$, it follows that for all $x\in[0,1]$ and $f\in C^{(1)}$

$$e^{i\lambda}f\big(\tau(x)\big) = \big(\varphi(f)\big)(x).$$

If F is the identity mapping of $[0,1]$ onto itself, we have

$$\tau(x) = e^{-i\lambda}\big(\varphi(F)\big)(x).$$

One then easily establishes the remaining statement of the theorem.

2. The algebra $AC([0,1])$. Let V denote the closed unit ball of the space $L^\infty([0,1])$ provided with the weak-star topology. It is well known that for this topology V is compact ([5], p. 228). We then let W denote the compact space $[0,1]\times V$. Given $f\in AC$, we define $\tilde{f}\in C(W)$ by

$$\tilde{f}(x,\alpha) = f(x)+\int_0^1 f'(s)\bar{\alpha}(s)\,ds, \qquad (x,\alpha)\in W,$$

and state the following lemma:

LEMMA 2.1. *The mapping $f\to\tilde{f}$ establishes a linear and norm-preserving correspondence between AC and the closed subspace S of $C(W)$, $S = [\tilde{f}: f\in AC]$.*

Next, for $(x,\alpha)\in W$, we define the continuous linear functional $L_{(x,\alpha)}$ on AC by

$$L_{(x,\alpha)}(f) = f(x)+\int_0^1 f'(s)\bar{\alpha}(s)\,ds, \qquad f\in AC.$$

It follows, as in the previous section, that the extreme points of the unit ball U^* of AC^* constitute a subset of $[e^{i\eta}L_{(x,\alpha)}: \eta\in[-\pi,\pi], (x,\alpha)\in W]$. Moreover, it is clear that if $L_{(x,\alpha)}$ is extreme in U^*, then α must be extreme in the unit ball of L^∞, i. e. $|\alpha| = 1$ almost everywhere on $[0,1]$ ([2], p. 138).

Now for a given point x in $[0,1]$ we denote by α_x the L^∞ function which takes the value 1 on $[0,x)$ (if this interval is non-void) and takes the value -1 on $(x,1]$ (if this latter set is non-void).

LEMMA 2.2. *For all x in $[0,1]$ and Θ in $(-\pi/2,\pi/2)$, the functional $L_{(x,e^{i\Theta}\alpha_x)}$ is an extreme point of the unit ball in AC^*.*

Proof. Given $x\in[0,1]$, we define $h_{x,0}\in AC$ by

$$h_{x,0}(x) = 1,$$
$$h'_{x,0}(y) = 1, \qquad y\in(0,x),$$
$$h'_{x,0}(y) = -1, \qquad y\in(x,1).$$

Since $L_{(x,\alpha_x)}(h_{x,0}) = \|h_{x,0}\|$, and $|L_{(y,\beta)}(h_{x,0})| < \|h_{x,0}\|$ for $(y,\beta)\in W$, $(y,\beta)\neq(x,\alpha_x)$, the result of de Leeuw previously cited shows that $L_{(x,\alpha_x)}$ is extreme.

Moreover, if M is any real constant, the function $h_{x,0}+Mi$ peaks in modulus at x. Thus if $\theta\in(-\pi/2,\pi/2)$, we can find a function $h_{x,\Theta}\in AC$, where $h_{x,\Theta}$ is of the form $e^{i\Theta}(h_{x,0}+Mi)$ for some real constant M, such that $L_{(x,e^{i\Theta}\alpha_x)}(h_{x,\Theta}) = \|h_{x,\Theta}\|$, and $|L_{(y,\beta)}(h_{x,\Theta})| < \|h_{x,\Theta}\|$, for $(y,\beta)\in W$, $(y,\beta)\neq(x,e^{i\Theta}\alpha_x)$. Thus $L_{(x,e^{i\Theta}\alpha_x)}$ is also extreme.

Suppose that φ is an isometry of AC. We may now easily establish the following lemma:

LEMMA 2.3. *The image by φ of the constant function 1 of AC is a constant function $e^{i\lambda}$, $\lambda\in[-\pi,\pi]$.*

Proof. Let x be any point of $[0,1]$. Then, for all $\theta\in(-\pi/2,\pi/2)$, the fact that $L_{(x,e^{i\Theta}\alpha_x)}$ is an extreme point of U^* implies that $\varphi^*L_{(x,e^{i\Theta}\alpha_x)}$ is a functional of the form $e^{i\eta}L_{(y,\beta)}$, some $\eta\in[-\pi,\pi]$, $(y,\beta)\in W$. Thus $|L_{(x,e^{i\Theta}\alpha_x)}(\varphi(1))| = |\varphi^*L_{(x,e^{i\Theta}\alpha_x)}(1)| = 1$, so that $|\big(\varphi(1)\big)(x)| = 1$ and

$$\int_0^1 |\big(\varphi(1)\big)'(s)|\,ds = 0.$$

Hence, for all y in $[0,1]$, $\big(\varphi(1)\big)(y) = e^{i\lambda}$, with λ fixed in $[-\pi,\pi]$.

For x in $[0,1]$, and θ in $(-\pi/2, \pi/2)$, we will denote by $e^{i\lambda}L_{(y_{x,\theta},\beta_{x,\theta})}$ the functional $\varphi^* L_{(x,e^{i\Theta}\alpha_x)}$. (Note that λ is fixed in $[-\pi, \pi]$, independent of x and θ.) We wish to show that if x is any given point of $[0,1]$, then for all $\theta \in (-\pi/2, \pi/2)$, $y_{x,\theta} = y_{x,0}$, and $\beta_{x,\theta} = e^{i\theta}\beta_{x,0}$. These facts are established by the following three lemmas.

LEMMA 2.4. *If $x \in [0,1]$, $\theta \in (-\pi/2, \pi/2)$ and E is a subset of $[0,1]$, open in $[0,1]$, which contains $y_{x,\theta}$, there exists an $h \in AC$ such that $\varphi^* L_{(x,e^{i\Theta}\alpha_x)}(h) = \|h\|$, and*

$$\max_{z\in([0,1]-E)} |h(z)| < |h(y_{x,\theta})|.$$

Proof. We employ the concept of a T-set introduced by Myers [4]. If B is any Banach space, then a subset T of B maximal with respect to the property that for every finite set $[f_1, \ldots, f_n]$ contained in T,

$$\Big\|\sum_{j=1}^{n} f_j\Big\| = \sum_{j=1}^{n} \|f_j\|$$

is called a *T-set*.

Thus we let $T(x, \theta)$ denote the subset of AC consisting of all f in AC such that $L_{(x,e^{i\Theta}\alpha_x)}(f) = \|f\|$. Clearly norm is an additive function on finite subsets of $T(x, \theta)$, and consideration of the function $h_{x,\theta}$ of Lemma 2.2 shows that $T(x, \theta)$ is maximal with respect to this property. A useful equivalent characterization of $T(x, \theta)$ is the following:

$$T(x, \theta) = [f \in AC: \|f + h_{x,\theta}\| = \|f\| + \|h_{x,\theta}\|].$$

Since φ^{-1} is an isometry, the set

$$\varphi^{-1}\big(T(x, \theta)\big) = [\varphi^{-1}(f): f \in T(x, \theta)]$$

is a T-set of AC, which admits the characterizations

$$\varphi^{-1}\big(T(x, \theta)\big) = [g \in AC: \varphi^* L_{(x,e^{i\Theta}\alpha_x)}(g) = \|g\|]$$
$$= [g \in AC: \|g + \varphi^{-1}(h_{x,\theta})\| = \|g\| + \|\varphi^{-1}(h_{x,\theta})\|].$$

We will assume that $y_{x,\theta}$ is an interior point of $[0,1]$, as the following argument may readily be modified if $y_{x,\theta} = 0$, or $y_{x,\theta} = 1$. Thus there exists an open interval (a, b) such that $y_{x,\theta} \in (a, b) \subseteq E$. Then $\varphi^{-1}\big(T(x, \theta)\big)$ contains an element g_1 which is non-constant on $(a, y_{x,\theta}]$. For supposing the contrary, and letting χ denote the characteristic function of $(a, y_{x,\theta}]$, we obtain an immediate contradiction by defining $g \in AC$ to be given by

$$g(z) = e^{-i\lambda}\int_0^z \chi(s)\,ds,$$

and noting that $\|g + \varphi^{-1}(h_{x,\theta})\| = \|g\| + \|\varphi^{-1}(h_{x,\theta})\|$.

Thus there exists a $g_1 \in \varphi^{-1}\big(T(x, \theta)\big)$ and a point $c \in (a, y_{x,\theta})$ such that

$$g_1(c) \neq g_1(y_{x,\theta}) = e^{-i\lambda}\big(\max_{z\in[0,1]} |g_1(z)|\big).$$

Similarly there exists $g_2 \in \varphi^{-1}\big(T(x, \theta)\big)$ and a point $d \in (y_{x,\theta}, b)$ such that

$$g_2(d) \neq g_2(y_{x,\theta}) = e^{-i\lambda}\big(\max_{z\in[0,1]} |g_2(z)|\big).$$

Clearly the functions h_1 and h_2, defined by

$$h_1(z) = \begin{cases} g_1(c) & \text{for } z \leqslant c, \\ g_1(z) & \text{for } z \geqslant c, \end{cases}$$

$$h_2(z) = \begin{cases} g_2(z) & \text{for } z \leqslant d, \\ g_2(d) & \text{for } z \geqslant d, \end{cases}$$

belong to $\varphi^{-1}\big(T(x, \theta)\big)$. Then $h = h_1 + h_2 + e^{-i\lambda}$ has the desired properties.

LEMMA 2.5. *If $x \in [0,1]$, $\theta \in (-\pi/2, \pi/2)$, then $\beta_{x,\theta} = e^{i\theta}\beta_{x,0}$.*

Proof. We recall the function $h_{x,0}$ of Lemma 2.2, and note first of all that $\big(\varphi^{-1}(h_{x,0})\big)'$ vanishes on no set of positive measure. For suppose, to the contrary, that this function vanished on a set D of non-zero measure. Then for some positive integer k, at least one of the two sets $D \cap [0, y_{x,0} - 1/k]$, $D \cap [y_{x,0} + 1/k, 1]$ has non-zero measure. Choose such a set and denote it by A. By Lemma 2.4 there exists an h in $\varphi^{-1}\big(T(x, 0)\big)$, and an $\varepsilon > 0$, such that

$$\sup_{z\in A} |h(z)| < |h(y_{x,0})| - \varepsilon.$$

Next choose a measurable function G with

$$|G| = 1 \text{ on } A, \quad G = 0 \text{ on } [0,1]-A, \quad \int_0^1 G(s)\,ds = 0,$$

and such that $e^{i\lambda}G\bar{\beta}_{x,0}$ has non-zero imaginary part on some subset of A with positive measure. Now define $g \in AC$ by

$$g(z) = h(0) + \int_0^z \big(h'(s) + \varepsilon G(s)\big)\,ds.$$

Then clearly we have the relation $\|g + \varphi^{-1}(h_{x,0})\| = \|g\| + \|\varphi^{-1}(h_{x,0})\|$, but $e^{i\lambda}L_{(y_{x,0},\beta_{x,0})}(g) \neq \|g\|$, which contradicts the characterizations of the mapping $\varphi^{-1}\big(T(x, 0)\big)$.

Thus writing

$$\big(\varphi^{-1}(h_{x,0})\big)'(s) = \big|\big(\varphi^{-1}(h_{x,0})\big)'\big|(s)\,\beta(s)$$

defines β almost everywhere as a function on $[0,1]$ with $|\beta| = 1$, and it is evident that $\beta_{x,0} = e^{i\lambda}\beta$. Finally, recalling that for $\theta \in (-\pi/2, \pi/2)$, the function $h_{x,\theta}$ of Lemma 2.2 is of the form $e^{i\theta}(h_{x,0}+Mi)$ for some real constant M, it follows that $\beta_{x,\theta} = e^{i\theta}\beta_{x,0}$.

LEMMA 2.6. *If* $x \in [0,1]$, $\theta \in (-\pi/2, \pi/2)$, *then* $y_{x,\theta} = y_{x,0}$.

Proof. Given $\theta \in (-\pi/2, \pi/2)$, let E be an open neighborhood of $y_{x,\theta}$ in $[0,1]$. By Lemma 2.4 there is an h in AC such that $\varphi^* L_{(x,e^{i\theta}\alpha_x)}(h) = \|h\|$, and

$$\max_{z\in([0,1]-E)} |h(z)| < |h(y_{x,\theta})| - \varepsilon$$

for some positive ε. We then have

$$\|h\| = L_{(x,e^{i\theta}\alpha_x)}(\varphi(h)) = (\varphi(h))(x) + e^{-i\theta}\int_0^1 (\varphi(h))'(s)\,\overline{\alpha}_x(s)\,ds,$$

so it is clear that for θ_1 sufficiently close to θ, $y_{x,\theta_1} \in E$.

Thus the mapping of $(-\pi/2, \pi/2)$ into $[0,1]$ given by $\theta \to y_{x,\theta}$ is continuous, and the image of $(-\pi/2, \pi/2)$ under this map is hence a connected subset of $[0,1]$. It then follows readily that this image is a singleton.

We now define a mapping τ of $[0,1]$ into $[0,1]$ setting

$$\tau(x) = y_{x,0},$$

where $y_{x,0}$ is determined as above by

$$\varphi^* L_{(x,\alpha_x)} = e^{i\lambda} L_{(y_{x,0},\beta_{x,0})}.$$

THEOREM 2.7. *Let* φ *be an isometry of* AC. *Then, for* $f \in AC$,

$$(\varphi(f))(x) = e^{i\lambda} f(\tau(x))$$

with $e^{i\lambda} = \varphi(1)$. *Moreover,* τ *is the function* $e^{-i\lambda}\varphi(F)$, *where* F *is the identity mapping of* $[0,1]$ *onto itself:* $F(x) = x$ *for* $x \in [0,1]$.

Proof. Let x belong to $[0,1]$. We first suppose that f is an element of AC with $f(x) = 0$. Then, for all $\theta \in (-\pi/2, \pi/2)$,

$$\int_0^1 f'(s)\,\overline{\alpha}_x(s)\,ds = e^{i\theta} L_{(x,e^{i\theta}\alpha_x)}(f) = e^{i\theta}\varphi^* L_{(x,e^{i\theta}\alpha_x)}(\varphi^{-1}(f))$$
$$= e^{i(\theta+\lambda)} L_{(\tau(x),e^{i\theta}\beta_{x,0})}(\varphi^{-1}(f)),$$

so that $(\varphi^{-1}(f))(\tau(x)) = 0$.

For arbitrary $f \in AC$, define $g(y)$ by

$$g(y) = f(y) - f(x), \quad y \in [0,1].$$

Then

$$0 = (\varphi^{-1}(g))(\tau(x)) = (\varphi^{-1}(f))(\tau(x)) - f(x)(\varphi^{-1}(1))(\tau(x))$$
$$= (\varphi^{-1}(f))(\tau(x)) - e^{-i\lambda} f(x).$$

Replacing f by $\varphi(f)$, we find that for $x \in [0,1]$ and $f \in AC$,

$$e^{i\lambda} f(\tau(x)) = (\varphi(f))(x).$$

If F is the identity mapping of $[0,1]$ onto itself, we have

$$\tau(x) = e^{-i\lambda}(\varphi(F))(x)$$

and the theorem is proved.

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Reçu par la Rédaction le 1. 4. 1964