On the convergence of orthogonal series of polynomials

by

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1. Introduction. Young [5] proved a test for the convergence of trigonometric series formulated by means of generalized variation. The aim of this paper is to formulate and to prove this test in the case of a possibly large class of polynomial orthogonal series. First we give the definition of generalized \( \Phi \)-variation of a function, as introduced by Wiener [4] and generalized in various directions by Young [5], and Musielak and Orlicz [3].

Let \( \Phi(w) \) be a continuous function defined for \( w \geq 0 \), strictly increasing, \( \Phi(0) = 0 \), \( \Phi(w) \to \infty \) as \( w \to \infty \), and let \( f(x) \) be a real-valued function defined in the interval \( [a, b] \). Young defined \( \Phi \)-variation of the function \( f(x) \) by the formula

\[
V_{\Phi}[f, [a, b]] = \sup_{\Pi} \sum_{i=1}^{n-1} \Phi|f(x_{i+1}) - f(x_i)|,
\]

where \( \Pi \) runs over all partitions \( a = x_0 < x_1 < \ldots < x_n = b \) of the interval \( [a, b] \). He modified the classical theorem on limits of Stieltjes integrals, and applying this modification he proved the following test for the convergence of trigonometric Fourier series (\cite{5}, p. 610):

If a function \( f(x) \) continuous in \( [-\pi, \pi] \) has a bounded \( \Phi \)-variation in this interval, where

\[
\lim_{n \to +\infty} \Phi(n) = 1 \text{ with an } a < \frac{1}{2},
\]

then the trigonometric Fourier series of \( f(x) \) is convergent to \( f(x) \) at every point \( x \in (-\pi, \pi) \).

Remark. The theorem is quoted here in a slightly weaker form than the original formulation by Young, namely we assume \( f(x) \) to be continuous in the whole interval. However, this case seems to be the most interesting one; moreover, limiting ourselves to continuous functions, we need not mention special properties of functions of bounded generalized variation.

(\( ^{(*)} \) This problem was raised by W. Orlicz.)
Let \( g(x) \) be a positive integrable function in \([a, b]\). We shall deal with a system of polynomials \( \{p_n(x)\} \) orthonormal in \([a, b]\) with respect to \( g(x) \) as a weight-function, i.e., such that
\[
\int_a^b p_n(x) p_k(x) g(x) \, dx = \begin{cases} 1 & \text{for } n = k, \\ 0 & \text{for } n \neq k. \end{cases}
\]

Here \( p_n(x) \) denotes a polynomial of degree \( n \), with a positive coefficient of \( x^n \). If \( S_n(x) \) denotes the \( n \)-th partial sum of the Fourier series of a function \( f(x) \) with respect to the system \( \{p_n(x)\} \), it is well known that
\[
S_n(x) = \int_a^b f(t) K_n(t, x) \, dt,
\]
where
\[
K_n(t, x) = \sum_{k=0}^{n} p_k(t) p_k(x).
\]
is called the kernel of the integral (1.1).

In the problems of convergence of Fourier series, the following summation formula of Christoffel and Darboux is of importance:
\[
\sum_{k=0}^{n} p_k(t) p_k(x) = \frac{a_n}{a_{n+1}} \frac{p_n(x) p_{n+1}(t) - p_n(t) p_{n+1}(x)}{t - x},
\]
where \( a_n, a_{n+1} \) are the positive coefficients of \( x^n, x^{n+1} \) in polynomials \( p_n(x), p_{n+1}(x) \), respectively. It is known that \( a_n/a_{n+1} \leq \max(|a_i|, |b_i|) \) (see [1], p. 33). In the sequel \( \mathcal{F}(u) \) will be a continuous function defined for \( u \geq 0 \), strictly increasing, \( \mathcal{F}(0) = 0, \mathcal{F}(u) \to \infty \) as \( u \to \infty \), and such that
\[
\mathcal{F}(u_1) + \mathcal{F}(u_2) - \mathcal{F}(u_1 + u_2)
\]
for arbitrary \( u_1, u_2 \geq 0 \). Moreover, we shall write
\[
g_n(t, x) = \int_a^b K_n(t, u) g(u) \, du \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

In our further considerations we shall limit ourselves to the interval \([-1, 1]\), since the general case is reduced to this one by the substitution \( t = \frac{1}{2} \frac{1 + x - a}{b - a} \).

2. We now formulate Young's test for convergence in the case of orthonormal polynomial series.

**Theorem.** Let \( \{p_n(x)\} \) be a system of polynomials orthonormal in \([-1, 1]\) with respect to the weight-function \( g(x) \), and let \( f(x) \) be defined in \([-1, 1]\). We suppose that \((*)\)

\[
(2.1) \quad \text{there exist constants } \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 > 0 \text{ such that}
\]

\[
0 < g(x) \leq \frac{\alpha_1}{1 - x^2} + \frac{\alpha_2}{(1 - x^2)^2}, \quad \|p_n(x)\| \leq \frac{\alpha_3}{(1 - x^2)^{\alpha_4}}
\]

for all \( x \in (-1, 1) \) and \( n = 0, 1, 2, \ldots \).

\[
(2.2) \quad f(x) \text{ is continuous in } [-1, 1],
\]

\[
(2.3) \quad \text{there exists a constant } C < \frac{1}{2} \text{ such that } f(x) \text{ is of bounded } \mathcal{F}\text{-variation in } [-1, 1],
\]

where

\[
(2.4) \quad \Phi(u) \sim \exp(-u^{-\gamma}) \quad \text{as} \quad u \to 0+.\]

Then
\[
f(x) = \sum_{n=0}^{\infty} b_n p_n(x) \quad \text{for every} \quad x \in (-1, 1),
\]

where
\[
b_n = \int_{-1}^{1} f(t) p_n(t) g(t) \, dt.
\]

Remark. It is easy to give an example of a function which is of infinite \( \mathcal{F}\)-variation for all \( \Phi(u) = u^p, p > 1 \), but satisfies conditions (2.3) and (2.4).

3. **Proof.** We shall prove the Theorem basing ourselves on the following lemmas.

**Lemma 3.1.** If the system \( \{p_n(x)\} \) satisfies condition (2.1) and \([a, b] \subset (-1, 1)\), then
\[
\int_{\xi_1}^{\xi_2} p_n(x) p_m(x) g(x) \, dx = O\left( \frac{1}{|m - n|} \right)
\]
uniormly with respect to \( \xi_1, \xi_2 \), where \( a < \xi_1 < \xi_2 < b \).

The proof of this lemma may be found in [2], p. 91-92.

**Lemma 3.2** ([3], p. 602, Theorem 5.5). Let \( f(t), g(t) \) be of bounded generalized variation of the type \( \mathcal{P}_1, \mathcal{P}_2 \), respectively, and let \( f, g \) have no common points of discontinuity. Moreover, let \( \mathcal{P}_1, \mathcal{P}_2 \) be the functions inverses to \( \mathcal{F}_1, \mathcal{F}_2 \). We suppose that
\[
\sum_{n=0}^{\infty} \frac{1}{n} \mathcal{P}_1 \left( \frac{1}{n} \right) \mathcal{P}_2 \left( \frac{1}{n} \right) < \infty.
\]

Then the Riemann-Stieltjes integral \( \int_{-1}^{1} f(t) d\mathcal{F}(t) \) exists.

**Lemma 3.3.** Let the following conditions be satisfied:

\[
(3.1) \quad g_n(t) (n = 1, 2, \ldots) \text{ have uniformly bounded } \mathcal{P}\text{-variation, where } \mathcal{P} \text{ is a convex function, and } F(t) \text{ is of bounded } \mathcal{F}\text{-variation in } [a, b].
\]

**Study Mathematics XXV**
(3.2) \( G_n(t) \) (\( n = 1, 2, \ldots \)) and \( F(t) \) are continuous in \([a, b]\), \( |G_n(t)| \leq M \) for \( n = 1, 2, \ldots \).

(3.3) \( \sum_{n=1}^{\infty} \psi(1/n)\psi(1/n) < \infty \), where \( \psi \) are the functions inverse to \( G, F \), respectively.

(3.4) \( G_n(t) \to G(t) \) as \( n \to \infty \) for \( t \in (a, \tau) \) and \( t \in (\tau, b) \) where \( \tau \) is a given point in \((a, b)\).

Then
\[
\lim_{n \to \infty} \int_a^b G_n(t) dF(t) = \int_a^b G(t) dF(t).
\]

The proof of this lemma is obtained by a slight modification of the proof of Theorem (6.2), p. 606, [5].

**Lemma 3.4.** Let \( \psi(u) \) be a continuous, strictly increasing function, defined for \( u > 0 \), \( \psi(0) = 0 \), \( \psi(u) \to \infty \) as \( u \to \infty \), satisfying (1.4). Given any \( \delta > 0 \), we then have
\[
V_\psi(t, x), -1 \leq t \leq 1 \leq C(\delta) \left[ 1 + \sum_{m=1}^{\infty} \psi \left( \frac{b}{m} \right) \right] \text{ for } -1 + \delta \leq x \leq 1 - \delta,
\]
where \( V_\psi(t, x) \) are defined by (1.5), and \( (\cdot, \cdot) \) are constants depending only on \( \delta \).

Proof of Lemma 3.4. Let \( a \in [-1+\delta, 1-\delta] \), where \( \delta > 0 \). Then \([a+\delta/2, a-\delta/2] \subset \mathbb{R} \) for sufficiently large \( n \). The points \(-1+\delta/2, -1+\delta, -1, -1-\delta, 1-\delta/2, 1, 1+\delta/2 \) divide the interval \([-1, 1]\) into seven subintervals. On the other hand, let us divide the interval \([-1, 1]\) by means of the points
\[
a_m = \frac{x + m}{n}
\]
where \( m \) takes such integer values that \( a_m \in [-1, 1] \).

We now give some auxiliary estimations:

(i) If either \((a, \beta) \in [-1, -1+\delta/2] \) or \((a, \beta) \in [1-\delta/2, 1] \) then
\[
\int_a^b |K_n(t, x)| \psi(t) dt \leq \gamma_1(\delta).
\]

Applying the Christoffel-Darboux summation formula for polynomials \( \psi(x) \), we obtain
\[
\int_a^b |K_n(t, x)| \psi(t) dt
\leq \frac{a_m}{a_{m+1}} |\psi(x)| \int_{a_m}^{a_{m+1}} |P_{m+1}(t)| \psi(t) dt + \frac{a_m}{a_{m+1}} \int_{a_m}^{a_{m+1}} |\psi(t)| \int_{a_m}^{a_{m+1}} |P_{m+1}(t)| \psi(t) dt.
\]

Since \( \psi(1+\delta, 1-\delta) \), there exists by (2.1) a constant \( h_1 = h_1(\delta) \) such that \( |\psi(\alpha)| < h_1(\delta) \) for every index \( n \). Changing the interval of integration to the whole \([-1, 1]\) and applying \( |t-a| \geq \delta/2 \) and Schwarz's inequality
\[
\int_a^b |P_n(t)| \psi(t) dt \leq \int_a^b |P_n(t)| \psi(t) \psi(t) dt \leq \int_a^b |P_n(t)| \psi(t) dt \leq \int_a^b |t-a|^2 \psi(t) dt \leq \gamma_1(\delta),
\]
we find that the integral
\[
\int_a^b |K_n(t, x)| \psi(t) dt
\]
is bounded by a constant \( \gamma_1 \) dependent on \( \delta \). The second part of (i) is obtained in a similar way.

(ii) If \((a, \beta) \in [a+\delta/2, a+\delta/2] \), then
\[
\int_a^b |K_n(t, x)| \psi(t) dt < \gamma_1(\delta).
\]

Indeed, we have \(-1+\delta/2 < a+\delta/2 \leq t \leq a+\delta/2 < 1 - \delta/2 \) for sufficiently large \( n \). Hence, by (2.1),
\[
\int_a^b |K_n(t, x)| \psi(t) dt \leq \int_a^b \sum_{m=1}^{a_m} \int_{a_m}^{a_{m+1}} |P_n(t)| \psi(t) dt \leq h_1(\delta) \int_{a_m}^{a_{m+1}} \psi(t) dt \leq h_1(\delta) \int_{a_m}^{m+1} \psi(t) dt \leq \gamma_1(\delta).
\]

(iii) Let \((a, \beta) \in [a_m, a_{m+1}] \), where \( m \) is such an integer that \( m \neq 0 \) and \( a_m \in [-1+\delta, 1-\delta] \).

Then
\[
\int_a^b |K_n(t, x)| \psi(t) dt \leq \gamma_1(\delta)/m.
\]

Indeed, since \(-1+\delta/2 < a+\delta/2 < m/n < t \leq a+\delta/2 < 1 - \delta/2 \), the last inequality follows by applying the summation formula for polynomials \( P_n(x) \) and the inequalities
\[
|t-a| \geq \frac{m}{n} \quad \text{for } m > 0,
\]
\[
\frac{m}{n} \quad \text{for } m < 0.
\]

In the case of \( m > 0 \) we have
\[
\int_a^b |K_n(t, x)| \psi(t) dt
\leq \frac{a_m}{a_{m+1}} |\psi(x)| \int_{a_m}^{a_{m+1}} |P_{m+1}(t)| \psi(t) dt + \frac{a_m}{a_{m+1}} \int_{a_m}^{a_{m+1}} |\psi(t)| \int_{a_m}^{a_{m+1}} |P_{m+1}(t)| \psi(t) dt.
\]

and the proof is finished.
In the case of \( m < 0 \) analogous estimations hold.

(iv) If either \((a, b) \in [a_n, 1 - \delta/2] \) for a fixed \( m > 0 \), or \((a, b) \in [-1 - \delta/2, a_n] \) for a fixed \( m < 0 \), then

\[
\int_a^b K_n(t, x) \phi(t) \, dt \leq \gamma_1(\delta) \frac{1}{|m|}.
\]

Indeed, by (3.3) we have

\[
\int_a^b K_n(t, x) \phi(t) \, dt = \frac{a_n}{a_{n+1}} p_n(x) \int_a^{a_{n+1}} \phi(t) \, dt - \frac{a_n}{a_{n+1}} p_{n+1}(x) \int_{a_{n+1}}^{b} \phi(t) \, dt.
\]

Supposing \( m > 0 \), let us apply the mean-value theorem to the first of the integrals. Taking into account the inequality \( m/|a - b| \), we get

\[
\int_a^b p_{n+1}(t) \phi(t) \, dt = \frac{1}{|a - b|} \int_a^b p_{n+1}(t) \phi(t) \, dt \leq \frac{m}{m} \int_a^b p_{n+1}(t) \phi(t) \, dt,
\]

where \( \alpha < \beta < \delta \). By lemma 3.1,

\[
\int_a^b p_{n+1}(t) \phi(t) \, dt = O\left(\frac{1}{n+1}\right).
\]

The second integral is estimated analogously. Hence

\[
\int_a^b K_n(t, x) \phi(t) \, dt \leq \gamma_1(\delta) \frac{1}{|m|}.
\]

In the case of \( m < 0 \), the argument is analogous to the above one.

We now turn to the proof of lemma 3.4. We take a partition \(-1 = t_0 < t_1 < \ldots < t_N = 1\) of the interval \([-1, 1]\) and consider the sum

\[
(3.5) \quad \sigma = \sum_{j=1}^N \|g_n(t_j, x) - g_n(t_{j-1}, x)\| = \sum_{j=1}^N \| \int_{t_{j-1}}^{t_j} K_n(t, x) \phi(t) \, dt \|.
\]

Let

\[
I_1 = \left(-1, 1 - \frac{\delta}{2}\right), \quad I_2 = \left(-1 + \frac{\delta}{2}, 1 - \frac{\delta}{2}\right), \quad I_3 = \left(-1, 1 - \frac{\delta}{2}\right), \quad I_4 = \left(-1 + \frac{\delta}{2}, 1\right).
\]

We group the intervals \((t_{j-1}, t_j)\) in three classes, namely: (i) \((t_{j-1}, t_j)\) belongs

1. to the first class if \((t_{j-1}, t_j) \subseteq I_1 \cup I_2 \cup I_3\),
2. to the second one if \((t_{j-1}, t_j) \subseteq I_1 \cup I_4\),
3. to the third one if \((t_{j-1}, t_j)\) contains at least one of the points \(-1 + \delta/2, x - 1/n, x + 1/n, 1 - \delta/2\).

Denoting by \(\sigma_1, \sigma_2, \text{ and } \sigma_3\) the sums in (3.5) extended over intervals \((t_{j-1}, t_j)\) belonging to the first, the second and the third class, respectively, we now prove all sums \(\sigma_1, \sigma_2, \sigma_3\) to be bounded; this will give the boundedness of the sum (3.5).

As regards \(\sigma_1\), applying (1.4) and the estimations (i), (ii), we obtain

\[
\sigma_1 \leq \Psi \left( \int_{-1}^{1 - \delta/2} |K_n(t, x)| \phi(t) \, dt \right) + \Psi \left( \int_{-1 + \delta/2}^{-1} |K_n(t, x)| \phi(t) \, dt \right)
\]

\[
+ \Psi \left( \int_{1 - \delta/2}^{1} |K_n(t, x)| \phi(t) \, dt \right) \leq \gamma_1(\delta).
\]

In order to estimate the sum \(\sigma_2\) we divide the indices \(r\) from the second class again in two subclasses, denoting by \(r'\) such \(r\) that \((t_{j-1}, t_j) \subseteq (a_n, b_{n+1})\) and by \(r''\) such \(r\) that \(t_{j-1} < a_m < t_j\) for some \(m\). Denoting the respective sums by \(\sigma_2'\) and \(\sigma_2''\), we have \(\sigma_2 = \sigma_2' + \sigma_2''\), and by (1.4) and the estimation (iii) we get

\[
\sigma_2' = \sum_{r' \in (a_n, b_{n+1})} \Psi \left( \int_{t_{j-1}}^{t_j} K_n(t, x) \phi(t) \, dt \right) \leq \sum_{r' \in (a_n, b_{n+1})} \Psi \left( \int_{t_{j-1}}^{t_j} |K_n(t, x)| \phi(t) \, dt \right)
\]

\[
\leq \sum_{m=1}^N \Psi \left( \int_{a_m}^{b_m} |K_n(t, x)| \phi(t) \, dt \right) \leq 2 \sum_{m=1}^N \Psi \left( \gamma_1(\delta) \frac{1}{m} \right).
\]

In order to estimate the sum \(\sigma_2''\), let us note that to every \(m\) there exists at most one value \(r''\) such that

\[
(3.6) \quad t_{j-1} < a_m < t_j.
\]

We limit ourselves only to such intervals \((t_{j-1}, t_j)\) which are contained in \(I_4\), since the sum of intervals contained in \(I_2\) is estimated analogously. Denoting the first interval of the type \((t_{j-1}, t_j)\) on the right-hand side of the point \(x + 1/n\) by \((\tau_1, \tau_2)\), we obtain by the estimation (iv)

\[
\int_{t_{j-1}}^{t_j} K_n(t, x) \phi(t) \, dt \leq \gamma_1(\delta) \frac{1}{x + 1/n}.
\]

Now writing \((\tau_3, \tau_4)\) for the next interval of the type \((t_{j-1}, t_j)\) on the right-hand side of \((\tau_1, \tau_2)\) and applying (3.6), we obtain \(\sigma_2'' \leq \gamma_1(\delta) \frac{2}{x + 1/n} \).

Proceeding further in the same way, we get the inequality

\[
\sigma_2'' \leq \sum_{m=1}^N \Psi \left( \gamma_1(\delta) \frac{1}{m} \right) \leq 2 \sum_{m=1}^N \Psi \left( \gamma_1(\delta) \frac{1}{m} \right).
\]
We now estimate $a_1$; it is easily seen that $a_1$ contains at most four terms. Let $(a, \beta)$ denote the interval containing, say, the point $-1 + \delta/2$. We write the integral $\int_1^a K_n(t, x) g(t) dt$ in the form

$$\int_1^a K_n(t, x) g(t) dt = \int_1^{a_1} K_n(t, x) g(t) dt + \int_{a_1}^{\beta} K_n(t, x) g(t) dt.$$ 

Hence

$$\left| \int_1^a K_n(t, x) g(t) dt \right| \leq \int_1^{a_1} \left| K_n(t, x) g(t) dt \right| + \int_{a_1}^{\beta} \left| K_n(t, x) g(t) dt \right|.$$

The first integral is bounded by a constant independent of $x, n$ in virtue of (i). By (iv), the second integral is estimated by a constant dependent only on $\delta$ in case where $\beta \leq a_1$. If $\beta > a_1$, we must estimate the integrals of the form

$$\int_{a_1}^{\beta} K_n(t, x) g(t) dt.$$

But the second integral is estimated by (iv) if $a_1 \leq 1 - \delta/2$, and by (iv) and (i) if $\beta > 1 - \delta/2$. Hence

$$\left| \int_a^1 K_n(t, x) g(t) dt \right| < \gamma_4(\delta)$$

and this shows $a_1$ to be bounded by a constant dependent on $\delta$. Thus lemma 3.3 is proved completely.

4. We now proceed to the proof of the Theorem. Formula (1.1) and the definition of the functions $g_n(t, x)$ give

$$S_n(x) = \int_{-1}^1 f(t) dt [g_n(t, x)].$$

Integrating (4.1) by parts, we obtain

$$S_n(x) = f(1) - \int_{-1}^1 g_n(t, x) df(t).$$

In particular, let the function $F$ be defined for small $u > 0$ by formula $F(u) = u/(\ln u)^{1+\epsilon}$, where $\epsilon > 0$, and let it be defined for other $u > 0$ arbitrarily but in such a way that $F(u)$ satisfies condition (1.4) and is convex. We now apply lemma 3.3; we obtain

$$S_n(\zeta) \sim \Phi(1 - u^{1+\epsilon}) (u \to 0+);$$

$$\Psi(u) \sim -\frac{u}{\ln u}^{1+\epsilon}, \quad \Psi(u) \sim -\frac{u}{\ln u}^{1+\epsilon}, \quad \tau_o = \pi,$$

$$G(t) \equiv G(t, x) = \begin{cases} 0 & \text{for } -1 \leq t < \pi, \\ 1 & \text{for } \pi < t \leq 1. \end{cases}$$

It is easily seen that the assumptions of lemma 3.3 are satisfied; property (3.4) follows from the generalized Riemann-Lebesgue theorem. Hence, by (4.2), we obtain

$$S_n(x) \to f(1) - \int_{-1}^1 G(t, x) df(t) = f(1) - \int_{-1}^1 df(t) = f(x),$$

as $n \to \infty$, and the proof of the Theorem is thus completed.

References


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