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Supplement to my paper "On simultaneous extension of continuous functions"

by

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The purpose of this supplement is the following improvement of the Main Theorem in [3]:

THEOREM. *Let X be a closed linear subspace of the space $C(T)$ of all complex-valued continuous functions on a compact metric space T and let S be a closed subset of T such that*

(B) *for every f in $C(S)$ and for every Δ in $C(T)$ with $\Delta(t) > 0$ for t in T and $\Delta(s) > |f(s)|$ for s in S there is in X an extension F of f such that $\Delta(t) > |F(t)|$ for t in T.*

Then there is a linear operator of extension (¹) $L\colon C(S)\to X$ *with* $\|L\| = 1$.

In [3] it is proved that there exists for every $\delta > 0$ a linear operator of extension $L_\delta\colon C(S)\to X$ with $\|L_\delta\| < 1+\delta$; the existence of a linear operator of extension with norm one is proved only under an additional condition (G) (see [3], p. 287). In fact condition (G) is used only in the proof of Lemma 7 ([3], p. 299) which, in short, states that for every linear operator of extension $L\colon E_\mu\to X$ with the norm $\|L\|$ sufficiently near to one there is a linear operator of extension $L'\colon E_\mu\to X$ with $\|L'\| = 1$ such that the norm $\|L-L'\|$ is small, where E_μ is the finite dimensional subspace of $C(S)$ spanned on a peak partition of unity $\mu = (\mu_j)_{j=1}^{N_\mu}$ (see [3], p. 290-291). In this supplement we shall show that condition (G) is superfluous by proving an analogous lemma (Proposition I) based only on assumption (B). Our method enables us to obtain some improvement of a result of Bishop [1] and Glicksberg [2] concerning interpolation with no increase in norm (Proposition II).

PROPOSITION I. *Let T be a compact Hausdorff space, let S be a closed subset of T, let X be a closed subspace of $C(T)$ and let* (B) *be satisfied. Then for every peak partition of unity $\mu = (\mu_j)_{j=1}^{N_\mu}$ in $C(S)$ and for every linear operator of extension $L\colon E_\mu\to X$ with $\|L\| < 1+\sigma$ $(\sigma > 0)$ there is a linear operator of extension $L'\colon E_\mu\to X$ with $\|L'\| = 1$ such that $\|L-L'\| \leqslant 3\sigma$.*

(¹) We retain the terminology and notation of [3].

We shall need the following lemma:

LEMMA. *Under the assumptions of Proposition 1 for arbitrary functions* $f_1, f_2, \ldots, f_n$ $(n = 1, 2, \ldots)$ *in* $C(S)$ *and for every* Δ *in* $C(T)$ *such that* $\Delta(t) > 0$ *for* t *in* T *and* $\Delta(s) > \sum_{k=1}^{n} |f_k(s)|$ *for* s *in* S *there exist functions* $F_1, F_2, \ldots, F_n$ *in* X *such that* F_k *is an extension of* f_k $(k = 1, 2, \ldots, n)$ *and* $\Delta(t) > \sum_{k=1}^{n} |F_k(t)|$.

Proof. We shall prove this Lemma by induction. For $n = 1$ this is condition (B). Let us suppose that the assertion of the Lemma is true for $n = m-1 \geqslant 1$. Let $f_1, f_2, \ldots, f_m$ and Δ have properties as in the assumption of the Lemma for $n = m$. By the induction hypothesis there are functions $F_1, F_2, \ldots, F_{m-1}$ in X such that F_k is an extension of f_k $(k = 1, 2, \ldots, m-1)$ and

$$\Delta_m(t) = \Delta(t) - \sum_{k=1}^{m-1} |F_k(t)| > 0 \qquad (t \in T).$$

Since (by the assumptions and the induction hypothesis)

$$\Delta_m(s) = \Delta(s) - \sum_{k=1}^{m-1} |f_k(s)| > |f_m(s)| \qquad (s \in S),$$

there exists (by (B)) in X an extension F_m of f_m such that

$$\Delta_m(t) = \Delta(t) - \sum_{k=1}^{m-1} |F_k(t)| > |F_m(t)| \qquad (t \in T).$$

Hence

$$\Delta(t) - \sum_{k=1}^{m} |F_k(t)| > 0 \qquad (t \in T).$$

Proof of Proposition I. We shall construct functions $F_j^{(n)}$ in X $(n = 1, 2, \ldots; j = 1, 2, \ldots, N_\mu)$ and Δ_n in $C(T)$ $(n = 2, 3, \ldots)$ with the following properties:

(1) $F_j^{(1)} = (1+\sigma)^{-1} L\mu_j \quad (j = 1, 2, \ldots, N_\mu)$,

(2) $F_j^{(n)}$ is an extension of

$$\frac{\sigma}{1+\sigma} 2^{-n+1} \mu_j \quad (j = 1, 2, \ldots, N_\mu;\ n = 2, 3, \ldots),$$

(3) $\Delta_n(t) = \min\left[\frac{\sigma}{1+\sigma} 2^{-n+2}, 1 - \sum_{k=1}^{n-1} \sum_{j=1}^{N_\mu} |F_j^{(k)}(t)|\right] > 0$ for t in T $(n = 2, 3, \ldots)$,

(4) $\sum_{j=1}^{N_\mu} |F_j^{(n)}(t)| < \Delta_n(t)$ for t in T $(n = 2, 3, \ldots)$.

The definition of $F_j^{(1)}$ $(j = 1, 2, \ldots, N_\mu)$ is unique. According to Lemma 3 in [3]

$$\|L\| = \sup_{t \in T} \sum_{j=1}^{N_\mu} |L\mu_j(t)| = (1+\sigma) \sup_{t \in T} \sum_{j=1}^{N_\mu} |F_j^{(1)}(t)| < 1+\sigma.$$

Hence $\Delta_2(t) > 0$ for t in T. Now, let us suppose that we have defined $F_j^{(k)}$ and Δ_{k+1} for $1 \leqslant k < m$ and for $j = 1, 2, \ldots, N_\mu$, satisfying the conditions (1)-(4). By the induction hypothesis and the fact that $\mu_j(s) \geqslant 0$ for s in S $(j = 1, 2, \ldots, N_\mu)$ and $\sum_{j=1}^{N_\mu} \mu_j \equiv 1$ we get

$$1 - \sum_{k=1}^{m-1} \sum_{j=1}^{N_\mu} |F_j^{(k)}(s)| = 1 - \sum_{j=1}^{N_\mu} \mu_j(s) \left(1 - 2^{-m+2} \frac{\sigma}{1+\sigma}\right)$$

$$= 2^{-m+2} \frac{\sigma}{1+\sigma} \qquad \text{for} \quad s \in S.$$

Hence

$$\Delta_m(s) > 2^{-m+1} \frac{\sigma}{1+\sigma} \sum_{j=1}^{N_\mu} \mu_j(s) \qquad (s \in S).$$

Thus, according to the Lemma, there exist functions $F_j^{(m)}$ in X $(j = 1, 2, \ldots, N_\mu)$ such that $F_j^{(m)}$ is an extension of

$$2^{-m+1} \frac{\sigma}{1+\sigma} \mu_j \qquad (j = 1, 2, \ldots, N_\mu)$$

and

$$\sum_{j=1}^{N_\mu} |F_j^{(m)}(t)| < \Delta_m(t) \leqslant 1 - \sum_{k=1}^{m-1} \sum_{j=1}^{N_\mu} |F_j^{(k)}(t)| \qquad (t \in T).$$

Thus

$$\Delta_{m+1}(t) = \min\left[\frac{\sigma}{1+\sigma} 2^{-m+1}, 1 - \sum_{k=1}^{m} \sum_{j=1}^{N_\mu} |F_j^{(k)}(t)|\right] > 0 \qquad \text{for } t \text{ in } T.$$

Hence the correctness of the induction definition of $(F_j^{(n)})$ and (Δ_{n+1}) $(j = 1, 2, \ldots, N_\mu;\ n = 1, 2, \ldots)$ is proved.

Let c_j $(j = 1, 2, \ldots, N_\mu)$ be arbitrary complex numbers. Let us set

(5) $L'f = \sum_{j=1}^{N_\mu} c_j \sum_{k=1}^{\infty} F_j^{(k)}$ for $f = \sum_{j=1}^{N_\mu} c_j \mu_j \in E_\mu$.

By (3) and (4) we have

$$\sum_{n=1}^{\infty} \|F_j^{(n)}\| \leqslant \|F_j^{(1)}\| + \sum_{n=2}^{\infty} \|\Delta_n\| \leqslant \|F_j^{(1)}\| + \frac{\sigma}{1+\sigma} \sum_{n=2}^{\infty} 2^{-n+2} < +\infty$$

$$(j = 1, 2, \ldots, N_\mu).$$

Thus L' is a linear operator from E_μ into X. Since L is a linear operator of extension, from (1) and (2) we get

$$L'\mu_j(s) = \frac{1}{1+\sigma} L\mu_j(s) + \sum_{k=2}^{\infty} F_j^{(k)}(s)$$

$$= \mu_j(s)\left[\frac{1}{1+\sigma} + \frac{\sigma}{1+\sigma}\sum_{k=2}^{\infty} 2^{-k+1}\right] = \mu_j(s) \quad \text{for} \quad s\in S \quad (j = 1, 2, \ldots, N_\mu).$$

Hence L' is a linear operator of extension. According to Lemma 3 in [3] and (3) we have

$$\|L'\| = \sup_{t\in T} \sum_{j=1}^{N_\mu} |L'\mu_j(t)| \leqslant \sum_{j=1}^{N_\mu}\sum_{k=1}^{\infty} |F_j^{(k)}(t)| \leqslant 1.$$

Thus $\|L'\| = 1$. If $f = \sum_{j=1}^{N_\mu} c_j\mu_j$, then $\|f\| = \max_{1\leqslant j\leqslant N_\mu} |c_j|$ (cf. [3], Lemma 3). Hence, by (1), (3), (4)

$$\|(L-L')f\| \leqslant \left\|\left(L - \frac{1}{1+\sigma}L\right)f\right\| + \left\|\sum_{j=1}^{N_\mu} c_j \sum_{k=2}^{\infty} F_j^{(k)}\right\|$$

$$\leqslant \sigma\frac{\|L\|}{1+\sigma}\|f\| + \max_{1\leqslant j\leqslant N_\mu} |c_j| \sum_{k=2}^{\infty} \|\varDelta_k\|$$

$$\leqslant \left(\frac{\sigma}{1+\sigma}\|L\| + \frac{\sigma}{1+\sigma}\sum_{k=2}^{\infty} 2^{-k+2}\right)\|f\| \leqslant 3\sigma\|f\|.$$

Thus $\|L-L'\| \leqslant 3\sigma$.

Proposition II. *Under the notation and the assumptions of Proposition I, for every f in $C(S)$ and for every $\varDelta$ in $C(T)$ with $\varDelta(t) > 0$ for t in T and $\varDelta(s) \geqslant |f(s)|$ for $s\in S$ and for an arbitrary closed G_δ subset S_0 of T with $S_0 \supset S$ there is in X an extension F of f such that $\varDelta(t) \geqslant |F(t)|$ for t in T and $\varDelta(t) > |F(t)|$ for t in $T\setminus S_0$.*

Proof. Since S_0 is a closed G_δ subset of a compact Hausdorff space T, there exists a function K in $C(T)$ such that $K(s) = 1$ for $s\in S_0$ and $0 < K(t) < 1$ for $t\in T\setminus S_0$. Let us set $\tilde{\varDelta} = \varDelta\cdot K$. Now, in a way analogous to that used in the proof of Proposition I we define functions $F^{(n)}$ in X and $\varDelta_n$ in $C(T)$ $(n = 1, 2, \ldots)$ such that

(6) $F^{(n)}$ is an extension of $2^{-n}f$ $\quad (n = 1, 2, \ldots)$,

(7) $\varDelta_1 = \frac{3}{4}\tilde{\varDelta}$,

(8) $\varDelta_n(t) > |F^{(n)}(t)|$ for $t\in T$ $(n = 1, 2, \ldots)$,

(9) $\varDelta_{n+1}(t) = \min\left(2^{-n+1}\|\varDelta\|, \tilde{\varDelta}(t) - \sum_{k=1}^{n} |F^{(k)}(t)|\right) > 0$ for $t\in T$ $(n = 1, 2, \ldots)$.

Now we put $F = \sum_{n=1}^{\infty} F^{(n)}$. By (8) and (9) the series $\sum_{n=1}^{\infty}\|F^{(n)}\| < \infty$. Thus the series $\sum_{n=1}^{\infty} F^{(n)}$ converges uniformly to $F\in X$. Hence, by (6), F is an extension of f. By (9) we have

$$\tilde{\varDelta}(t) - \sum_{k=1}^{n} |F^{(k)}(t)| > 0$$

for t in T and for $n = 1, 2, \ldots$ Hence $|F(t)| \leqslant \tilde{\varDelta}(t)$ for $t\in T$. Thus $|F(t)| \leqslant \varDelta(t)$ for $t\in T$ and $|F(t)| < \varDelta(t)$ for $t\in T\setminus S_0$, q. e. d.

Corollary. *Under the assumptions of Proposition* I *(in particular under the assumptions of the theorem of Bishop* [1]*) for every f in $C(S)$ there is an extension F of f in X with $\|f\| = \|F\|$.*

Moreover, F can be chosen so that $|F(t)| < \|f\|$ for $t\in T\setminus S$ if and only if S is a G_δ.

This Corollary under the assumption that X is an algebra has been obtained in another way by Glicksberg [2], see also [4].

References

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