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The preservation of Lipschitz spaces under singular integral operators

by

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The purpose of this note is to extend a result of Calderón and Zygmund ([3], p. 262, Theorem 11), to show that for periodic functions, Lipschitz (Hölder) continuity is preserved by singular integral operators whose kernels satisfy a Dini condition. It was shown in [3] that singular integral operators whose kernels satisfy a Hölder condition of order β would preserve Hölder continuity of order α when $\beta > \alpha$.

Let x be a point in Euclidean n space E_n. If $|x| \neq 0$ let x' be the projection of x onto the unit sphere Σ of E_n. That is, $x' = x/|x|$ and $|x'| = 1$. With Calderón and Zygmund we consider kernels $K(x)$ of the form $K(x) = \Omega(x')|x|^{-n}$ where $\Omega(x')$ is a complex valued function on Σ satisfying two conditions:

a) $\int\limits_{x'\in\Sigma} \Omega(x')\,dx' = 0$.

b) $\Omega(x')$ is a continuous function on Σ and there is a monotone increasing, non-negative function $\omega(t)$, $t > 0$, such that $\omega(t) \geqslant t$ and $|\Omega(r) - \Omega(s)| \leqslant \omega(|r-s|)$ for $r, s \in \Sigma$ and so that the Dini condition, $\int_0^1 \big(\omega(t)/t\big)\,dt < \infty$, is satisfied.

For a full discussion of these properties and their implications the reader is refered to [3], p. 249-252, and [6], p. 468-473.

Let the fundamental torus T_n be defined by

$$T_n = \{x = (x_1, x_2, \ldots, x_n) \in E_n\colon -1/2 < x_i \leqslant 1/2,\ i = 1, 2, \ldots, n\}.$$

We will study here functions defined on T_n, or what is the same thing, their periodic extensions to E_n. That is, we say $f(x)$ is *periodic* if $f(x+k) = f(x)$ where k is any lattice point in E_n.

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For a kernel $K(x)$ as above define for $x \in E_n$

$$K^*(x) = K(x) + \sum_{k\neq 0} \big(K(x+k) - K(k)\big) = K(x) + \bar{K}(x),$$

where the summation is over the non-zero lattice points of E_n. It is well known that the sum $\bar{K}(x)$ converges absolutely and uniformly to a bounded function for $x \in T_n$. Furthermore, we may assume (by subtracting a constant from $\bar{K}(x)$) that

$$(*) \qquad \int_{T_n} K^*(x)\,dx = 0,$$

where the integral is taken in the principal value sense. (See [3] for details.)

For a function $f(x)$ defined on T_n, assumed once and for all to be extended periodically to E_n, let

$$f_\varepsilon^*(x) = \int_{T_n} f(x-z) K_\varepsilon^*(z)\,dz,$$

where $K_\varepsilon^*(z) = K_\varepsilon(z) + \bar{K}(z)$, $K_\varepsilon(z) = K(z)$ if $|z| \geqslant \varepsilon$ and is equal to zero otherwise.

We now define Banach spaces of functions satisfying Lipschitz (Hölder) conditions. For functions $f \in L_p(T_n)$, $1 \leqslant p \leqslant \infty$ (that is, $f \in L_p$ if f is measurable and $\big[\int_{T_n} |f|^p\big]^{1/p} = \|f\|_p < \infty$ if $1 \leqslant p < \infty$ and $\operatorname{ess\,sup}_{x\in T_n} |f(x)| = \|f\|_\infty < \infty$ for $p = \infty$) and such that

$$\|f(x+h) - f(x)\|_p \leqslant C|h|^\alpha, \quad 0 < \alpha < 1,$$

where C is independent of h, we say that $f \in \Lambda_\alpha^p$ and "norm" Λ_α^p with $\|f\|_\alpha^p = \|f\|_p + \operatorname{ess\,sup}_{h\in T_n} \|f(x+h) - f(x)\|_p |h|^{-\alpha}$.

THEOREM. *If $f \in \Lambda_\alpha^p$, $1 \leqslant p \leqslant \infty$, $0 < \alpha < 1$, then f_ε^* converges in the L_p norm to a function $f^* \in \Lambda_\alpha^p$ and $\|f^*\|_\alpha^p \leqslant A\|f\|_\alpha^p$, A independent of f.*

Remarks. For $1 < p < \infty$ the theorem is an immediate consequence of the fact that the "conjugate" operator $Tf = f^*$ is a bounded translation invariant map from L_p into itself for $1 < p < \infty$. (See [3], p. 250.)

Considerably more than the above theorem holds (see [4], Chapter 10, and Chapter 1 for background material, or [5] where some results are stated without proof). In particular, we infer that the maps we are considering preserve all the Lipschitz spaces $\Lambda(\alpha; p, q)$, α real, $1 \leqslant p, q \leqslant \infty$, for distributions on T_n/E_n with a substitute result holding over E_n. The essential idea, however, is contained in the theorem stated here, which can be proved without the introduction of the cumbersome machinery of [4].

The idea of our proofs is to modify the argument of Calderón and Zygmund so as to use a second difference characterization of Λ_α^p classes.

Proof of the theorem. For definiteness we will do the $p = \infty$ case. The other cases are proved in essentially the same way.

Suppose $f \in \Lambda_\alpha^p$. We have

$$f_\varepsilon^*(x) = \int_{z\in T_n} \big(f(x-z) - f(x)\big) K_\varepsilon^*(z)\,dz$$

using $(*)$ above.

Since $\|f(x-z) - f(x)\|_\infty \leqslant \|f\|_\alpha^\infty |z|^\alpha$ we see that

$$\|f_\varepsilon^*\|_\infty \leqslant M\|f\|_\alpha^\infty \int_{z\in T_n} |z|^\alpha |z|^{-n}\,dz = O(1) \quad \text{as} \quad \varepsilon \to 0$$

for an appropriate M. Similarly for $0 < \varepsilon_1 \leqslant \varepsilon_2$ and small enough

$$\|f_{\varepsilon_1}^* - f_{\varepsilon_2}^*\|_\infty \leqslant M\|f\|_\alpha^\infty \int_{\varepsilon_1 \leqslant |z| < \varepsilon_2} |z|^{\alpha-n}\,dz = O(1), \quad \varepsilon_1, \varepsilon_2 \to 0.$$

Therefore f_ε^* tends, in the L_∞ norm to a function f^* which is in L_∞ and clearly the constant M is independent of f and $\|f^*\|_\infty \leqslant M\|f\|_\alpha^\infty$.

It will then suffice to show that for all $|h|$ small enough, $\|f^*(x+2h) - 2f^*(x) + f^*(x-2h)\|_\infty \leqslant M\|f\|_\alpha^\infty |h|^\alpha$ for a suitable M independent of f. (See [1] for a combinatorial proof of this well known fact that for $0 < \alpha < 1$ the conditions above on the first and second differences are equivalent.) We assume, for convenience that $|h| \leqslant 1/6$. We have

$$f_\varepsilon^*(x) = \int_{\substack{|z|<3|h| \\ z\in T_n}} \big(f(x-z) - f(x)\big) K_\varepsilon(z)\,dz + \int_{\substack{|z|\geqslant 3h \\ z\in T_n}} \big(f(x-z) - f(x+h)\big) K_\varepsilon(z)\,dz + \int_{z\in Tn} \big(f(x-z) - f(x+h)\big) \bar{K}(z)\,dz = I_1 + I_2 + I_3,$$

using conditions a) and $(*)$.

Clearly

$$|I_1| \leqslant M\|f\|_\alpha^\infty \int_{|z|<3|h|} |z|^{\alpha-n}\,dz \leqslant M'\|f\|_\alpha^\infty |h|^\alpha.$$

In I_2 and I_3 we change variables and obtain

$$I_2 + I_3 = \int_{\substack{|z-h|\geqslant 3|h| \\ z\in T_n+h}} \big(f(x+h-z) - f(x+h)\big) K_\varepsilon(z-h)\,dz + \int_{z\in T_n+h} \big(f(x+h-z) - f(x+h)\big) \bar{K}(z-h)\,dz,$$

where $T_n + h = \{x : x = y + h,\ y \in T_n\}$.

If in the first integral we replace the domain of integration by $|z| \geqslant 3|h|$, $z \in T_n$, and in the second by $z \in T_n$, we have

$$I_2+I_3 = \int\limits_{\substack{|z|\geqslant 3|h| \\ z\in T_n}} \big(f(x+h-z)-f(x+h)\big)K_\varepsilon(z-h)\,dz + \int\limits_{z\in T_n} \big(f(x+h-z)-f(x+h)\big)\bar{K}(z-h)\,dz + g(x,h),$$

where $g(x,h)$ is an error term such that $\|g(x,h)\|_\infty \leqslant M'\|f\|_\alpha^\infty |h|^\alpha$ with M' independent of f and ε.

We therefore have (up to an error term)

$$f_\varepsilon^*(x) = \int\limits_{\substack{|z|\geqslant 3|h| \\ z\in T_n}} \big(f(x+h-z)-f(x+h)\big)K_\varepsilon(z-h)\,dz + \int\limits_{z\in T_n} \big(f(x+h-z)-f(x+h)\big)\bar{K}(z-h)\,dz.$$

Making obvious substitutions we obtain also that

$$f_\varepsilon^*(x) = \int\limits_{\substack{|z|\geqslant 3|h| \\ z\in T_n}} \big(f(x-h-z)-f(x-h)\big)K_\varepsilon(z+h)\,dz + \int\limits_{z\in T_n} \big(f(x-h-z)-f(x-h)\big)\bar{K}(z+h)\,dz,$$

$$f_\varepsilon^*(x+2h) = \int\limits_{\substack{|z|\geqslant 3|h| \\ z\in T_n}} \big(f(x+h-z)-f(x+h)\big)K_\varepsilon(z+h)\,dz + \int\limits_{z\in T_n} \big(f(x+h-z)-f(x+h)\big)\bar{K}(z+h)\,dz,$$

$$f_\varepsilon^*(x-2h) = \int\limits_{|z|\geqslant 3|h|} \big(f(x-h-z)-f(x-h)\big)K_\varepsilon(z-h)\,dz + \int\limits_{z\in T_n} \big(f(x-h-z)-f(x-h)\big)\bar{K}(z-h)\,dz,$$

all with their appropriate error terms.

We add and gather terms and find

$$f_\varepsilon^*(x+2h)-2f_\varepsilon^*(x)+f_\varepsilon^*(x-2h) = \int\limits_{\substack{|z|\geqslant 3|h| \\ z\in T_n}} \big(f(x+h-z)-f(x+h)+f(x-h)- -f(x-h-z)\big)\big(K_\varepsilon(z+h)-K_\varepsilon(z-h)\big)\,dz + \int\limits_{z\in T_n} \big(f(x+h-z)-f(x+h)+f(x-h)-f(x-h-z)\big)\big(\bar{K}(z+h)- -\bar{K}(z-h)\big)\,dz + l(x,h) = P+Q+l(x,h),$$ where $\|l(x,h)\|_\infty \leqslant M'\|f\|_\alpha^\infty |h^\alpha|$.

An easy calculation shows that

$$\|Q\|_\infty \leqslant M'\|f(x+2h)-f(x)\|_\infty \leqslant M'\|f\|_\alpha^\infty |h|^\alpha,$$

$$\|P\|_\infty \leqslant 2\|f(x+2h)-f(x)\|_\infty \int\limits_{\substack{|z|\geqslant 3|h| \\ z\in E_n}} |K_\varepsilon(z+h)-K_\varepsilon(z-h)|\,dz.$$

There is a constant $C>0$, which depends on K, such that $|K_\varepsilon(z+h) -K_\varepsilon(z-h)| \leqslant (1+1/C)\,\omega(C|h|/|z|)z|^{-n}$ whenever $|z| \geqslant 3|h|$. (See [2], p. 95.) Therefore

$$\|P\|_\infty \leqslant M'\|f\|_\alpha^\infty |h|^\alpha \int\limits_{3|h|}^{\infty} \omega(C|h|/t)\,dt/t = M''\|f\|_\alpha^\infty |h|^\alpha \int\limits_0^{C/\varepsilon} \big(\omega(t)/t\big)\,dt = M'''\|f\|_\alpha^\infty |h|^\alpha$$

by our assumption b) on $\omega(t)$.

Since $f_\varepsilon^*(x+2h)-2f_\varepsilon^*(x)+f_\varepsilon^*(x-2h)$ converges in the norm to $f^*(x+2h)-2f^*(x)+f^*(x-2h)$ and our estimates are independent of ε we see that

$$\|f^*(x+2h)-2f^*(x)+f^*(x-2h)\|_\infty \leqslant M\|f\|_\alpha^\infty |h|^\alpha$$

for some M independent of h and f and so $f^* \in \Lambda_\alpha^\infty$ and $\|f^*\|_\alpha^\infty \leqslant A\|f\|_\alpha^\infty$ for some A independent of f.

This completes the proof.

References

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