The preservation of Lipschitz spaces under singular integral operators

by

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The purpose of this note is to extend a result of Calderón and Zygmund ([3], p. 262, Theorem 11), to show that for periodic functions, Lipschitz (Hölder) continuity is preserved by singular integral operators whose kernels satisfy a Dini condition. It was shown in [3] that singular integral operators whose kernels satisfy a Hölder condition of order $\beta$ would preserve Hölder continuity of order $a$ when $\beta > a$.

Let $x$ be a point in Euclidean $n$ space $\mathbb{E}_n$. If $|x| \neq 0$ let $x'\Sigma$ be the projection of $x$ onto the unit sphere $\Sigma$ of $\mathbb{E}_n$. That is, $x' = x/|x|$ and $|x'| = 1$. With Calderón and Zygmund we consider kernels $K(x)$ of the form $K(x) = \mathcal{O}(x')|x'|^{-n}$ where $\mathcal{O}(x')$ is a complex valued function on $\Sigma$ satisfying two conditions:

a) $\int_{x'\Sigma} \mathcal{O}(x')d\sigma' = 0$.

b) $\mathcal{O}(x')$ is a continuous function on $\Sigma$ and there is a monotone increasing, non-negative function $\omega(t)$, $t > 0$, such that $\omega(t) \geq t$ and $|\mathcal{O}(x') - \mathcal{O}(y)| \leq \omega(|x - y|)$ for $r, s \in \Sigma$ and so that the Dini condition,

$$\int_0^t \omega(t')dt' < \infty,$$

is satisfied.

For a full discussion of these properties and their implications the reader is referred to [3], p. 249-232, and [5], p. 468-473.

Let the fundamental torus $T_n$ be defined by

$$T_n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{E}_n: -1/2 < x_i < 1/2, \ i = 1, 2, \ldots, n\}.$$

We will study here functions defined on $T_n$, or what is the same thing, their periodic extensions to $\mathbb{E}_n$. That is, we say $f(x)$ is periodic if $f(x + k) = f(x)$ where $k$ is any lattice point in $\mathbb{E}_n$.

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For a kernel $K(x)$ as above define for $x \in E_n$

$$K^*(x) = K(x) + \sum_{h \in \mathbb{Z}^n} (K(x+h) - K(h)) = K(x) + \overline{K}(x),$$

where the summation is over the non-zero lattice points of $E_n$. It is well known that the sum $K(x)$ converges absolutely and uniformly to a bounded function for $x \in T_n$. Furthermore, we may assume (by subtracting a constant from $\overline{K}(x)$) that

$$\int_{T_n} K^*(x) dx = 0,$$

where the integral is taken in the principal value sense. (See [3] for details.)

For a function $f(x)$ defined on $T_n$ assumed once and for all to be extended periodically to $E_n$, let

$$f^*_n(x) = \int_{T_n} f(x-s)K^*_n(s) ds,$$

where $K^*_n(x) = K_n(x)+\overline{K}(x)$, $K_n(x) = K(x)$ if $|x| \geq s$ and is equal to zero otherwise.

We now define Banach spaces of functions satisfying Lipshitz (Hölder) conditions. For functions $f \in \mathcal{L}(T_n)$, $1 < p < \infty$ (that is, $f \in \mathcal{L}_p$), if $f$ is measurable and $\int_{T_n} |f(x)|^p dx = \|f\|_p < \infty$ if $1 < p < \infty$ and ess sup $|f(x)| = \|f\|_\infty < \infty$ for $p = \infty$ and such that

$$\|f(x+h)-f(x)\|_p \leq C|h|^a, \quad 0 < a < 1,$$

where $C$ is independent of $h$, we say that $f \in \mathcal{L}^a_p$ and "norm" $\|f\|^a_p$ with $\|f\|^a_p = \|f\|_p + \text{ess sup} |f(x+h)-f(x)|_p|h|^{-a}$.\footnote{\textcopyright ICAM} \cite{1}

**Theorem.** If $f \in \mathcal{L}^a_p$, $1 < p < \infty$, $0 < a < 1$, then $f^*_n$ converges in the $L^p$ norm to a function $f^* \in \mathcal{L}^a_p$ and $\|f^*_n\|^a_p \leq M\|f\|^a_p$, $A$ independent of $f$.

**Remarks.** For $1 < p < \infty$ the theorem is an immediate consequence of the fact that the "conjugate" operator $Tf = f^*$ is a bounded translation invariant map from $L^p$ into itself for $1 < p < \infty$. (See [3], p. 250.)

Considerably more than the above theorem holds (see [4], Chapter 10, and Chapter 1 for background material, or [5] where some results are stated without proof). In particular, we infer that the maps we are considering preserve all the Lipschitz spaces $\mathcal{A}(s; p, q)$, a real, $1 \leq p, q \leq \infty$, for distributions on $T_n/E_n$ with a substitute result holding over $E_n$.

The essential idea, however, is contained in the theorem stated here, which can be proved without the introduction of the cumbersome machinery of [4].

The idea of our proofs is to modify the argument of Calderón and Zyigmund so as to use a second difference characterization of $\mathcal{A}^a_p$ classes.

**Proof of the theorem.** For definiteness we will do the $p = \infty$ case. The other cases are proved in essentially the same way.

Suppose $f \in \mathcal{L}^a_p$. We have

$$f^*_n(x) = \int_{T_n} (f(x) - f(x-s))K^*_n(s) ds$$

using (a) above.

Since $||f(x)| - f(x)||_\infty \leq ||f^*_n||_\infty$ we see that

$$||f^*_n||_\infty \leq M ||f^*_n||_\infty \int_{T_n} |x|^a dx = O(1)$$

as $x \to 0$

for an appropriate $M$. Similarly for $0 < \varepsilon_1 \leq \varepsilon_2$ and small enough

$$||f^*_n - f^*_n||_\infty \leq M ||f^*_n||_\infty \int_{T_n} |x|^a dx = O(1),$$

$\varepsilon_1 \to 0$.

Therefore $f^*_n$ tends, in the $L^p$ norm to a function $f^*$ which is in $L^p$ and clearly the constant $M$ is independent of $f$ and $||f^*_n||_p \leq M ||f^*_n||_p$.

It will then suffice to show that for all $|h|$ small enough, $||f^*(x+2h) - 2f^*(x+h) + f^*(x-2h)||_\infty \leq M ||f^*_n||_\infty |h|^a$ for a suitable $M$ independent of $f$. (See [1] for a combinatorial proof of this well known fact that for $0 < a < 1$ the conditions above on the first and second differences are equivalent.) We assume, for convenience that $|h| < \varepsilon/6$. We have

$$f^*_n(x) = \int_{T_n} (f(x) - f(x-s))K^*_n(s) ds + \int_{T_n} (f(x-s) - f(x-h))K^*_n(s) ds +$$

$$+ \int_{T_n} (f(x-h) - f(x+h))K^*_n(s) ds = I_1 + I_2 + I_3,$$
If in the first integral we replace the domain of integration by \( |s| \geq 3|h| \), and in the second by \( s \mathcal{T}_h \), we have

\[
I_1 + I_3 = \int_{|s| \geq 3|h|} (f(s+h-z) - f(s+h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s+h-z) - f(s+h)) \mathcal{K}(z) \, ds + g(s, h),
\]

where \( g(s, h) \) is an error term such that \( |g(s, h)|_{\infty} \leq M \|f\|_{\infty}^n \) with \( M \) independent of \( f \) and \( s \).

We therefore have (up to an error term)

\[
f_1^*(s) = \int_{|s| \geq 3|h|} (f(s+h-z) - f(s+h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s+h-z) - f(s+h)) \mathcal{K}(z) \, ds.
\]

Making obvious substitutions we obtain also that

\[
f_2^*(s) = \int_{|s| \geq 3|h|} (f(s+h-z) - f(s-h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s-h-z) - f(s-h)) \mathcal{K}(z) \, ds,
\]

\[
f_3^*(s+2h) = \int_{|s| \geq 3|h|} (f(s+h-z) - f(s-h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s+h-z) - f(s-h)) \mathcal{K}(z) \, ds,
\]

\[
f_4^*(s-2h) = \int_{|s| \geq 3|h|} (f(s-h-z) - f(s-h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s-h-z) - f(s-h)) \mathcal{K}(z) \, ds,
\]

all with their appropriate error terms.

We add and gather terms and find

\[
f_1^*(s+2h) - 2f_1^*(s) + f_2^*(s-2h) = \int_{|s| \geq 3|h|} (f(s+h-z) - f(s+h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s+h-z) - f(s-h)) \mathcal{K}(z) \, ds
\]

\[
- f(s-h-z) \mathcal{K}(s+h) \mathcal{K}(s-h) \, ds + \int_{s \mathcal{T}_h} (f(s+h-z) - f(s+h)) \mathcal{K}(z) \, ds + \int_{s \mathcal{T}_h} (f(s+h-z) - f(s-h)) \mathcal{K}(z) \, ds
\]

\[
= \mathcal{K}(s-h) dz + I(s, h) = P + Q + I(s, h), \quad \text{where} \quad |I(s, h)|_{\infty} \leq M \|f\|_{\infty}^n.
\]

An easy calculation shows that

\[
|Q|_{\infty} \leq M \|f(s+2h) - f(s)\|_{\infty} \leq M \|f\|_{\infty}^n,
\]

\[
|P|_{\infty} \leq 2 \|f(s+2h) - f(s)\|_{\infty} \int_{s \mathcal{T}_h} |\mathcal{K}(s+h) - \mathcal{K}(s-h)| \, ds.
\]

There is a constant \( C > 0 \), which depends on \( \mathcal{K} \), such that \( |\mathcal{K}(s+h) - \mathcal{K}(s-h)| \leq (1 + 1/C) \omega(C|h|/|s|)|s|^{-n} \) whenever \( |s| \geq 3|h| \). (See [2], p. 96.) Therefore

\[
|P|_{\infty} \leq M \|f\|_{\infty}^n \int_{s \mathcal{T}_h} \omega(C|h|/|s|) \, ds = M \|f\|_{\infty}^n \int \omega(t) \, dt = M \|f\|_{\infty}^n \frac{C}{n}
\]

by our assumption b) on \( \omega(t) \).

Since \( f_1^*(s+2h) - 2f_1^*(s) + f_2^*(s-2h) \) converges in the norm to \( f_1^*(s+2h) - 2f_1^*(s) + f_2^*(s-2h) \) and our estimates are independent of \( s \), we see that

\[
|f_1^*(s+2h) - 2f_1^*(s) + f_2^*(s-2h)|_{\infty} \leq M \|f\|_{\infty}^n
\]

for some \( M \) independent of \( h \) and \( f \) and so \( f \in A^n \) and \( \|f\|_{\infty} \leq A \|f\|_{\infty}^n \) for some \( A \) independent of \( f \).

This completes the proof.

References


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