THEOREM C'. *If the series* (1.1) *in Theorem* C *is of power series type, then the function* F *has at* x_0 *an* r*-th Peano unsymmetric derivative in* L^p, $p<\infty$, *equal to* s.

The proof parallels that of Theorem C. The generalization obviously adds nothing to our Theorem 2.

2. The conclusion of Theorem 1 holds if the hypothesis is replaced by the following one: at each point $x \in E$ we have (1.4) with O instead of o (the polynomial $P(t)$ may then, of course, be of degree $k-1$ or less). The proof remains unchanged if we note that the conclusion of the lemma on p. 91 remains unchanged if we replace the o in (2.3) and (2.4) by O, provided $\alpha>1$.

References

[1] J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometric series*, Fund. Math. 26 (1936), p. 1-43.

[2] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. 20 (1961), p. 171-225.

[3] J. Marcinkiewicz, *Sur les séries de Fourier*, Fund. Math. 27 (1927), p. 38-69.

[4] A. Zygmund, *Trigonometric series*, vols. I and II, Cambridge 1959.

[5] — *Sur la différentiabilité de fonctions*, Séminaire Lelong, Paris 1961.

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On a theorem of Mackey, Stone and v. Neumann

by

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The canonical commutation rules of the quantum mechanics of a system with N degrees of freedom have the following form:

$$\begin{aligned} Q_nP_m-P_mQ_n &= i\delta_{nm}, \\ P_nP_m-P_mP_n &= 0, \qquad (m,n=1,2,\ldots,N) \\ Q_nQ_m-Q_mQ_n &= 0, \end{aligned}$$

where Q_n, P_n $(n=1,2,\ldots,N)$ are self-adjoint operators in the Hilbert space H. For $\hat{x}\in E^N$, $x\in E^{N*}$ we define

$$U(\hat{x}) = \exp\Big(i\sum_{n=1}^{N}\hat{x}_nQ_n\Big), \qquad V(x)=\exp\Big(i\sum_{n=1}^{N}x_nP_n\Big).$$

The commutation rules were put by H. Weyl in the following correct form:

$$\begin{aligned} U(\hat{x}_1)\,U(\hat{x}_2) &= U(\hat{x}_1+\hat{x}_2), \\ V(x_1)\,V(x_2) &= V(x_1+x_2), \\ V(x)\,U(\hat{x}) &= U(\hat{x})\,V(x)\exp\Big(i\sum\hat{x}_nx_n\Big). \end{aligned}$$

Let us notice that $x\to\exp\big(i\sum\hat{x}_nx_n\big)$ is a character of the group E^N. We assume that the algebra generated by the operators $U(\hat{x})$ is cyclic. This assumption means that this algebra has a simple spectrum. In the language of physics we say that the operators Q_n form a complete set of commuting observables (cf. [6], p. 122). From these assumptions it follows that there exists so called *Schrödinger representation* of the operators Q and P. This means that there exists an isomorphism

$$H \xrightarrow{I} L^2(E^N)$$

such that

$$(IQ_nI^{-1}\varphi)(y) = y_n\varphi(y), \qquad (IP_nI^{-1}\varphi)(y) = -i\frac{\partial}{\partial y_n}\varphi(y)$$

for $\varphi\in C_0^\infty(E^N)\subset L^2(E^N)$.

We can rewrite these equations in the Weyl form:

$$\big(IU(\hat{x})I^{-1}\varphi\big)(y) = \varphi(y)\exp\Big(i\sum \hat{x}_n y_n\Big),$$
$$\big(IV(x)I^{-1}\varphi\big)(y) = \varphi(x+y)$$

for $\varphi \in L^2(E^N)$.

This theorem (called Stone and v. Neumann theorem) was proved in [7] and [8]. It was generalized by G. Mackey (in [5]) for the case where E^N was replaced by any separable locally compact Abelian group.

H. Araki (in [1]) and I. M. Gelfand (in [3]), using Bochner theorem for nuclear spaces, determined all cyclic representations of the commutation rules for systems with infinite number of degrees of freedom (quantum theory of fields).

In the present note we show that Araki-Gelfand method allows to obtain a simple proof of a variant of the Mackey theorem.

Let G be a separable locally compact Abelian group, $\hat{G}$ the character group of G, and $(\hat{x}, x)$ the value of the character $\hat{x} \in \hat{G}$ at the point $x \in G$.

THEOREM (Mackey). *Let H be a separable Hilbert space and let (U, H) and (V, H) be a strongly continuous unitary representation of G and of $\hat{G}$ in the space H respectively such that*

1. $V(x)U(\hat{x}) = (\hat{x}, x)U(\hat{x})V(x)$.
2. *There exists a cyclic vector $h_0 \in H$:*

$$[\{U(\hat{x})h_0 : \hat{x} \in \hat{G}\}] = H, \qquad (h_0 \mid h_0) = 1.$$

Then there exists an isomorphism $H \ni h \leftrightarrow h[\cdot] \in L^2(G)$, the Hilbert spaces H and $L^2(G)$ such that

$$\big(U(\hat{x})h\big)[y] = (\hat{x}, y)h[y], \qquad \big(V(x)h\big)[y] = h[xy].$$

Proof. Put $E(\hat{x}) = \big(U(\hat{x})h_0 \mid h_0\big)$. One checks that $E(\hat{e}) = 1$ ($\hat{e}$ is the unity of $\hat{G}$), $E(\cdot)$ is positive definite continuous function on $\hat{G}$. Applying the generalized Bochner theorem (see [4]) we have

$$E(\hat{x}) = \big(U(\hat{x})h_0 \mid h_0\big) = \int_G (\hat{x}, y)\,d\mu(y),$$

where μ is positive measure on G.

Let $h \in H$ and $h = \sum_{i=1}^{k} c_i U(\hat{x}_i)h_0$. Let us put $h(y) = \sum_{i=1}^{k} c_i(\hat{x}_i, y)$. One can easily check that

$$\int_G h_1(y)\overline{h_2(y)}\,d\mu(y) = (h_1 \mid h_2).$$

From the assumption 2 and from the completeness of the set of characters it follows that the mapping $h \to h(\cdot)$ can be extended to an isomorphism of the Hilbert spaces H and $L^2(G, \mu)$. From the definition it follows that

$$\big(U(\hat{x})h\big)(y) = (\hat{x}, y)h(y) \quad \text{for } \mu\text{-almost all } y.$$

(From $h = h_1$ it follows that $h(y) = h_1(y)$ for μ-almost all y.)

Now let us consider the operators $V(x)$. In view of the assumption 1 we have

$$\big(V(x)h\big)(y) = \Big(\sum c_i V(x)\,U(\hat{x}_i)h_0\Big)(y) = \Big(\sum c_i(\hat{x}_i, x)\,U(\hat{x}_i)\,V(x)h_0\Big)(y)$$
$$= \sum c_i(\hat{x}_i, x)(\hat{x}_i, y)\big(V(x)h_0\big)(y) = \sum c_i(\hat{x}_i, xy)\big(V(x)h_0\big)(y)$$
$$= h(xy)\big(V(x)h_0\big)(y).$$

The mapping $x \to \big(V(x)h_0\big)(\cdot)$ isa continuous (hence measurable) function on G. Its value belongs to $L^2(G, \mu)$. Then (cf. [2], p. 198) there exists a measurable function on the product

$$G \times G \ni (x, y) \to a(x, y) \in C^1$$

such that $\big(V(x)h_0\big)(y) = a(x, y)$ for all x and μ-almost all y. Therefore

$$\big(V(x)h\big)(y) = h(xy)\,a(x, y).$$

Setting in this equations $h = V(z)h_0$ and $h(y) = a(z, y)$ we get

$$a(xz, y) = a(z, xy)\,a(x, y).$$

This equation is true for all x, z and μ-almost all y. Setting $z = x^{-1}$ we get (because $h_0(y) = 1$) $1 = a(x^{-1}, xy)\,a(x, y)$ and $a(x, y) \neq 0$ for μ-almost all y.

Let us write

$$N = \{(x, z, y) \in G^3 : a(xz, y) \neq a(z, xy)\,a(x, y)\},$$
$$N' = \{(x, y) \in G^2 : a(x, y) = 0\}.$$

The sets N and N' are measurable, since $a(\cdot, \cdot)$ is measurable. Let $\chi_N(x, z, y)$ and $\chi_{N'}(x, y)$ be characteristic functions of the set N and N' respectively. Obviously,

$$\int_G \chi_N(x, z, y)\,d\mu(y) = 0, \qquad \int_G \chi_{N'}(x, y)\,d\mu(y) = 0.$$

Using Fubini theorem we have

$$\int_G d\mu(y) \int_{G^2} \chi_N(x, z, y)\,d\mu(x)\,dz = \int_{G^2} d\mu(x)\,dz \int_G \chi_N(x, z, y)\,d\mu(y) = 0,$$
$$\int_G d\mu(y) \int_G \chi_{N'}(x, y)\,d\mu(x) = \int_G d\mu(x) \int_G \chi_{N'}(x, y)\,d\mu(y) = 0$$

(dz refers to Haar measure on G). Then there exists a $y_0 \in G$ such that

$$\int_{G^2} \chi_N(x, z, y_0)\, d\mu(x)\, dz = 0, \qquad \int_G \chi_{N'}(x, y_0)\, d\mu(x) = 0.$$

These equations mean that $a(xz, y_0) = a(z, xy_0)a(x, y_0)$ for μ-almost all x and almost all z and $a(x, y_0) \neq 0$ for μ-almost all x. From these relations it follows that

$$a(z, xy_0) = \frac{a(xz, y_0)}{a(x, y_0)} \quad \text{for } \mu\text{-almost all } x \text{ and almost all } z.$$

Since the operators $V(x)$ are unitary, the measure $\mu(x)$ and its translation $\mu(xy_0)$ are equivalent (cf. [3]). Setting $y = xy_0$ and

$$K(y) = a(yy_0^{-1}, y_0)$$

we get the equation

$$a(z, y) = \frac{K(zy)}{K(y)} \quad \text{for almost all } z \text{ and } \mu\text{-almost all } y.$$

Now for each $h \in H$ we put

$$h[y] = h(y)K(y).$$

Then

$$\big(U(\hat{x})h\big)[y] = K(y)\big(U(\hat{x})h\big)(y) = K(y)(\hat{x}, y)h(y) = (\hat{x}, y)h[y],$$
$$\big(V(x)h\big)[y] = K(y)\big(V(x)h\big)(y) = K(y)h(xy)a(x, y)$$
$$= h(xy)K(xy) = h[xy].$$

(The last equation was proved for almost all x and μ-almost all y. From the continuity of the mapping $x \to V(x)h$ it follows that it is true for all x and μ-almost all y).

We have

$$(h_1 | h_2) = \int h_1(y)\overline{h_2(y)}\, d\mu(y) = \int h_1[y]\overline{h_2[y]}\, dy, \quad \text{where} \quad dy = \frac{d\mu(y)}{|K(y)|^2}.$$

Since the operators $V(x)$ are unitary, the measure dy is invariant; therefore dy is the Haar measure on the group G. This proves that $h[\cdot] \in L^2(G)$. The proof is complete.

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References

[1] H. Araki, *Hamiltonian formalism and the canonical commutation relations in quantum field theory*, Journal of Math. Physics 1 (1960), p. 492-504.

[2] N. Dunford and J. T. Schwartz, *Linear operators, Part I. General theory*, New York 1958.

[3] И. М. Гельфанд и Н. Я. Виленкин, *Некоторые применения гармонического анализа*, Москва 1961.

[4] И. М. Гельфанд, Д. А. Райков и Г. Е. Шилов, *Коммутативные нормированные кольца*, Москва 1960.

[5] G. W. Mackey, *A theorem of Stone and v. Neumann*, Duke Math. Jour. 16 (1949), p. 313.

[6] K. Maurin, *Metody przestrzeni Hilberta*, Warszawa 1959.

[7] J. v. Neumann, *Die Eindeutigkeit der Schrödingerschen Operatoren*, Mathematische Annalen 104 (1931), p. 570.

[8] M. H. Stone, *Linear transformations in Hilbert space, III. Operational methods and group theory*, Proc. National Acad. of Sciences of the USA 16 (1930), p. 172.

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