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Groups whose regular representation weakly contains all unitary representations

by

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In the theory of representations of locally compact groups one of the central problem is to find and describe the decompositions of a given unitary representation into irreducible ones. A procedure of such decomposition originated by Mautner [11] has been developed by many authors (cf. e. g. [12] for references). The method is to represent the Hilbert space of the representation in the form of a direct integral of Hilbert spaces and to define on each of the components a unitary representation of the group, this being irreducible for almost all components. It is a natural question to ask which of the unitary representations of a group can be built up out of the irreducible representations obtained by decomposing a given unitary representation as the building blocks. Postponing the precise definition to the next section, we say that a unitary representation T is *weakly contained* in a unitary representation S if T can be "build up" out of the irreducible representations that occur in a decomposition of T into irreducible representations. It is clear that the representations which are weakly contained in the left (right) regular representation deserve a special attention. They form the "principal series of unitary representations" to be distinguished from the "complementary series of unitary representations" which are not weakly contained in the left (right) regular representation.

In this paper we are concerned with the class (R) of groups which have only the principal series of unitary representations. The first who called attention to the class (R) was Godement, cf. [6], who has shown that in order that a group belong to the class (R) it is necessary and sufficient that its trivial unit representation be weakly contained in the left regular representation. Then Yoshizawa [14] has proved that the discrete non-Abelian free group does not belong to the class (R). In 1955 Takenouchi [13] proved that if a locally compact group G has the property that the factor group of G by the component of the unity is compact, the G belongs to (R) if and only if it has property (C) of Yamabe [1]. Recently Fell [3] has build up a theory of weak containment of unitary representations of a group in terms of which the problem of characterizing the representations which are weakly contained in the regular representation as well as the problem of characterizing the groups of the class (R) has found a very natural formulation. Fell himself has shown that in the case of the group of $n\times n$ complex unimodular matrices the regular representation of G weakly contains the representations of the principal non-degenerate series (in the sense of Gelfand and Naimark [5]), and no others. We are going to show that in the case of discrete groups the class (R) coincides with the class of groups with the full Banach mean value and, as the matter of fact, we present a considerable reduction of the problem of characterizing the class (R) in general, e. g. we show that in the general case any group of the class (R) has a left invariant Banach mean value defined on the class of all essentially bounded measurable functions on the group. The class of groups for which a left invariant Banach mean value exists is well described in the case of discrete groups (cf. [2] and [4]). In the case of locally compact groups many results known for the discrete case remain to be true (cf. [1] and section 2 of this paper).

The paper is organized as follows. After a preliminary section 1, section 2 is devoted to the study of invariant Banach mean value defined on various classes of bounded measurable functions on a locally compact group. We show that the theorems concerning subgroups, factor groups and extensions have their analogies in the general case. Section 3 presents two conditions which are generalizations of Følner's conditions formulated by Følner [4] for discrete groups and, in this case, proved to be necessary and sufficient for a group to exist an invariant Banach mean value on it. Section 4 is devoted to the proof of the fact that if a locally compact group G satisfies the first of the generalized Følner conditions, then it belongs to the class (R). In the four theorems of section 5, called auxiliary theorems, all reasonings needed in the proof of the main theorem are virtually contained. The first theorem evaluates the maximum of a certain symmetric quadratic form on the class of characteristic functions. The idea of the theorem is due to Kesten [8] and the proof makes the utmost use of his result. Two of the remaining theorems are to the effect that a certain condition on the norms of operators being images of integrable functions by representations of the group implies the existence of a left-invariant Banach mean value on the group, provided the group is unimodular. Finally the last theorem of this section shows that this

(1) This result can be easily deduced from the main theorem of this paper and theorem 4.1. In fact, any group of the class (C) satisfies (F_1) and any group which is not in (C) contains a free subgroup with two generators.

condition is implied by the requirement that the group belongs to the class (R). This, after a simple application of a lemma due to Takenouchi and a result of Gleason, leads to the main theorem and its corollary in section 6.

1. Preliminaries. All groups considered here will be locally compact topological groups. If M is a measurable subset of a group G, by $|M|$ we denote the left-invariant Haar measure of M and by ds the differential of the left-invariant Haar measure. The Radon-Nikodym derivate of the left-invariant Haar measure with respect to the right-invariant Haar measure is denoted by ϱ.

By $L_p(G)$, $1\leqslant p<\infty$, we denote the space of the measurable functions on the group G the p-th power of which is integrable with respect to the left-invariant Haar measure, the corresponding norms being denoted by $\|\cdot\|_p$. By $L_\infty(G)$ we denote the space of essentially bounded measurable functions on G and, accordingly, the norm in $L_\infty(G)$ is denoted by $\|\cdot\|_\infty$. By $L(G)$ we denote the set of the measurable essentially bounded functions on G which vanishes outside compact sets.

Let $C(G)$ denote the space of continuous bounded functions on G.

We say that a function φ on a group G is *uniformly continuous* if for any $\varepsilon>0$ there is an open neighbourhood U of the unity of G such that $|\varphi(stu)-\varphi(t)|<\varepsilon$ for any $t\in G$ and any $s,u\in U$. The space of uniformly continuous bounded functions on a group G is denoted by $C_0(G)$. It is clear that $C_0(G)$ is a closed subspace of the space $L_\infty(G)$.

For any element $s\in G$ by T_s and ${}_sT$, respectively, we denote the operations on functions on G defined by

$$(T_s\varphi)(t)=\varphi(ts),\qquad ({}_sT\varphi)(t)=\varphi(st).$$

As an immediate consequence of the definition of uniformly continuous function we obtain

1.1. *If m is a bounded linear functional on $C_0(G)$, then for a fixed function $\varphi\in C_0(G)$ the functions*

$$\varphi_1(s)=m(T_s\varphi),\qquad \varphi_2(s)=m({}_sT\varphi)$$

are continuous bounded functions on G.

For any set A we denote by χ_A the characteristic function of A. If A is a Borel subset of G such that $0<|A|<\infty$, then by e_A we denote the function

$$e_A=\frac{1}{|A|}\chi_A.$$

By a Borel measure μ on a group G we mean a finite countably additive non-negative measure on the class of Borel subsets of G. A measure is called *regular* if for any Borel set A

$$\mu(A) = \inf_{U \supset A} \mu(U),$$

where U are open sets. For a Borel measure μ in G we write $\mu^{\sim}$ for the measure defined by

$$\int f(s)\, d\mu(s) = \int f(s^{-1})\, d\mu^{\sim}(s)$$

for any $f \in C(G)$. A Borel measure is called *symmetric* if

$$\mu = \mu^{\sim}.$$

The convolution

$$\int f(t)\, g(t^{-1}s)\, dt$$

of two functions $f \in L_{p_1}(G)$, $g \in L_{p_2}(G)$, provided it exists, is denoted by $f*g$. Similarly, if μ is a Borel measure, we define the convolutions $\mu*f$ and $f*\mu$, $f \in L_p$, as

$$[\mu*f](s) = \int f(t^{-1}s)\, d\mu(t) \quad \text{and} \quad [f*\mu](s) = \int f(st^{-1})\, d\mu(t).$$

We have

$$\|\mu*f\|_p \leqslant \mu(G)\, \|f\|_p, \qquad 1 \leqslant p \leqslant \infty. \tag{1.1}$$

Let μ be a Borel measure in G. Then μ defines a bounded operator A_μ on the Hilbert space $L_2(G)$ by the formula

$$A_\mu \xi = \mu * \xi, \qquad \xi \in L_2(G).$$

We have $A_\mu^* \xi = \mu^{\sim} * \xi$, where by A^* we denote the operator conjugate to A. Given a group G, the group algebra of G is the space $L_1(G)$ with the multiplication $x*y$ and the involution

$$x^*(s) = \overline{x(s^{-1})}\, \varrho^{-1}(s).$$

For any element x of $L_p(G)$ we write $x^{\sim}$ for the function

$$x^{\sim}(s) = \overline{x(s^{-1})}.$$

By a *unitary representation* of a group G we mean a continuous homomorphism of G into the group of unitary operators of a Hilbert space. If

$$T\colon s \to T_s, \qquad s \in G,$$

is a unitary representation of G and ξ is a vector of the Hilbert space of the representation, then

$$\varphi(s) = (T_s \xi, \xi) \tag{1.2}$$

is a positive definite function on G. A positive definite function on the group G which is of the form (1.2), where ξ is a vector in the Hilbert space of the representation T is said to be *associated* with the representation T.

By a *representation of a Banach *-algebra* $\mathscr{A}$ we mean a homomorphism $x \to T_x$ of the algebra $\mathscr{A}$ into the algebra of all bounded operators of a Hilbert space $\mathscr{H}(T)$ such that $(T_x)^* = T_{x^*}$ for all $x \in \mathscr{A}$.

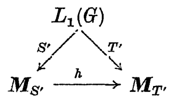
There is a one-to-one correspondence $T \to T'$ between the set of all unitary representations T of a group G and all representations T' of the group algebra $L_1(G)$ which are nowhere trivial in the sense that the linear union of the ranges of the T'_x, $x \in L_1(G)$ is dense in $\mathscr{H}(T')$. The correspondence between T and T' is defined by the requirement that

$$(T'_x \xi, \eta) = \int x(s)(T_s \xi, \eta)\, ds$$

for all $x \in L_1(G)$ and $\xi, \eta \in \mathscr{H}(T)$.

For any representation T' of $L_1(G)$ we denote by $\boldsymbol{M}_{T'}$, the least Banach subalgebra of the algebra of all bounded operators of $\mathscr{H}(T')$ which contains all the operators T'_x, $x \in L_1(G)$.

Following Fell [3], we say that a unitary representation T of a group G is *weakly contained* in a unitary representation S of G if there is a continuous homomorphism $h\colon \boldsymbol{M}_{S'} \to \boldsymbol{M}_{T'}$ of the Banach $*$-algebra $\boldsymbol{M}_{S'}$ onto the Banach $*$-algebra $\boldsymbol{M}_{T'}$, such that the diagram



is commutative.

As has been shown by Fell [3] this is equivalent to the requirement that any positive function associated with the representation T' is the limit of a net of linear combinations with positive coefficients of positive definite functions associated with S which is convergent uniformly on compact sets.

By a *left-regular representation* R of a group G we mean the representation of G into the bounded operators of $L_2(G)$ given by the formula

$$R\colon s \to {}_{s^{-1}}T.$$

The corresponding representation of $L_1(G)$ is given by

$$R'_x \xi = x * \xi, \qquad \xi \in L_2(G).$$

We say that a group G *belongs to the class* (R) if any unitary representation of G is weakly contained in the left-regular representation.

The connection of the notion of weak containment as defined above and the one which is suggested by the "containment" of a representation T in the decomposition of a representation S into the direct integral of representations can be seen by means of the following theorem due to Fell.

We say that a representation T is *weakly contained* in the set of representations $\{S\}$, if any positive definite function associated with T is the limit of a net of finite sums of positive definite functions associated with the representations belonging to the set $\{S\}$ which is convergent uniformly on compact sets.

THEOREM (Fell). *If a unitary representation T is a direct integral of unitary representations*

$$T = \int \oplus S_\varrho \sqrt{d\varrho},$$

then T is weakly contained in the set $\{S_\varrho\}$ and any of the representations S_ϱ is weakly contained in T.

2. Invariant Banach mean values. In this section we are concerned with the notion of an invariant Banach mean value defined on various classes of functions on a locally compact group. This notion has been invastigated by Dixmier [1], who has given some necessary and sufficient conditions for the existence of an invariant Banach mean value. Moreover, he proved under some restrictive assumptions the standard theorems on factor group, subgroup and extension known for the case of discrete groups. Here we recall one of his theorems and for another we get free of this restrictive assumption in another one making use of a lemma due to Mackey.

Let G be a locally compact group. Let Φ be a closed linear subspace of $L_\infty(G)$ with the property that

(a) if $\varphi \in \Phi$, then $\overline{\varphi} \in \Phi$,

(b) if $\varphi \in \Phi$, then $T_s\varphi \in \Phi$ and ${}_sT\varphi \in \Phi$.

Let Φ^* denote the subspace of real functions contained in Φ.

By a *left-invariant Banach mean value* on Φ we mean a linear functional m which has the following properties:

(i) $m\varphi = m_sT\varphi$ *for any* $s \in G$ *and* $\varphi \in \Phi$,

(ii) *if* $\varphi \in \Phi^*$, *then* $\operatorname{ess\,inf}_{s\in G} \varphi(s) \leqslant m\varphi \leqslant \operatorname{ess\,sup}_{s\in G} \varphi(s)$.

A Banach mean value is called *right-invariant* if instead of (i)

(i′) $m\varphi = mT_s\varphi$ *for any* $s \in G$ *and* $\varphi \in \Phi$

holds. We say that a Banach mean value is *invariant* if both (i) and (i′) are valid.

In [2] Dixmier gives the following condition necessary and sufficient for the existence of a left-invariant Banach mean value on Φ:

(D) *if* $\varphi_1, \ldots, \varphi_n \in \Phi^*$ *and* $a_1, \ldots, a_n \in G$, *then*

$$\operatorname*{ess\,inf}_{s\in G} \sum_{i=1}^{n} \big(\varphi_i(s) - \varphi_i(a_i s)\big) \leqslant 0 .$$

The following simple fact has been noticed by C. Ryll-Nardzewski:

LEMMA 2.1. *The existence of a left-invariant mean value on $C(G)$ is equivalent to the existence of a left-invariant Banach mean value on $L_\infty(G)$.*

In fact, let $U\varphi = \psi$ denote the operation from $L_\infty(G)$ into $C(G)$ defined by the formula

$$\psi(s) = \frac{1}{|A|}\int_A \varphi(st)\,dt, \qquad 0 < |A| < \infty .$$

If m_0 is a left-invariant Banach mean value on $C(G)$, then putting $m\varphi = m_0(U\varphi)$ we obtain a left-invariant Banach mean value on $L_\infty(G)$.

THEOREM 2.2. *The existence of a left-invariant Banach mean value on $C(G)$ implies the existence of an invariant Banach mean value on $C_0(G)$.*

Proof. Let $\varphi \in C_0(G)$. Then, by (1.1), the function

$$m(T_{t^{-1}}\varphi) = \psi(t),$$

where m is a left-invariant Banach mean value on $C(G)$, is continuous and bounded. We put

$$\overline{m}\varphi = m\psi .$$

It is easy to verify that $\overline{m}$ is an invariant Banach mean value on $C_0(G)$.

THEOREM 2.3 (Dixmer). *If G has a left-invariant Banach mean value on $L_\infty(G)$ and if H is a closed subgroup of G, then G/H has a left-invariant Banach mean value on $L_\infty(G/H)$.*

THEOREM 2.4. *If H is a closed subgroup of a separable group G and G has a left-invariant Banach mean value on $L_\infty(G)$, then H has a left-invariant Banach mean value on $L_\infty(H)$.*

Proof. The proof is a consequence of the following lemma due to Mackey [10]:

LEMMA. *If G is a separable group, H a closed subgroup of G, then there exists a Borel set B in G which intersects each right H-coset in exactly one point.*

By this Lemma and the theorem of Lusin and Susin (cf. e. g. [9], p. 398), the function σ which maps a point $s \in G$ onto the point $Hs \cap B$ is a Borel function. (In fact, σ is the superposition of the continuous function $h\colon G \to G/H$ and the function $(h \mid B)^{-1}$ which is a Borel function because it is one-to-one and is the inverse function of the continuous function h on the Borel set B). Therefore, by Theorem 2 of [9], p. 285, the function $t(\sigma(s))^{-1}$ is a Borel function of two variables (t, s) and, consequently, $s(\sigma(s))^{-1}$ is a Borel function. It maps G into H; for, by definition of σ, for any $s \in G$ we have $s = t\sigma(s)$ for some $t \in H$, so $s(\sigma(s))^{-1} = t$. Similarly, $\sigma(ts) = \sigma(s)$ for any $t \in H$, $s \in G$. Let $\varphi \in C(H)$. Then

$$\psi(s) = \varphi\big(s(\sigma(s))^{-1}\big)$$

is a bounded Borel function on G. We put

$$\overline{m}\varphi = m\psi.$$

To see that in fact m is a left-invariant Banach mean value on $C(H)$ we note that, if $a \in H$,

$${}_aT\varphi\big(s(\sigma(s))^{-1}\big) = \varphi\big(a(s\sigma(s))^{-1}\big) = \varphi\big(as(\sigma(as))^{-1}\big) = {}_aT\psi(s).$$

Hence, by Lemma 2.2, the theorem is proved.

THEOREM 2.5. *Let G be a group, H a normal subgroup of it. Suppose that H and G/H have left-invariant Banach mean values m_1 and m_2 on $C(H)$ and $C(G/H)$, respectively. Then G has a left-invariant Banach mean value on $C_0(G)$.*

Proof. If $\varphi \in C_0(G)$, then the function of G into $C_0(G)$ defined by

$$t \to {}_tT\varphi$$

s continuous. Moreover, if $\overline{m}_1$ is a continuous functional on $C_0(G)$ defined by $m_1\varphi = m_1(\chi_H\varphi)$, then $m_1({}_tT\varphi) = \psi(t)$ is a continuous bounded function on G which is constant on the cosets $Hs = sH$, $s \in G$. In fact,

$${}_aT\psi(t) = \psi(at) = m_{1at}T\varphi = m_{s1}\{\varphi(ats)\}, \qquad s \in H.$$

If $at = ta'$ for $a, a' \in H$, then

$${}_aT\psi(t) = m_{s1}\{\varphi(ta's)\} = m_{s1}\{\varphi(ts)\} = \psi(t).$$

Therefore ψ defines in the unique way a continuous bounded function $\hat{\psi}$ on G/H. We put

$$m\varphi = m_2\hat{\psi}.$$

It is the matter of simple computation to verify that, in fact, m is a left-invariant mean on $C_0(G)$.

PROBLEM. *Does the existence of a left-invariant Banach mean value on $C_0(G)$ imply the existence of a left-invariant Banach mean value on $C(G)$ (and hence on $L_\infty(G)$)?*

3. Generalized Folner's conditions. We now present two conditions which are generalizations of Følner's conditions (cf. [4]) formulated by Følner for discrete groups and, in this case, proved to be necessary

and sufficient for the existence of a left-invariant Banach mean value on a group. In the case of topological groups a straightforward generalization of Følner's conditions just by substituting the words "compact" for "finite" and "the Haar measure of a set" for "the number of the elements of a set" though is a consequences of our condition, does not seem to be satisfactory as far as the condition (F_2) is concerned. The author is unable to deduce from such generalization the existence of left-invariant Banach mean value on $L_\infty(G)$, the fact which will be proved to be a consequence of (F_2).

Let A be a finite set in a group G. Denote by μ_A the measure on G defined by

$$\mu_A(X) = \textit{the number of elements of } A \cap X.$$

Let $G_A^{(n)} = A \times \underbrace{G \times \ldots \times G}_{n\ \text{times}}$. The product measure in $G_A^{(n)}$

$$\mu_A \times |\cdot| \times \ldots \times |\cdot|$$

will be denoted by $\mu_A^{(n)}$.

(F_1) *For any compact set A in G and any $\varepsilon > 0$ there exists a Baire subset E with finite positive measure such that*

$$|E \cap aE| > (1-\varepsilon)|E|$$

holds for any $a \in A$.

(F_2) *There is a positive number k such that for any finite subset A in G and any Baire subset Q of $G_A^{(n)}$ with compact closure and positive $\mu_A^{(n)}$ measure there is a Baire set E with positive finite measure in G such that*

$$\frac{1}{\mu_A^{(n)}(Q)} \int_Q |E \cap t_1^{-1}\ldots t_n^{-1}at_n\ldots t_1E|\, d\mu_A^{(n)}(a, t_1, \ldots, t_n) \geqslant k|E|.$$

Remark. One sees that (F_1) implies (F_2). The converse implication, which is known to be true for discrete groups (cf. [4]) has not been proved for the general case as yet. It seems very likely that using Følner's ideas one can prove that if a locally compact group has a left invariant Banach mean value, then it satisfies (F_1). This would settled down the problem of the characterization of the class (R) in general.

4. Sufficiency of (F_1). We have

THEOREM 4.1. *If a group G satisfies* (F_1), *then* $G \in (R)$.

Proof. Let T be any representation of $L_1(G)$ and let R be the left-regular representation. Denote by M_R and M_T the rings of operators R_x, T_x, with $x \in L_1(G)$. Since R is one-to-one, there exists a homomorphism h

such that the diagram

$$(4.1)\qquad \begin{array}{ccc} & L_1(G) & \\ {}^{R}\swarrow & & \searrow{}^{T} \\ M_R & \xrightarrow{\ h\ } & M_T \end{array}$$

is commutative. If we prove that the fact that G satisfies (F_1) implies that h is continuous in the operator norm of M_R and M_T, then h can be extended to the continuous homomorphism of M_R onto M_T and the theorem follows. To prove this it is sufficient to show that

$$(4.2)\qquad \|R_x\| \geqslant \|T_x\|$$

for any $x \in L_1(G)$ of the form $x = y*y^*$. In fact, since $\|R_{y*y^*}\| = \|R_y\|^2$ and similarly $\|T_{y*y^*}\| = \|T_y\|^2$, we see that (4.2) will be then proved for any x in $L_1(G)$, and this shows that h is continuous.

Inequality (4.2) will be easily derived from the following

LEMMA. *If G satisfies* (F_1), *then any continuous positive definite function φ on G can be approximated uniformly on compact sets by functions $u*u^\sim$, where $u \in L$.*

Proof. Suppose $\varphi \equiv 1$. Let A be a compact set, ε a positive number. Then, by (F_1), there exists a Baire set E with finite positive measure such that

$$|E \cap aE| > (1-\varepsilon)|E|$$

holds for any $a \in A$. This means that if $f_E = |E|^{-1/2}\chi_E$, then

$$|1-f_E*f_E^\sim(a)| = \left|1-\frac{1}{|E|}\int \chi_E(t)\chi_E(a^{-1}t)\,dt\right|$$
$$= \left|1-\frac{1}{|E|}\int \chi_E(t)\chi_{aE}(t)\,dt\right| = \left|1-\frac{1}{|E|}|E \cap aE|\right| < \varepsilon$$

for any $a \in A$.

If φ is an arbitrary continuous positive definite function, then for the compact set A and $\varepsilon > 0$

$$(4.3)\qquad |\varphi(a)-\varphi\cdot f_E*f_E^\sim(a)| < \varepsilon\varphi(e) \quad \text{for all } a \in A.$$

By a theorem of Godement (cf. [6]) the product of two positive definite functions φ and $f_E*f_E^\sim$ is a positive definite function and, since φ is bounded and $f_E*f_E^\sim \in L_2(G)$, we have $\varphi\cdot f_E*f_E^\sim \in L_2(G)$. But any positive definite function which belongs to $L_2(G)$ is of the form $w*w^\sim$, where $w \in L_2(G)$ (cf. [6]). Since L is dense in $L_2(G)$, there exists a u in L such that

$$|w*w^\sim(s)-u*u^\sim(s)| < \varepsilon \quad \text{for all } s \in G,$$

whence, by (4.3), the lemma follows.

Now to obtain the result we could have used the theorem of Fell that was mentioned in section 1, but a simple and straight-forward reasoning is at hand.

We take a y in $L_1(G)$ and a positive number ε and we select a vector ξ in $\mathscr{H}(T)$ such that $(\xi, \xi) = 1$ and

$$\|T_{y*y^*}\| - \varepsilon \leqslant (T_{y*y^*}\xi, \xi).$$

The function $(T_s\xi, \xi) = \varphi(s)$ is a positive definite function on G, so by the lemma, we can find a function $u \in L$ such that

$$\left|\int \varphi(s)y*y^*(s)\,ds - \int u*u^\sim(s)y*y^*(s)\,ds\right| < \varepsilon$$

and

$$|u*u^\sim(e) - \varphi(e)| < \varepsilon.$$

We then have

$$\|T_{y*y^*}\| - \varepsilon \leqslant (T_{y*y^*}\xi, \xi) = \int (T_s\xi, \xi)\,y*y^*(s)\,ds$$
$$\leqslant \int u*u^\sim(s)\,y*y^*(s)\,ds + \varepsilon = \int y*u(s)\cdot\overline{y*u(s)}\,ds + \varepsilon$$
$$= \|y*u\|_2^2 + \varepsilon \leqslant \|R_y\|^2\|u\|_2^2 + \varepsilon = \|R_{y*y^*}\|\,\|u\|_2^2 + \varepsilon.$$

But $\|u\|_2^2 = u*u^\sim(e) \leqslant \varphi(e) + \varepsilon = (\xi, \xi) + \varepsilon = 1 + \varepsilon$, whence

$$\|T_{y*y^*}\| \leqslant \|R_{y*y^*}\|(1+\varepsilon) + 2\varepsilon,$$

whence, since ε is arbitrary positive number, (4.2) follows.

5. Auxiliary theorems. In this section we are going to consider first the following situation. Let G be a unimodular group, μ a symmetric measure on G. Then μ defines a bounded operator A_μ on $L_2(G)$ by the formula

$$A_\mu\xi = \mu*\xi, \qquad \xi \in L_2(G),$$

which, in virtue of the fact that μ is symmetric, is Hermitian. Hence putting $\lambda(\mu) = \|A_\mu\|$, we have

$$(5.1)\qquad \lambda(\mu) = \sup_\xi (A\xi, \xi)\|\xi\|_2^{-2},$$

where the supremum is taken over all $\xi \in L_2(G)$. Let R be the left-regular representation of G. The following Lemma holds:

LEMMA 5.1. *Suppose G has the property that for any non-negative function $x \in L_1(G)$*

$$\|R_x\| = \|x\|_1,$$

then

$$\lambda(\mu) = \mu(G).$$

In fact, by (1.1) we have $\lambda(\mu) \leqslant \mu(G)$. On the other hand, for any e_U we have

$$\lambda(\mu) = \sup_{\|\xi\|_2\leqslant 1, \|\eta\|_2\leqslant 1} (\mu*\xi, \eta) \geqslant \sup_{\|\xi\|_2\leqslant 1, \|\eta\|_2\leqslant 1} (\mu*e_U*\xi, \eta) = \lambda(\mu_0),$$

where $d\mu_0(s) = (\mu*e_U)(s)ds$. (The inequality is valid because $\|e_U*\xi\|_2 \leqslant \|e_U\|_1\|\xi\|_2 = \|\xi\|_2$.) But μ_0 is a measure absolutely continuous with respect to the Haar measure and, moreover,

$$\|\varkappa\|_1 = \mu_0(G) = \int\int e_U(st^{-1})d\mu(t)ds = \mu(G), \tag{5.2}$$

where $\varkappa$ is the Radon-Nikodym derivate of μ_0. Hence, by assumption,

$$\|\varkappa\|_1 = R_\varkappa = \lambda(\mu_0),$$

whence, by (5.2), $\lambda(\mu_0) = \mu(G)$, and the inequality $\lambda(\mu) \geqslant \mu(G)$ follows.

Now our aim is to find the evaluation of the expression on the right-hand side of (5.1) when ξ are restricted to characteristic functions of Baire sets with compact closure and positive measure. The required evaluation is given by the following

THEOREM 5.1. *Let μ be a symmetric regular measure on a group G such that $\mu(G) \leqslant 1$. If for any Baire set E with compact closure and positive Haar measure*

$$\frac{1}{|E|}(\mu*\chi_E, \chi_E) \leqslant k < 1, \tag{5.3}$$

then

$$\lambda(\mu) \leqslant 4k(1+2k^{-1})^{2/3}.$$

The proof of theorem 5.1 is based on a similar theorem formulated and proved by Kesten [8] for discrete groups, i. e. when the quadratic form

$$(\mu*\xi, \xi) = \Phi_\mu(\xi)$$

is an ordinary symmetric quadratic form with non-negative coefficients.

LEMMA 5.2 (Kesten). *If (b_{ij}) is a symmetric substochastic N by N matrix, that is $b_{ij} = b_{ji} \geqslant 0$,*

$$\sum_{i=1}^{N} b_{ij} \leqslant 1 \tag{5.4}$$

such that for any set $S \subset \{1, 2, \ldots, N\}$ of m indices $(1 \leqslant m \leqslant N)$

$$m^{-1}\sum_{i,j\in S} b_{ij} \leqslant k < 1, \tag{5.5}$$

then

$$\sup_{\sum_{i=1}^{n}|x_i|^2=1} \sum_{i,j=1}^{N} b_{ij}x_ix_j \leqslant 4k(1+2k^{-1})^{2/3}.$$

Roughly speaking, the idea of the proof is to approximate the kernel of the form $\Phi_\mu(\xi)$ by finite matrices which are to play the role of the matrix (b_{ij}). In a few simple lemmas, which follow, we present a procedure of approximating regular measures and Baire functions on a locally compact group by discrete measures and simple functions, respectively. The proofs are easy and use the standard technique but, since the lemmas are too special to be easily derived from text-books results of this kind, we sketch the proofs here.

Let G be a non-discrete locally compact group [2], X an open set with compact closure in G. For simplicity sake we assume that $|X|$ is a rational number.

LEMMA 5.3. *There exists a "rational basis" of open Baire sets in G, that is a basis $\mathbf{B}$ of open Baire sets such that $|U|$ is a rational number for any $U \in \mathbf{B}$.*

The proof of the lemma is a copy of the proof of the well-known fact that the Haar measure of a non-discrete group is convex, i. e. that if $A \subset C$ are two Borel sets and b is a real number such that $|A| \leqslant b \leqslant |C|$, then there exists a Borel set B with the property that $A \subset B \subset C$ and $|B| = b$. One sees that for an open set C the set B can be selected among open Baire sets.

Let ν' denote the product measure in $X\times X$ of the corresponding Haar measures and let μ' be another regular, symmetric Borel measure in $X\times X$ ("symmetric" means that $\int f(s,t)d\mu'(s,t) = \int f(t,s)d\mu'(s,t)$ for any continuous function f on $X\times X$).

LEMMA 5.4. *Let $A_1, \ldots, A_r$ be a finite family of Baire subsets of $X\times X$. For any $\delta > 0$ and any open neighbourhood U of the unity there exists a finite family $V_1, \ldots, V_n$ of Baire subsets of X such that*

(i) *$V_i \cap V_j = \emptyset$ if $i \neq j$,*

(ii) *$|V_i| = |V_j|$ for any $i, j = 1, 2, \ldots, n$,*

(iii) *$\bigcup_{i=1} V_i = X$,*

(iv) *for any $i = 1, 2, \ldots, n$ there is an element s of G such that $sU \supset V_i$,*

(v) *for any $k = 1, 2, \ldots, r$ there is a set E_k of the form*

$$E_k = \bigcup_{(i,j)\in S_k} V_i\times V_j \tag{5.7}$$

such that

$$\nu'(A_k \Delta E_k) < \delta, \quad \mu'(A_k\Delta E_k) < \delta. \tag{5.8}$$

(2) The assumption that G is not discrete is needed for the proofs only. The lemmas trivially remain true when G is discrete.

Proof. Let $\boldsymbol{B}$ be a rational basis in G. For any A_k, $k = 1, 2, \ldots, r$, we select an open set U_k such that $U_k \supset A_k$ and

$$\nu'(U_k \setminus A_k) < \frac{\delta}{2}, \qquad \mu'(U_k \setminus A_k) < \frac{\delta}{2}.$$

Let $U_k = \bigcup_s Q_s^{(k)}$, where

(5.9) $$Q_s^{(k)} = I_s^{(k)} \times J_s^{(k)} \quad \text{with} \quad I_s^{(k)}, J_s^{(k)} \in \boldsymbol{B}.$$

Let $E_k = \bigcup_{s=1}^{n_k} Q_s^{(k)}$ with

$$\nu'(U_k \setminus E_k) < \frac{\delta}{2}, \qquad \mu'(U_k \setminus E_k) < \frac{\delta}{2}.$$

Clearly

$$\nu'(A_k \Delta E_k) < \delta, \qquad \mu'(A_k \Delta E_k) < \delta.$$

Let

(5.10) $$I_1^{(1)}, J_1^{(1)}, \ldots, I_{n_r}^{(r)}, J_{n_r}^{(r)}$$

be all the sets that appear in (5.9) for $k = 1, 2, \ldots, r$. The Haar measure of any of the sets (5.10) is a rational number, consequently, there exists a family $V_1, \ldots, V_n$ of Baire subsets of X which satisfy (i)-(iv) and such that any of the sets (5.10) is the union of some of the V_i's. Consequently E_k is of the form (5.7) and the lemma follows.

LEMMA 5.5. *Let $f_1, \ldots, f_m$ be a family of continuous functions on $G \times G$ vanishing outside $X \times X$. Then for any $\eta > 0$ there exists a family of Baire sets $V_1, \ldots, V_n$ which satisfy* (i)-(iv) *and such that*

(5.11) $$\Big| \int_{X \times X} f_k(s,t)\, d\nu'(s,t) - \sum_{i,j=1}^{n} f_k(s_i, s_j) \nu'(V_i \times V_j) \Big| < \eta,$$

(5.12) $$\Big| \int_{X \times X} f_k(s,t)\, d\mu'(s,t) - \sum_{i,j=1}^{n} f_k(s_i, s_j) \mu'(V_i \times V_j) \Big| < \eta$$

for any selection $s_i \in V_i$, $i = 1, 2, \ldots, n$.

Proof. Let

$$\max_{s \in X,\, k=1,\ldots,r} |f_k(s)| \leqslant M.$$

We pick up an ε such that

$$0 < \varepsilon < \eta \big(2 \max(\mu'(X), \nu'(X)) + 1\big)^{-1}$$

and let U be an open neighbourhood of the unity of G such that

(5.13) $$|f_k(s,t) - f_k(s',t')| < \varepsilon \text{ if only } s^{-1}s', t^{-1}t' \in U.$$

We divide the interval $\langle -M, M \rangle$ into disjoint intervals $\{I\}$ each of the length $\leqslant \varepsilon$ and we consider the family

$$A_1^1, \ldots, A_{u_1}^1, \ldots, A_1^r, \ldots, A_{u_r}^r$$

of the counter-images of the intervals $\{I\}$ by the functions $f_1, \ldots, f_r$, respectively. By means of Lemma 5.4 for

$$0 < \delta < \varepsilon \big(M \max(u_1, \ldots, u_r)\big)^{-1}$$

and the neighbourhood U we select a family $V_1, \ldots, V_n$ which satisfies (i)-(v). We have

(5.14) $$\int_{X \times X} f_k(s,t)\, d\mu'(s,t) = \sum_{m=1}^{u_k} \int_{A_m^k} f_k(s,t)\, d\mu'(s,t).$$

But

(5.15) $$\Big| \int_{A_m^k} f_k(s,t)\, d\mu'(s,t) - \int_{E_m^k} f_k(s,t)\, d\mu'(s,t) \Big| < \delta M,$$

where E_m^k is of the form (5.7) and satisfies both inequalities (3.8) (with $A_k = A_m^k$). But since for any $V_i \times V_j$ that appear in (3.7) we may suppose $V_i \times V_j \cap A_m^k \neq 0$ and since (iv) is satisfied, we see that (5.13) and the definition of A_m^k imply

$$|f_k(s,t) - f_k(s',t')| < 2\varepsilon$$

if only (s,t) and (s',t') belong to E_m^k. Consequently,

$$\Big| \int_{E_m^k} f_k(s,t)\, d\mu'(s,t) - \sum_{i,j \in S_m^k} f(s_i, s_j) \mu'(V_i \times V_j) \Big| < 2\varepsilon \mu'(E_m^k)$$

for any $s_i \in V_i$, $i = 1, 2, \ldots, n$. Hence, by (5.15) and (5.14),

$$\Big| \int_{X \times X} f_k(s,t)\, d\mu'(s,t) - \sum_{i,j=1}^{n} f_k(s_i, s_j) \mu'(V_i \times V_j) \Big|$$
$$< 2\varepsilon \mu'(X) + \delta M u_k < \varepsilon\big(2\mu'(X) + \varepsilon\big).$$

The proof of (5.12) is analogous.

Proof of Theorem 5.1. For the measure μ and an $\varepsilon > 0$ we take a continuous real function ξ vanishing outside a compact set such that $\|\xi\|_2 \leqslant 1$ and

$$\lambda(\mu) - (\mu * \xi, \xi) < \varepsilon$$

(the existence of such a function follows immediately from the fact that μ is symmetric and the family of continuous functions vanishing outside compact sets is dense in $L_2(G)$). Let X be an open set with compact closure that contains the support of ξ. On $X\times X$ we define a measure μ' be the formula

$$\int f(s,t)\,d\mu'(s,t) = \int f(s^{-1}t,t)\,d\mu(s)\,dt,$$

where f is any bounded continuous function on $X\times X$. Clearly μ' is symmetric. Consider two continuous functions $f_1(s,t) = \xi(s)\xi(t)$ and $f_2(s,t) = \xi^2(s)$ defined on $X\times X$. For these two functions, the measures μ' and $\nu' = |\cdot|\times|\cdot|$ and $\eta = \varepsilon$ we apply Lemma 5.5. Let $V_1,\ldots,V_n$ be the family of Baire subsets of X the existence of which the lemma asserts.

We put

$$b_{ij} = \mu'(V_i\times V_j)(|V_i|\,|V_j|)^{-1/2}, \tag{5.16}$$

$$\xi_i = \xi(s_i)[(1+\varepsilon)|V_i|]^{-1/2}, \quad \text{where} \quad s_i\in V_i. \tag{5.17}$$

It is easy to verify that the matrix (b_{ij}) satisfies conditions of Lemma 5.2. Only (5.4) and (5.5) need proofs. We have

$$\sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n}\mu'(V_i\times V_j)|V_j|^{-1} = |V_j|^{-1}\int \chi_{X\times V_j}(s,t)\,d\mu'(s,t)$$

$$= |V_j|^{-1}\int \chi_{X\times V_j}(s^{-1}t,t)\,d\mu(s)\,dt$$

$$= |V_j|^{-1}\int \chi_X(s^{-1}t)\,d\mu(s)\,\chi_{V_j}(t)\,dt = \mu(X)\leqslant 1.$$

To verify (5.5) we select a set $S\subset\{1,2,\ldots,n\}$ of m indices and write

$$m^{-1}\sum_{i,j\in S} b_{ij} = m^{-1}\sum_{i,j\in S}\mu'(V_i\times V_j)(|V_i|\,|V_j|)^{-1/2}. \tag{5.18}$$

If $E = \bigcup_{i\in S} V_i$, then, by (5.18),

$$m^{-1}\sum_{i,j\in S} b_{ij} = \frac{1}{|E|}\mu'(E\times E) = \frac{1}{|E|}\int \chi_E(s^{-1}t)\,\chi_E(t)\,d\mu(s)\,dt < k.$$

Further, we note that

$$\sum_{i=1}^{n}\xi_i^2 \leqslant 1. \tag{5.19}$$

In fact,

$$\sum_{i=1}^{n}\xi_i^2 = \frac{1}{1+\varepsilon}\sum_{i=1}^{n}\xi_i^2(s_i)|V_i|$$

$$= \frac{1}{1+\varepsilon}\frac{1}{|X|}\sum_{i,j=1}^{n}\xi_i^2(s_i)|V_i|\,|V_j| \leqslant \frac{1}{1+\varepsilon}\left(\frac{1}{|X|}\int_{X\times X}\xi_i^2(s)\,d\nu'(s,t)+\varepsilon\right)\leqslant 1.$$

We have

$$(\mu*\xi,\xi) = \int \xi(s)\xi(t)\,d\mu'(s,t) \leqslant \sum_{i,j=1}^{n}\xi(s_i)\xi(s_j)\,d\mu'(V_i\times V_j)+\varepsilon$$

$$\leqslant \sum_{i,j=1}^{n}\xi(s_i)\big((1+\varepsilon)|V_i|\big)^{-1/2}\xi(s_j)\leqslant\big((1+\varepsilon)|V_j|\big)^{-1/2}\mu'(V_i\times V_j)(|V_i|\,|V_j|)^{1/2}+\varepsilon$$

$$= \sum_{i,j=1}^{n}\xi_i\,\xi_j\,b_{ij}+\varepsilon.$$

Hence, by Lemma 5.2 and (5.19) we obtain

$$(\mu*\xi,\xi)\leqslant 4k(1+2k^{-1})^{2/3}+\varepsilon,$$

which, in virtue of the fact that ε is an arbitrary positive number, completes the proof of theorem 5.1.

THEOREM 5.2. *If G is a unimodular group such that for any symmetric regular Borel measure μ we have*

$$\mu(G) = \lambda(\mu), \tag{5.20}$$

then G satisfies (F_2).

Proof. Let A be a finite subset of G and μ_A the measure defined by

$$\mu_A(X) = \text{the number of elements of } X\cap A.$$

Let $G_A^{(n)} = A\times\underbrace{G\times\ldots\times G}_{n\text{ times}}$, $\mu_A^{(n)}$ be the product measure

$$\mu_A\times|\cdot|\times\ldots\times|\cdot|$$

and Q a Baire subset with compact closure and positive $\mu_A^{(n)}$ measure in $G_A^{(n)}$. We have

$$I = \int_Q |E\cap t_n^{-1}\ldots t_1^{-1}at_1\ldots t_nE|\,d\mu_A^{(n)}(a,t_1,\ldots,t_n)$$

$$= \int_Q\int_G \chi_E(u)\chi_{t_n^{-1}\ldots t_1^{-1}at_1\ldots t_nE}(u)\,du\,d\mu_A^{(n)}(a,t_1,\ldots,t_n)$$

$$= \int_G d\mu_A(a)\int_{S_a(Q)} d(t_1,\ldots,t_n)\int_G \chi_E(u)\chi_E(t_n^{-1}\ldots t_1^{-1}a^{-1}t_1\ldots t_nu)\,du,$$

where $S_a(Q)$ is the intersection of Q and the cylinder $\{a\}\times G\times\ldots\times G$. Let

$$\chi_{S_{a(Q)}}(t_1,\ldots,t_n) = f(a,t_1,\ldots,t_n).$$

Then

$$(5.21)\qquad I = \int\limits_G d\mu_A(a) \int\limits_{G\times\ldots\times G} d(t_1,\ldots,t_n) \int\limits_G f(a,t_1,\ldots,t_n)\cdot$$
$$\cdot\chi_E(u)\chi_E(t_n^{-1}\ldots t_1^{-1}a^{-1}t_1\ldots t_n u)\,du$$
$$= \int\limits_G \int\limits_G \chi_E(a^{-1}u)\chi_E(u)\,du \int\limits_{G\times\ldots\times G} f(t_1,\ldots,t_n a t_n^{-1}\ldots t_1^{-1}, t_1,\ldots,t_n)\cdot$$
$$\cdot d(t_1,\ldots,t_n)\,d\mu_A(a).$$

Let

$$(5.22)\qquad \nu(X) = \frac{1}{\mu_A^{(n)}(Q)} \int\limits_X \int\limits_{G\times\ldots\times G} f(t_1\ldots t_n a t_n^{-1}\ldots t_1^{-1}, t_1,\ldots,t_n)\cdot$$
$$\cdot d(t_1,\ldots,t_n)\,d\mu_A(a),$$

where X is a Baire subset in G. Clearly μ is a discrete measure with the support A in G. Moreover,

$$\mu_A^{(n)}(Q)\nu(G) = \int\limits_G \int\limits_{G\times\ldots\times G} f(t_1\ldots t_n a t_n^{-1}\ldots t_1^{-1}, t_1,\ldots,t_n)\,d(t_1,\ldots,t_n)\,d\mu_A(a)$$
$$= \int\limits_{G\times\ldots\times G}\int\limits_G f(a,t_1,\ldots,t_n)\,d(t_1,\ldots,t_n)\,d\mu_A(a)$$
$$= \int\limits_G \int\limits_{G\times\ldots\times G} \chi_{S_a(Q)}(t_1,\ldots,t_n)\,d(t_1,\ldots,t_n)\,d\mu_A(a) = \mu_A^{(n)}(Q).$$

Hence $\nu(G) = 1$.

Now suppose that for any $k>0$ there exists an n and a Baire subset Q of $G_A^{(n)}$ with compact closure and positive $\mu_A^{(n)}$ measure such that for any Baire subset E of G with compact closure and positive measure we have

$$\frac{1}{\mu_A^{(n)}(Q)} \int\limits_Q |E \cap t_n^{-1}\ldots t_1^{-1}at_1\ldots t_n E|\,d\mu^{(n)}(a,t_1,\ldots,t_n) < k|E|.$$

Let k be a positive number such that

$$4k(1+2k^{-1})^{2/3} < 1.$$

Then, by (5.21) and (5.22),

$$\frac{1}{|E|}(\nu*\chi_E, \chi_E) < k$$

holds for any Baire subset E of G with compact closure and positive measure. Consequently

$$(\nu*\chi_E,\chi_E) = (\chi_E, \nu*\chi_E) = (\nu*\chi_E, \chi_E) < |E|k,$$

which shows that the inequality

$$\frac{1}{|E|}\left(\frac{\nu+\nu^\sim}{2}*\chi_E, \chi_E\right) < k$$

is valid for any Baire set E with compact closure and positive measure. Thus, by Theorem 5.1,

$$\lambda\left(\frac{\nu+\nu^\sim}{2}\right) < 1,$$

which contradicts (5.20) because

$$\frac{\nu+\nu^\sim}{2}(G) = 1.$$

THEOREM 5.3. *If G is unimodular and satisfies* (F_2), *then there exists a left-invariant Banach mean value on* $L_\infty(G)$.

Proof. We verify that G satisfies condition (D). For a finite set $A \subset G$ and a real function $\varphi_a \in L_\infty$, $a\in A$, we write

$$\psi(s) = \sum_{a\in A}\varphi_a(s) - \varphi_a(as).$$

Suppose

$$\operatorname*{ess\,inf}_{s\in G}\psi(s) \geqslant \delta.$$

By means of (F_2) we are going to define two sequences

$$E_1, E_2, \ldots$$

of Baire subsets with compact closure and positive measure of G and

$$Q_0, Q_1, \ldots,$$

where Q_n is a Baire subset with compact closure of $G_A^{(n)}$ and measure $\mu^{(n)}(Q_n) > 0$.

We put $Q_0 = A$.

Suppose that for an $n \geqslant 0$ the sets $E_1,\ldots,E_n, Q_0,\ldots,Q_n$ are already defined. Then by means of (F_2) we find a set E_{n+1} such that

$$\frac{1}{\mu_A^{(n)}(Q_n)} \int\limits_{Q_n} |E_{n+1} \setminus t_n^{-1}\ldots t_1^{-1}at_1\ldots t_n E_{n+1}|\,d\mu_A^{(n)}(a,t_1,\ldots,t_n) \leqslant (1-k)|E_{n+1}|.$$

We put

$$Q_{n+1} = \{(a,t_1,\ldots,t_{n+1}) : (a,t_1,\ldots,t_n)\in Q_n$$

and

$$t_{n+1}\in E_{n+1}\setminus t_n^{-1}\ldots t_1^{-1}at_1\ldots t_n E_{n+1}\}.$$

We see that

$$\mu_A^{(n+1)}(Q_{n+1}) = \int_{Q_n} |E_n \setminus t_n^{-1}\dots t_1^{-1}at_1\dots t_nE_n| d\mu_A^{(n)}(t_n, \dots, t_1, a) \leqslant (1-k)|E_{n+1}|\mu_A^{(n)}(Q_n).$$

Hence

(5.23) $\mu_A^{(n)}(Q_n) \leqslant (1-k)|E_n|\mu_A^{(n-1)}(Q_{n-1})$ for any $n = 1, 2, \dots$

Now we define a sequence of functions

$$\psi_0(s) = \psi(s), \quad \dots, \quad \psi_n(s) = \frac{1}{|E_n|}\int_{E_n} \psi_{n-1}(ts)\,dt, \quad \dots$$

It is clear that

(5.24) $$\operatorname*{ess\,inf}_{s\in G} \psi_n(s) \geqslant \delta.$$

On the other hand, we shall prove that

(5.25) $$\|\psi_n\|_\infty \leqslant \frac{\mu_A^{(n)}(Q_n)}{|E_1|\dots|E_n|}\cdot 2K, \quad \text{where} \quad K = \max_{a\in A}\|\varphi_a\|_\infty.$$

Hence, by (5.23),

$$\|\psi_n\|_\infty \leqslant 2K(1-k)^n \quad \text{for any} \quad n = 1, 2, \dots$$

Hence, since $0 < k \leqslant 1$,

$$\lim_{n=\infty} \|\psi_n\|_\infty = 0,$$

which, by (5.24), shows that $\delta \leqslant 0$, as required.

Thus all that remains is to prove (5.25). This is a consequence of the following equation:

(5.26) $$\psi_n(s) = \frac{1}{|E_1|\dots|E_n|}\int_{Q_n} [\varphi_a(t_1\dots t_ns) - \varphi_a(at_1\dots t_ns)]d\mu_A^{(n)}(a, t_1, \dots, t_n).$$

The proof of (5.26) is by induction on n. For $n = 0$ it is obvious. Suppose it is true for an $n \geqslant 0$. Then

$$\psi_{n+1}(s) = \frac{1}{|E_1|\dots|E_n||E_{n+1}|}\int_{E_{n+1}}\int_{Q_n} [\varphi_a(t_1\dots t_nt_{n+1}s) - \varphi_a(at_1\dots t_nt_{n+1}s)]\,d\mu_A^{(n)}(a, t_1, \dots, t_n)\,dt_{n+1}$$

$$= \frac{1}{|E_1|\dots|E_{n+1}|}\int_{Q_n}\int_{E_{n+1}} [\varphi_a(t_1\dots t_nt_{n+1}s) - \varphi_a(at_1\dots t_nt_{n+1}s)]\,dt_{n+1}\,d\mu_A^{(n)}(a, t_1, \dots, t_n).$$

For a fixed $(a, t_1, \dots, t_n)\in Q_n$ we see that if

$$t_{n+1}\in E_{n+1} \cap t_n^{-1}\dots t_1^{-1}at_1\dots t_nE_{n+1},$$

then $\varphi_a(t_1\dots t_nt_{n+1}s)$ cancels with $\varphi_a(at_1\dots t_nt'_{n+1})$ for a $t'_{n+1}\in E_{n+1}$. Therefore

$$\psi_{n+1}(s) = \frac{1}{|E_1|\dots|E_{n+1}|}\int_{Q_n}\int_{E_{n+1}\setminus t_n^{-1}\dots t_1^{-1}at_1\dots t_nE_{n+1}} [\varphi_a(t_1\dots t_{n+1}s) - \varphi_a(at_1\dots t_{n+1}s)]\,d\mu_A^{(n)}(a, t_1, \dots, t_n)\,dt_{n+1},$$

which, by the definition of Q_{n+1}, completes the proof of (5.26) and, at the same time, the proof of Theorem 5.3.

THEOREM 5.4. *If a group G belongs to the class (R), then for any non-negative function $x\in L_1(G)$ we have*

(5.27) $$\|x\|_1 = \|R_x\|,$$

where R denotes the left-regular representation.

Proof. It is clear that if $G\in(R)$, then for any representation T of $L_1(G)$ we have

$$\|R_x\| \geqslant \|T_x\| \quad \text{for any} \quad x\in L_1(G).$$

Thus, in particular, if T is the identity representation

$$T: x \to \int x(s)\,ds,$$

we obtain

$$\|R_x\| \geqslant \|x\|_1 \text{ for any non-negative function } x\in L_1(G).$$

But the converse inequality

$$\|x\|_1 \geqslant \|R_x\|$$

is always true, whence (5.27) follows.

6. Conclusion. Now all the pieces are before us; all that remains is to put them together to obtain

MAIN THEOREM. *If a group G belongs to the class (R), then there exists a left-invariant Banach mean value on $L_\infty(G)$.*

Proof. First we show that it is sufficient to prove the theorem for unimodular groups. In fact, the following lemma is due to Takenouchi (cf. 13, Lemma 3.5):

LEMMA. *Suppose a group H contains two subgroups K and L such that*

$$H = KL \quad \text{and} \quad K \cap L = \{e\}$$

and L is normal in G. Then, if $K, L\in(R)$, then $H\in(R)$.

Suppose the theorem is proved for unimodular groups. Let G be an arbitrary group. Then, as has been noticed by Gleason (cf. [7]), there exists a unimodular group $H = GR$, where R is the group of real numbers, is normal in H and $G \cap R = \{e\}$. Hence, by the lemma, $H \epsilon (R)$. Thus, since H is unimodular, there exists a left-invariant Banach mean value on H and consequently, there is a left-invariant Banach mean value on $L_\infty(G)$, since G is a homomorphic image of H (cf. Theorem 2.3).

Now suppose G is unimodular and let $G \epsilon (R)$. Then, by Theorem 5.4, $\|x_1\| = \|R_x\|$ for any non-negative function $x \epsilon L_1(G)$. Hence, by Lemma 5.1, $\lambda(\mu) = \mu(G)$ for any regular Borel measure μ in G. Hence, by Theorem 5.2, G satisfies (F_2), which in virtue of Theorem 5.3 implies the existence of a left-invariant Banach mean value on $L_\infty(G)$.

Corollary. *If G is a discrete group, then $G \epsilon (R)$ if, and only if, there is a left-invariant Banach mean value on G.*

In fact, if G is discrete, (F_1) is the ordinary Følner condition and, as such, is a consequence of the existence of a left-invariant mean on G. By theorem 4.1, (F_1) implies that $G \epsilon (R)$, the converse implication being a particular case of the main theorem.

For a generalization of the corollary to the case of locally compact groups see the remark in section 3.

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