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STUDIA MATHEMATICA, T. XXIV. (1964)

On formal trigonometrical series

by

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Using derivatives of infinite order we have extended in [5] the methods of the elementary theory of distributions due to J. Mikusiński and R. Sikorski (see [3] and [4]). The elements so obtained, which we call *ultra-distributions*, may be regarded as Fourier transforms of distributions of finite order.

This note deals with Fourier series expansions of ultra-distributions. For simplicity we restrict ourselves to the case of one variable. We show that each periodic ultra-distribution has a Fourier series converging to that ultra-distribution. The Fourier coefficients are defined by the classical formulas, which we interprete similarly as for periodic distributions (see [3], § 20). Moreover, each formal trigonometrical series, i. e. a series with no restrictions on the coefficients, converges in the sense of ultra-distributions and is the Fourier series of its sum. We also prove an analogue of the last result for generalized trigonometrical series.

Throughout the paper we use the notation and basic properties of ultra-distributions given in [5]. We recall briefly the definition of the convergence. Let $\varphi_t(x)$ be an ultra-distribution of x for each value of the parameter t and $\varphi(x)$ another ultra-distribution. We say that

$$\lim_{t\to t_0}\varphi_t(x) = \varphi(x),$$

if there exists an entire function $A(x)$, an integer ν, and continuous functions $\Phi_t(x)$, $\Phi(x)$, which are $O(|x|^{2\nu})$ as $|x|\to\infty$ and satisfy the following conditions:

(L_1) $\quad A\left(\frac{1}{i}D\right)\Phi_t(x) = \varphi_t(x), \qquad A\left(\frac{1}{i}D\right)\Phi(x) = \varphi(x),$

(L_2) For $t\to t_0$, $(1+x^2)^{-\nu}\Phi_t(x)$ converges to $(1+x^2)^{-\nu}\Phi(x)$ uniformly in R_1.

A series of ultra-distributions $\sum_{n=1}^{\infty}\varphi_n(x)$ *converges* to $\varphi(x)$ if the sequence of partial sums $\sigma_n(x) = \sum_{k=1}^{n} \varphi_k(x)$ converges to $\varphi(x)$. We say that $\varphi(x)$ is the *sum* of the series and we write

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x).$$

For sequences and series of numbers (constant functions) the above convergence coincides with the ordinary convergence.

Let now $\varphi(x)$ be an arbitrary ultra-distribution. Since $\varphi(x)$ is an infinite derivative of a slowly increasing continuous function, one can easily prove that there exists a primitive of $\varphi(x)$, i. e. an ultra-distribution $\psi(x)$ such that

$$D\psi(x) = \varphi(x).$$

Any two primitives of $\varphi(x)$ differ by a constant function. We adopt the following notation (introduced in [3] for distributions):

$$\int_a^b \varphi(x+t)\,dt = \psi(x+b) - \psi(x+a), \tag{1}$$

the expression on the right-hand side being independent of the choice of the primitive $\psi(x)$. If, in particular, the integral (1) is a continuous function, its value at $x = 0$ will be denoted by

$$\int_a^b \varphi(t)\,dt.$$

We say that the ultra-distribution $\varphi(x)$ is *periodic* with period 2π, i

$$\varphi(x+2\pi) = \varphi(x).$$

In what follows "periodic" will always mean "periodic with period 2π".

If $\varphi(x)$ is periodic, then the integral (1) with $a = -\pi$ and $b = \pi$ is a constant function, because its derivative is zero. Thus we may write

$$\int_{-\pi}^{\pi} \varphi(t)\,dt = \int_{-\pi}^{\pi} \varphi(x+t)\,dt. \tag{2}$$

Moreover, any convergent series of periodic ultra-distributions may be integrated term by term:

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \varphi_n(t)\,dt = \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \varphi_n(t)\,dt. \tag{3}$$

This is a consequence of equation (2) and the fact that, for every convergent series, the integration (1) may be carried out term by term.

THEOREM 1. *Every periodic ultra-distribution $\varphi(x)$ is the sum of its Fourier series*

$$\varphi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \tag{4}$$

where

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) e^{-int}\,dt, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{5}$$

If $A(x)$ is an entire function without real zeros and $\Phi(x)$ a continuous function such that

$$\varphi(x) = A\left(\frac{1}{i}D\right)\Phi(x), \tag{6}$$

then

$$\frac{a_n}{A(n)} \to 0 \quad as \quad |n| \to \infty. \tag{7}$$

Conversely, every formal trigonometrical series

$$\sum_{n=-\infty}^{\infty} a_n e^{inx} \tag{8}$$

converges to a periodic ultra-distribution $\varphi(x)$ and the coefficients a_n are determined by formula (5), *i. e.* (8) *is the Fourier series of $\varphi(x)$.*

Proof. A representation of the form (6) exists for every ultra-distribution. If $\varphi(x)$ is periodic, then the continuous function $\Phi(x)$ is also periodic, since

$$A\left(\frac{1}{i}D\right)[\Phi(x+2\pi) - \Phi(x)] = \varphi(x+2\pi) - \varphi(x) = 0$$

and $A(x) \neq 0$ implies

$$\Phi(x+2\pi) - \Phi(x) = 0.$$

Let us denote by b_n, $n = 0, \pm 1, \pm 2, \ldots$, the Fourier coefficients of $\Phi(x)$. Then

$$\Phi^*(x) = \sum_{n=-\infty}^{\infty} \frac{b_n}{1+n^2} e^{inx} \tag{9}$$

is a continuous function, the series on the right being uniformly convergent. Furthermore, if

$$A^*(x) = (1+x^2)A(x),$$

then

$$\varphi(x) = A\left(\frac{1}{i}D\right)\Phi(x) = A^*\left(\frac{1}{i}D\right)\Phi^*(x),$$

and taking into account the expansion (9) we infer that

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \frac{b_n}{1+n^2} A^*(n) e^{inx} = \sum_{n=-\infty}^{\infty} b_n A(n) e^{inx}. \tag{10}$$

On the other hand, an expansion of the form (4), if it exists, is unique; the coefficients are necessarily those defined by (5). In order to prove this we multiply both sides of (4) by e^{-imx} and integrate from $-\pi$ to π. In view of (3), the series may be integrated term by term. Thus we obtain

$$\int_{-\pi}^{\pi} \varphi(t) e^{-imt} dt = \sum_{n=-\infty}^{\infty} a_n \int_{-\pi}^{\pi} e^{i(n-m)t} dt = 2\pi a_m,$$

and so formula (5).

An application of the last argument to (10) leads to the desired equality (4). We also get condition (7), because

$$a_n = b_n A(n), \quad n = 0, \pm 1, \pm 2, \ldots,$$

and $b_n \to 0$ as $|n| \to \infty$.

Finally, let (8) be any formal trigonometrical series. Then there exists an entire function $A(x)$ such that

$$|a_n| < \frac{|A(n)|}{1+n^2}.$$

Hence it follows that the series

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{A(n)} e^{inx}$$

converges uniformly, say, to $\Phi(x)$. Consequently

$$\varphi(x) = A\left(\frac{1}{i}D\right)\Phi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

and, by what we have said before, a_n are the Fourier coefficients of $\varphi(x)$. The theorem is now established.

The Fourier coefficients (5) depend continuously on the ultra-distribution $\varphi(x)$. Thus we can draw from theorem 1 the following

COROLLARY. *The space of periodic ultra-distributions is isomorphic to the space of all sequences of complex numbers, provided with the normal topology* (see [2], p. 410).

Let now λ_n, $n = 0, \pm 1, \pm 2, \ldots$, be real numbers such that $\lambda_n \to \pm\infty$ as $n \to \pm\infty$.

THEOREM 2. *Every formal generalized trigonometrical series*

$$\sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n x} \tag{11}$$

converges to an ultra-distribution $\varphi(x)$, which is an infinite derivative of a uniformly almost periodic function. The coefficients are determined by the formula

$$a_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x+t) e^{-i\lambda_n(x+t)} dt, \tag{12}$$

$n = 0, \pm 1, \pm 2, \ldots$

Proof. The convergence of the series (11) can be proved in the same way as for λ_n integral. There exists an entire function $A(x)$ such that

$$|a_n| < \frac{|A(\lambda_n)|}{1+n^2} \tag{13}$$

and therefore the series

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{A(\lambda_n)} e^{i\lambda_n x}$$

converges uniformly in R_1; its sum $\Phi(x)$ is a uniformly almost periodic function (see e. g. [1], § 6). Hence, applying term by term the infinite derivative $A\left(\frac{1}{i}\right)D$, we obtain

$$\varphi(x) = A\left(\frac{1}{i}D\right)\Phi(x) = \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n x}, \tag{14}$$

which proves the first part of the theorem.

From (14) it follows that

$$\frac{1}{2T} \int_{-T}^{T} \varphi(x+t) e^{-i\lambda_m(x+t)} dt = a_m + r_m(x, T),$$

where

$$r_m(x, T) = \sum_{n\neq m} \frac{a_n \sin(\lambda_n T - \lambda_m T)}{\lambda_n T - \lambda_m T} e^{i(\lambda_n - \lambda_m)x}.$$

To complete the proof we have to show that

$$\lim_{T\to\infty} r_m(x, T) = 0. \tag{15}$$

For this purpose we set

$$R_m(x,T) = \sum_{n\neq m} \frac{a_n \sin(\lambda_n T - \lambda_m T)}{A(\lambda_n)(\lambda_n T - \lambda_m T)} e^{i(\lambda_n - \lambda_m)x}.$$

By virtue of (13), the last series converges uniformly in R_1. Therefore $R_m(x,T)$, as a function of x, is continuous and bounded. Also

$$A\left(\frac{1}{i}D_x\right) R_m(x,T) = r_m(x,T),$$

and, for $T\to\infty$, $R_m(x,T)$ converges uniformly to zero. This means that condition (15) is satisfied.

Remark. If $P_n(x)$, $n = 0, \pm 1, \pm 2, \dots$, are polynomials of degrees less than a given integer and λ_n are defined as before, then the series

$$\sum_{n=-\infty}^{\infty} P_n(x) e^{i\lambda_n x}$$

converges to an ultra-distribution $\varphi(x)$. Furthermore, there exists an entire function $F(x)$ such that

$$F\left(\frac{1}{i}D\right)\varphi(x) = 0. \tag{16}$$

Conversely, in the space of ultra-distributions each solution of equation (16) has the form

$$\varphi(x) = \sum_{n=-\infty}^{\infty} P_n(x) e^{i\lambda_n x}$$

where the exponential polynomials $P_n(x)e^{i\lambda_n x}$ satisfy the equation.

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On infinite derivatives of continuous functions

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Each distribution of L. Schwartz is locally a derivative of some order of a continuous function. The word "locally" is superfluous, if solely distributions of finite order are considered. This property is of great importance for the theory itself as well as for its applications. In particular, it is the starting point for some of the simplified approaches to the theory of distributions.

On the other hand, analytic functionals defined by L. Ehrenpreis [2], [3], and I. M. Gelfand-G. E. Šilov [4] in connection with Fourier transforms of rapidly increasing functions are not, in general, derivatives of continuous functions defined for real values of the arguments. The space D' of those functionals contains all tempered distributions and, in particular, all continuous slowly increasing functions. Furthermore, an operator of the form

$$A\left(\frac{1}{2\pi i}D\right) = \sum_k \frac{a_k}{(2\pi i)^{\sigma_k}} D^k, \tag{1}$$

where $k = (k_1, k_2, \dots, k_q)$, D^k is the partial differential operator of order $\sigma_k = k_1 + k_2 + \dots + k_q$, and $A(x) = \sum_k a_k x^k = \sum_{k_1,k_2,\dots,k_q=0}^{\infty} a_k x_1^{k_1} x_2^{k_2} \dots x_q^{k_q}$ is an entire function, carries each element of D' into an element of D'.

In this paper we are concerned with the subspace $D'_F \subset D'$ consisting of Fourier transforms of distributions of finite order. D'_F is closed with respect to the "infinite derivation" (1) and contains all tempered distributions. We prove that each element of D'_F is an infinite derivative of a continuous slowly increasing function (of real variables). The theorem can be extended to sequences converging in D'_F, and therefore enables us to extend in a natural way the methods used by J. Mikusiński and R. Sikorski in [7], [8], so as to obtain a sequential description of the elements of D'_F. We also characterize Fourier transforms of infinitely differentiable functions as "rapidly decreasing" elements of D'_F.