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On simultaneous extension of continuous functions

A generalization of theorems of Rudin-Carleson and Bishop

by

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Let us denote by $C_B(T)$ the space of all continuous mappings $f: T \to B$ from a topological space T into a linear topological space B. For $B = \mathscr{K}$, where $\mathscr{K}$ denotes the complex plane, we shall write $C(T)$ instead of $C_{\mathscr{K}}(T)$. Let S be a closed subset of T and let X and E be linear subspaces of $C_B(T)$ and $C_B(S)$ respectively and let X and E be linear topological spaces with certain topologies τ_X and τ_E respectively.

Definition 1. An operator $L: E \to X$ is said to be a *linear operator of extension* provided that the following conditions are satisfied:

(i) Lf is an extension of f, i. e. $f(s) = Lf(s)$ for every s in S and for every f in E,

(ii) $L(c_1 f_1 + c_2 f_2) = c_1 Lf_1 + c_2 Lf_2$ for $f_k \in E$ and $c_k \in \mathscr{K}$ $(k = 1, 2)$,

(iii) L is continuous with respect to the topologies τ_E and τ_X.

The first result concerning linear operators of extension is due to Borsuk [6], who considered the case where $X = C(T)$, T being a separable metric space. The result of Borsuk has been generalized by Dugundji [11] (see also Kakutani [18] and Michael [20]) to the case of the space $C_B(T)$, where B is a locally convex linear topological space. All these authors have considered only the case where $E = C_B(S)$ and $X = C_B(T)$. In the present paper we give certain sufficient conditions for S and a subspace X of $C(T)$ for the existence of linear operators of extension from $E = C(S)$ into X. Our conditions are closely related to some recent results of Bishop [1] and Glicksberg [14]. Obviously, in the general case, where X is a (proper) subspace of $C_B(T)$, the necessary condition for the existence of linear operators of extension from $C_B(S)$ into X is that $r_S X = C_B(S)$, where $r_S: C_B(T) \to C_B(S)$ is the operator of restriction of functions to S, i. e. $r_S F = f$ with $f(s) = F(s)$ for $s \in S$ and $F \in C_B(T)$. We shall show that this condition is also sufficient in the case where X is a Dirichlet algebra [1] on a compact metric space T. In this case there

[1] A subalgebra X of $C(T)$ is called a *Dirichlet algebra* provided that every real continuous function on T is uniformly approximable by real parts of functions in X (see [13] and [17], p. 54).

exists a linear operator of extension with the norm precisely one. The first non-trivial example is the algebra A of all continuous complex-valued functions on the unit circle of $\mathscr{K}$ which are the boundary values of holomorphic functions on the unit disc. As is shown by Rudin [24] and Carleson [7], $r_S A = C(S)$ if and only if S is a closed subset of the unit circle with Lebesgue measure 0. Therefore for every such subset S there is a linear operator of extension from $C(S)$ into A with the norm equal to 1, i. e. preserving the norm of every extended function. This fact enables us to show that the space A is isometrically universal for separable Banach spaces, i. e. every separable Banach space is isometrically isomorphic to a subspace of A. A slightly weaker result is given in [21]. Another example of an isometrically universal Banach space is the space $C(T)$ for any uncountable compact metric space T (cf. [8], p. 93, and [21]). However, we shall show that the space A is of a different isomorphic type from any space $C(T)$ for compact T.

The present paper consists of three sections. In Section 1 we formulate the main result and give some corollaries. Section 2 contains some unsolved problems and remarks. Section 3 is devoted to the proof of the main result. All the results of this paper remain valid if we replace everywhere the space of complex-valued functions by the space of real-valued functions.

1. Results

Unless otherwise stated, by T we shall denote an arbitrary compact metric space and by S a closed subset of T. The metric function on T will be denoted by $\varrho(\cdot,\cdot)$. Elements of $C(T)$ and $C(S)$ will be denoted by $F, G, H, \ldots$ and $f, g, h, \ldots$ respectively. If T is compact, then $C(T)$ and $C(S)$ are Banach spaces under the norms $\|F\| = \sup_{t\in T}|F(t)|$ and $\|f\| = \sup_{s\in S}|f(s)|$ respectively. In this case if X is a closed subspace of $C(T)$ and E is a closed subspace of $C(S)$ and $L: E\to X$ is a linear operator of extension, then

$$1 \leqslant \|L\| = \sup_{f\in E, \|f\|=1} \|Lf\| < +\infty.$$

A function K in $C(T)$ is said to be a *peak function* for a subset S of T provided that $|K(t)| < 1$ for $t\in T\setminus S$ and $K(s) = 1$ for $s\in S$.

MAIN THEOREM. *Let X be a closed subspace of the space $C(T)$ of all continuous complex-valued functions on a compact metric space T and let S be a closed subset of T. Let us suppose that*

(B) *for every f in $C(S)$ and for every Δ in $C(T)$ such that $\Delta(t) > 0$ for $t\in T$ and $\Delta(s) > |f(s)|$ for $s\in S$ there is in X an extension F of f with $|F(t)| < \Delta(t)$ for $t\in T$.*

Then for every $\delta > 0$ there exists a linear operator of extension L_δ: $C(S)\to X$ with $\|L_\delta\| < 1+\delta$.

Moreover, if

(G) *there exists in X a peak function K for S such that $K\cdot F\in X$ for every F in X* (²),

then there is a linear operator of extension $L: C(S)\to X$ with $\|L\| = 1$.

Assumption (B) may be justified by a recent result of Bishop [1], namely that if S is such a subset of T that

(R) every Baire measure ν on T orthogonal to X (i. e. such that $\int_T F(t)\nu(dt) = 0$ for $F\in X$) is identically zero on S (i. e. $\nu(S_1) = 0$ for every Baire subset S_1 of S),

then condition (B) is satisfied.

A subspace X of $C(T)$ is called an *algebra approximating in modulus* (briefly: an *a. m. algebra*) provided that for every $\varepsilon > 0$ and for every $\Delta\in C(T)$ with $\Delta(t) \geqslant 0$ for $t\in T$ there exists an f in X such that $\big||f(t)| - \Delta(t)\big| < \varepsilon$ for $t\in T$. The class of a. m. algebras has been introduced by Glicksberg ([14], p. 430); it contains the class of Dirichlet algebras as a proper subclass. Glicksberg [14] has shown that if X is an a. m. algebra on a compact metric space T and if S is a closed subset of T, then conditions (B), (G), (R) are equivalent to one another and equivalent to the fact that $r_S X = C(S)$. Thus in view of the Main Theorem we get

COROLLARY 1. *Let T be a compact metric space, let S be a closed subset of T and let X be an a. m. algebra on T (in particular let X be a Dirichlet algebra). Then a linear operator of extension $L: C(S)\to X$ with $\|L\| = 1$ exists if and only if one of the equivalent conditions* (B), (G), (R), *or $r_S X = C(S)$ holds.*

As is pointed in [1], for special Dirichlet algebras condition (R) is closely related to various generalizations of the classical theorem of F. and M. Riesz ([17], p. 47). Therefore, by Corollary 1, in view of the results of [5], [16], [10], [2], in which various general theorems of F. and M. Riesz have been proved, we obtain

COROLLARY 2. *In each of the following special cases there exists a linear operator of extension $L: C(S)\to X$ with $\|L\| = 1$:*

I. *$T = \{z\in\mathscr{K}: |z| = 1\}$, S is a closed subset of T of Lebesgue measure zero and $X = A$.*

II. *T is the tore $\{z\in\mathscr{K}: |z| = 1\}\times\{w\in\mathscr{K}: |w| = 1\}$, S is a closed subset of T with the surface area equal to zero and X is a closed subspace of $C(T)$ spanned on the functions $z^m w^n$ with (m, n) belonging to a sector of lattice points of opening greater than π.*

(²) Assumption (G) is superflous; see the remark at the end of this paper.

III. *$T = G$ is a metric compact abelian group, S its closed subset such that for each $s \in S$, $\{t \in R\colon s+\varphi(t) \in S\}$ has zero Lebesgue measure, where $\varphi\colon R \to G$ is a homeomorphism from the group R of real numbers into G induced* [3] *by a continuous homeomorphism $\psi\colon \hat{G} \to R$ ($\hat{G}$ denotes the associated dual group to G) and X consists of all continuous functions in $C(T)$ Fourier transforms of which vanish on the set $\{\sigma \in \hat{G}\colon \psi(\sigma) < 0\}$.*

IV. *T is the boundary of a simple connected open set U, S is a closed subset of T of ν_0-measure zero, where ν_0 is the measure induced by the Lebesgue measure on the unit circle* (see [2], Theorem 4) *and X consists of continuous functions on T which have continuous extensions to $T \cup U$ analytic on U.*

One may obtain further generalizations for the spaces of continuous functions on metric compact Abelian groups, on Riemann surfaces with boundary and on subsets of the complex plane more complicated than those in IV (using the results of [10], [27], [23] and [3]).

The condition $r_S = C(S)$ is obviously satisfied (by the Tietze extension theorem) in the case where $X = C(T)$ for on arbitrary closed subset S of T. Then Corollary 1 gives for compact metric spaces the result of Borsuk [6] mentioned above.

The next corollaries need the following simple proposition:

Proposition 1. *Suppose that T is a compact Hausdorff space and that S is a closed subset of T. If there exists a linear operator of extension $L\colon C(S) \to X \subset C(T)$ (with $\|L\| = 1$), then*

(a) *L is an (isometrical) isomorphism from $C(S)$ into X,*

(b) *the space X is the direct sum of its subspaces $Y_S = L(C(S))$ and $Z_S = \{F \in X : F(s) = 0 \text{ for } s \in S\}$,*

(c) *there are projections (= linear idempotent operators) from X onto Y_S and onto Z_S, i. e. Y_S and Z_S are complemented in X.*

Proof. (a) is an immediate consequence of the inequality

$$\|f\| = \sup_{s \in S} |f(s)| \leqslant \sup_{t \in T} |Lf(t)| = \|Lf\| \leqslant \|L\|\,\|f\| \quad \text{for} \quad f \in C(S).$$

(b) follows from the formula

$$F = Lr_S F + (F - Lr_S F) \quad \text{for} \quad F \text{ in } X.$$

(c) is a well-known consequence of (b) (see [12], p. 480).

Corollary 3. *Let X and T have the same meaning as in cases I-IV of Corollary 1 (in III suppose also that T is infinite). Then X contains a complemented subspace isometrically isomorphic to the space $C(\mathfrak{C})$ of all continuous complex-valued functions on the Cantor discontinuum $\mathfrak{C}$. Hence* (by [8], p. 93) *every separable Banach space is isometrically isomorphic to a subspace of X* (cf. [21]).

[3] i. e. $\varphi\colon R \to G$ is the unique mapping satisfying $\sigma(\varphi(t)) = e^{i\psi(\sigma)t}$ $(t \in R, \sigma \in \hat{G})$.

For the proof it is sufficient to note that under the assumptions of Corollary 3 there exists a closed subset S of T homeomorphic to $\mathfrak{C}$ and such that S, T and X satisfy the assumptions of Corollary 2 in each of cases I-IV and to apply Proposition 1 (a).

Corollary 4. *For an arbitrary closed subset S of the unit circle there exists a projection from the space A onto its subspace $A_S = \{F \in A : F(s) = 0$ for $s \in S\}$.*

Proof. If S is of positive Lebesgue measure, then according to the theorem on unicity (see [17], p. 52) A_S consists only of the zero function. In the other case we apply Corollary 2 (case I) and Proposition 1 (c).

It seems interesting to compare Corollaries 2 and 3 with the following

Proposition 2. *Let Q be an arbitrary compact Hausdorff space. Then the space A is not isomorphic to any complemented subspace of the space $C(Q)$.*

Proof. We shall use the following notation. By X/Y we shall denote the quotient space of the space X by its subspace Y. We shall write $X \sim X_1$ provided that the spaces X and X_1 are isomorphic. If X is a B-space, then X^* and X^{**} denote the first and the second conjugate spaces to X respectively. The symbols H^1, H^∞, A, L^1, L^∞ have the usual meaning (for the definitions cf. [17]).

We recall that a B-space X is said to have property P provided that for each of its isomorphic images X_1 in every B-space Z there is a projection from Z onto X_1 ([8], p. 94). It is well known that α) if X has property P and Y is a complemented subspace of X, then Y has property P [15], β) the second conjugate space to any space $C(Q)$ has property P ([15], [8], p. 95-106), γ) the space H^∞ does not have property P, because the natural embedding of H^∞ into L^∞ is not complemented in L^∞ ([24]; see also [17], p. 155). It follows from α) that if Y is a complemented subspace of X and X^{**} has property P, then Y^{**} also has this property. Hence to complete the proof it is sufficient to show that

δ) the space A^{**} does not have property P.

Let us put

$$A^\perp = \Big\{\nu \in [C(T)]^* : \int_T F(t)\,\nu(dt) = 0 \text{ for every } F \text{ in } A\Big\},$$

where $T = \{z : |z| = 1\}$. Then the theorem of F. and M. Riesz ([17], p. 47) implies that $A^\perp$ is isometrically isomorphic to the space

$$H_0^1 = \Big\{g \in L^1 : \int_0^{2\pi} g(t) e^{int}\,dt = 0 \text{ for } n = 0, 1, \ldots\Big\}.$$

We have

$$A^* \sim [C(T)]^*/A^\perp.$$

It follows immediately from [12], p. 132, that $[C(T)]^* \sim L^1 \times V_{\text{sing}}$, where V_{sing} denotes the space of all set functions singular (with respect to the Lebesgue measure) complex-valued countable additive defined on the class of all Baire subsets of T. Thus we have

$$A^* \sim (L^1 \times V_{\text{sing}})/H_0^1 \sim (L^1/H_0^1) \times V_{\text{sing}}.$$

Therefore

$$A^{**} \sim (L^1/H_0^1)^* \times (V_{\text{sing}})^* \sim H^\infty \times (V_{\text{sing}})^*,$$

because $(L^1/H_0^1)^* \sim H^\infty$ ([17], p. 137). Thus, by α) and γ), A^{**} does not have property P.

Corollary 5. *The space A is not isomorphic to any space $C(Q)$.*

2. Remarks and unsolved problems

1° It seems interesting to extend the Main Theorem to arbitrary metric spaces and to vector valued functions.

2° Recent results of Glicksberg [14] suggest the following problem. Let X be an arbitrary Dirichlet algebra on a compact metric space T and let S be such a subset of T that the restriction operator $r_S: X \underset{\text{into}}{\longrightarrow} C(S)$ has a closed range. Is it then true that there exists a linear operator of extension (with the norm equal to one) from $r_S X$ into X?

3° There are a few examples which show that the Borsuk-Dugundji theorem cannot be extended to arbitrary topological compact Hausdorff spaces (see [9], [19] and [20]). A very simple example given in [26] is the space βN with its closed subset $\beta N \setminus N$, where βN denotes the Čech-Stone compactification of a countable discrete set N. Then there is no linear operator of extension from the space $C(\beta N \setminus N)$ into $C(\beta N)$. Indeed, if it were not so, then, according to Proposition 1 (c), there would exist a projection from $C(\beta N) \sim m$ onto its subspace $Y = \{F \in C(\beta N): F(t) = 0 \text{ for } t \in \beta N \setminus N\} \sim c_0$, which would contradict a result of Philips (see [22]).

Consequently, the Main Theorem is not true if the assumption of metrisability is omitted.

3. Proof of the Main Theorem

3.1. Peak partitions of unit. We give

Definition 2. By a *peak partition of unit* in $C(S)$ we mean a finite collection $\lambda = (\lambda_i)_{i=1}^{N_\lambda}$ of functions in $C(S)$ satisfying the following conditions:

(1) $$\sum_{i=1}^{N_\lambda} \lambda_i(s) = 1 \quad \text{for} \quad s \text{ in } S,$$

(2) $$0 \leqslant \lambda_i(s) \leqslant 1 \quad \text{for } s \text{ in } S \ (i = 1, 2, \ldots, N_\lambda),$$

(3) the sets $V_\lambda^i = \{s \in S: \lambda_i(s) = 1\}$ are non-empty $(i = 1, 2, \ldots, N_\lambda)$.

Let us put

$$U_\lambda^i = \{s \in S: \lambda_i(s) > 0\} \quad (i = 1, 2, \ldots, N_\lambda),$$

$$d(\lambda) = \max_{1 \leqslant i \leqslant N_\lambda} \operatorname{diam} U_\lambda^i, \text{ where } \operatorname{diam} U_\lambda^i = \sup_{(s_1 s_2) \in U_\lambda^i \times U_\lambda^i} \varrho(s_1, s_2),$$

and let E_λ be a finite-dimensional subspace of $C(S)$ spanned by $\lambda_1, \lambda_2, \ldots, \lambda_{N_\lambda}$. If $p = (p_i)_{i=1}^{N_\lambda}$ is a finite system of points with $p_i \in V_\lambda^i$ $(i = 1, 2, \ldots, N_\lambda)$, then we define $\pi_\lambda^p: C(S) \to E_\lambda$, where

$$\pi_\lambda^p f = \sum_{i=1}^{N_\lambda} f(p_i) \lambda_i$$

for f in $C(S)$. It is easily seen that π_λ^p is a linear projection of the norm 1.

Lemma 1. *Let E be a finite-dimensional subspace of $C(S)$. Then for every $\varepsilon > 0$ there is $\eta = \eta(E, \varepsilon) > 0$ such that if μ is a peak partition of unit with $d(\mu) < \eta$, then $\|\pi_\mu^q f - f\| \leqslant \varepsilon \|f\|$ for every $f \in E$ and for every $q = (q_j)_{j=1}^{N_\mu}$ with $q_j \in V_\mu^j$ $(j = 1, 2, \ldots, N_\mu)$.*

Proof. Let us put $B_E = \{f \in E: \|f\| = 1\}$. Since E is of a finite dimension, B_E is compact and there exists an $\eta = \eta(E, \varepsilon) > 0$ such that if $\varrho(s_1, s_2) < \eta$, then $|f(s_1) - f(s_2)| < \varepsilon$ for f in B_E and for every s_1 and s_2 in S. Let μ be any peak partition of unit with $d(\mu) < \eta$ and let $q = (q_j)_{j=1}^{N_\mu}$ with $q_j \in V_\mu^j$ $(j = 1, 2, \ldots, N_\mu)$. By (1) and (2) we have

$$|f(s) - \pi_\mu^q f(s)| = \Big|\sum_{j=1}^{N_\mu} \big(f(s) - f(q_j)\big) \mu_j(s)\Big| \leqslant \sum_{j \in N_\mu(s)} |f(s) - f(q_j)| \mu_j(s),$$

where $N_\mu(s) = \{j: \mu_j(s) \neq 0\}$, $s \in S$. Let us observe that if $j \in N_\mu(s)$ then $s \in U_\mu^j$ and $\varrho(s, q_j) < \eta$ (because $d(\mu) < \eta$); therefore $|f(s) - f(q_j)| < \varepsilon$. Hence using (1) we get

$$|f(s) - \pi_\mu^q f(s)| \leqslant \sum_{j \in N_\mu(s)} |f(s) - f(q_j)| \mu_j(s) < \varepsilon \sum_{j=1}^{N_\mu} \mu_j(s) = \varepsilon \quad (s \in S).$$

Thus if $f \in B_E$, then $\|f - \pi_\mu^q f\| < \varepsilon$. Finally, by the homogeneity of the norm $\|\cdot\|$, we get $\|f - \pi_\mu^q f\| \leqslant \varepsilon \|f\|$ for $f \in E$, q. e. d.

Definition 3. Let λ and μ be peak partitions of unit and let $\varepsilon > 0$. We say that μ is *ε-subordinated* to λ provided that there is a finite system of points $q = (q_j)_{j=1}^{N_\mu}$ with $q_j \in V_\mu^j$ $(j = 1, 2, \ldots, N_\mu)$ such that

(4) there are indices $j_1, j_2, \ldots, j_{N_\lambda}$ with $q_{j_i} \in V_\lambda^i$ $(i = 1, 2, \ldots, N_\lambda)$,

(5) $$\|\pi_\mu^q f - f\| \leqslant \varepsilon \|f\| \quad \text{for every } f \text{ in } E_\lambda.$$

An operator $\pi_\mu^q : C(S) \to E_\mu$ satisfying (4) and (5) is called a *projection ε-subordinating μ to λ.*

LEMMA 2. *For every peak partition of unit λ in C(S) and for every positive number ε and δ there is a peak partition of unit μ which is ε-subordinated to λ and $d(\mu) < \delta$.*

Proof. Choose arbitrary points $p_i \in V_\lambda^i$ $(i = 1, 2, \ldots, N_\lambda)$ and put $\eta = \min\big(\delta, \frac{1}{2}\delta_1, \frac{1}{2}\eta(E_\lambda, \varepsilon)\big)$, where $\eta(E_\lambda, \varepsilon)$ has the same meaning as in Lemma 1 and $\delta_1 = \min\limits_{1 \leqslant i < k \leqslant N_\lambda} \varrho(p_i, p_k)$. Now let $q = (q_j)_{j=1}^{N_\mu}$ be a maximal system of points in S such that $\varrho(q_j, q_k) \geqslant \eta$ $(j \neq k)$ and for every i there is j_i with $p_i = q_{j_i}$ $(i = 1, 2, \ldots, N_\lambda)$. Since S is compact, the set q is of course finite. Let us set

$$(6) \qquad \varphi_j(s) = \begin{cases} 0 & \text{for} \quad \varrho(s, q_j) \geqslant \eta, \\ 2 - 2\dfrac{\varrho(s, q_j)}{\eta} & \text{for} \quad \dfrac{\eta}{2} \leqslant \varrho(s, q_j) < \eta, \\ 1 & \text{for} \quad \varrho(s, q_j) \leqslant \dfrac{\eta}{2}, \end{cases}$$

$$(7) \qquad \mu_j = \frac{\varphi_j}{\sum\limits_{\nu=1}^{N_\mu} \varphi_\nu} \qquad (j = 1, 2, \ldots, N_\mu).$$

Since q is a maximal set of points such that $\varrho(q_j, q_k) \geqslant \eta$ $(j \neq k)$, for every s in S there is an index $j(s)$ such that $\varrho(s, q_{j(s)}) < \eta$. Thus $\varphi_{j(s)}(s) \neq 0$ and $\sum\limits_{j=1}^{N_\mu} \varphi_j(s) > 0$ for every s in S. Hence formula (7) well-defines a continuous function on S (because φ_j are continuous). Obviously $\sum\limits_{j=1}^{N_\mu} \mu_j(s) = 1$ and $0 \leqslant \mu_j(s) \leqslant 1$ for $j = 1, 2, \ldots, N_\mu$ and for $s \in S$. It follows from (6) and (7) that

$$U_\mu^j = \{s \in S : \mu_j(s) \neq 0\} = \{s \in S : \varphi_j(s) \neq 0\} = \{s \in S : \varrho(s, q_j) < \eta\}.$$

Since $\varrho(q_j, q_k) \geqslant \eta$ $(j \neq k)$, we have $0 = \varphi_j(q_k) = \mu_j(q_k)$ for $j \neq k$ and therefore $\mu_j(q_j) = 1$ $(j = 1, 2, \ldots, N_\mu)$. Hence $\mu = (\mu_j)_{j=1}^{N_\mu}$ is a peak partition of unit with $d(\mu) = \min\limits_{1 \leqslant j \leqslant N_\mu} \operatorname{diam} U_\mu^j \leqslant 2\eta$. By the definition, $2\eta \leqslant \eta(E_\lambda, \varepsilon)$. Hence, by Lemma 1, $\|\pi_\mu^q f - f\| \leqslant \varepsilon \|f\|$ for f in E_λ. Since p_i are in q $(i = 1, 2, \ldots, N_\lambda)$, condition (4) is satisfied. Hence μ is ε-subordinated to λ, q. e. d.

LEMMA 3. *Let λ be a peak partition of unit in C(S) and let $L_\lambda : E_\lambda \to C(T)$ be an arbitrary linear operator of extension. Then*

$$(8) \qquad \|f\| = \max_{1 \leqslant i \leqslant N_\lambda} |c_i| \quad \textit{for } f = \sum_{i=1}^{N_\lambda} c_i \lambda_i \in E_\lambda \quad (c_i \in \mathscr{K};\ i = 1, 2, \ldots, N_\lambda),$$

$$(9) \qquad \|L_\lambda\| = \sup_{f \in E_\lambda; \|f\| = 1} \|L_\lambda f\| = \sup_{t \in T} \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)|.$$

Proof. Let $f = \sum\limits_{i=1}^{N_\lambda} c_i \lambda_i$. Then, by (1)-(3), we get

$$\max_{1 \leqslant i \leqslant N_\lambda} |c_i| = \max_{1 \leqslant i \leqslant N_\lambda} \sup_{s \in V_\lambda^i} \Big| \sum_{i=1}^{N_\lambda} c_i \lambda_i(s) \Big| \leqslant \|f\| = \sup_{s \in S} \Big| \sum_{i=1}^{N_\lambda} c_i \lambda_i(s) \Big| \leqslant \max_{1 \leqslant i \leqslant N_\lambda} |c_i|.$$

Thus (8) holds. Using (8) and conditions (i)-(iii), we have

$$\|L_\lambda f\| = \sup_{t \in T} \Big| \sum_{i=1}^{N_\lambda} c_i L_\lambda \lambda_i(t) \Big| \leqslant \max_{1 \leqslant i \leqslant N_\lambda} |c_i| \cdot \sup_{t \in T} \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)|$$
$$\leqslant \|f\| \sup_{t \in T} \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)|.$$

Therefore

$$\|L_\lambda\| \leqslant \sup_{t \in T} \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)|.$$

Now choose t_0 in T in such a way that

$$\sup_{t \in T} \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)| = \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t_0)|$$

and put

$$f_0 = \sum_{i=1}^{N_\lambda} c_i^0 \lambda_i, \quad \text{where} \quad c_i^0 = \begin{cases} 0 & \text{for} \quad L_\lambda \lambda_i(t_0) = 0, \\ L_\lambda \lambda_i(t_0) \cdot |L_\lambda \lambda_i(t_0)|^{-1} & \text{for} \quad L_\lambda \lambda_i(t_0) \neq 0. \end{cases}$$

Obviously $\|f_0\| = 1$ (because, by (i) and (3), $\sup\limits_{t \in T} \sum\limits_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)| \geqslant 1$). Thus

$$\|L_\lambda\| \geqslant \|L_\lambda f_0\| = \sup_{t \in T} \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)|.$$

Thus (9) holds, q. e. d.

3.2. Constructions of auxiliary extensions. We now prove

LEMMA 4. *Under assumption* (B), *for every g_1 and g_2 in $C(S)$ and for every extension $G_1 \in X$ of g_1 and for $\varepsilon > 0$ there is in X an extension G_2 of g_2 such that $\|G_1 - G_2\| \leqslant \|g_1 - g_2\| + \varepsilon$.*

Proof. Let us put $f = g_1 - g_2$. By (B), there exists in X an extension F of f with $\|F\| \leqslant \|f\| + \varepsilon$. We put $G_2 = G_1 - F$, q. e. d.

LEMMA 5. *Under assumption* (B), *for every non-negative function h in $C(S)$, for every $\eta > 0$ and $\delta_0 > 0$ there is a δ_1 with $0 < \delta_1 < \delta_0$ such that for every δ with $0 < \delta < \delta_1$ there is in X an extension H of h with the following properties:*

(10) $$\|H\| \leqslant \|h\| + \eta,$$

(11) $$|H(t)| - H(t) < \eta \quad \text{for} \quad t \in T,$$

(12) *if* $\varrho(t, S) \geqslant \delta$, *then* $|H(t)| < \eta$ *for* $t \in T$, *where* $\varrho(t, S) = \inf_{s \in S} \varrho(s, t)$,

(13) *if* $\varrho(t, s) < \delta$, *then* $h(s) - \operatorname{Re} H(t) > -\eta$ *for* $t \in T$ *and for* $s \in S$.

Proof. The case where $\|h\| = 0$ is trivial. Let us suppose in the sequel that $\|h\| > 0$. Let $H_0 \in X$ be an extension of h with $\|H_0\| < \|h\| + \frac{1}{4}\eta$ (such an extension exists by (B)). The uniform continuity of H_0 implies the existence of $\delta_2 > 0$ such that if $\varrho(t_1, t_2) < \delta_2$ then $|H_0(t_1) - H_0(t_2)| < \frac{1}{2}\eta$ for $(t_1, t_2) \in T \times T$. Let us put $\delta_1 = \min(\delta_0, \delta_2)$. Then for fixed δ with $0 < \delta < \delta_1$ we have

(14) if $\varrho(t, s) < \delta$, then $h(s) - \operatorname{Re} H_0(t) > -\frac{1}{2}\eta$ for $t \in T$ and for $s \in S$

(because $|h(s) - \operatorname{Re} H_0(t)| = |H_0(s) - \operatorname{Re} H_0(t)| = |\operatorname{Re} H_0(s) - \operatorname{Re} H_0(t)| \leqslant |H_0(s) - H_0(t)| < \frac{1}{2}\eta$ for $\varrho(t, s) < \delta_1$).

Choose an index n_0 such that

(15) $$(1+\delta)^{-n_0} < \min(\tfrac{1}{2}\eta \|H_0\|^{-1}, 1).$$

Let $H_n \in X$ be such extensions of h that

(16) $$|H_n(t)| < \Delta_n(t) = \big(1 + \varrho(t, S)\big)^{-n_0 - n}\big(|H_0(t)| + \tfrac{1}{4}\eta\big) \quad \text{for} \quad t \in T \quad (n = 1, 2, \ldots).$$

The existence of such H_n follows from (B).

Let us consider the sequence $(F_n) = (|H_n| - H_n)$. Since $|F_n(t)| < 2\Delta_n(t)$, $t \in T$, we have $\lim_n F_n(t) = 0$ for $t \in T \setminus S$. Since $h(s) \geqslant 0$ $(s \in S)$ and H_n are extensions of h, $F_n(s) = |H_n(s)| - H_n(s) = h(s) - h(s) = 0$ $(n = 1, 2, \ldots; s \in S)$. Thus $\lim_n F_n(t) = 0$ for every t in T. Moreover, by (16), $\|F_n\| \leqslant 2\|H_n\| \leqslant 2\|H_0\| + \frac{1}{2}\eta$. Hence, by [12], p. 265, the sequence (F_n) weakly converges to 0 in $C(T)$. Therefore, by a theorem of Mazur (cf. [12], p. 422), there exist non-negative numbers $(a_\nu)_{\nu=1}^{M}$ such that

$$\sum_{\nu=1}^{M} a_\nu = 1 \quad \text{and} \quad \Big\|\sum_{\nu=1}^{M} a_\nu F_\nu\Big\| < \tfrac{1}{4}\eta.$$

Let us set

(17) $$H = \sum_{\nu=1}^{M} a_\nu H_\nu.$$

Then we have

(18) $$H(s) = \sum_{\nu=1}^{M} a_\nu H_\nu(s) = \sum_{\nu=1}^{M} a_\nu h(s) = h(s) \quad \text{for} \quad s \text{ in } S,$$

(19) $$\|H\| \leqslant \sum_{\nu=1}^{M} a_\nu \|H_\nu\| < \Big(\sum_{\nu=1}^{M} a_\nu\Big)(\|H_0\| + \tfrac{1}{4}\eta) \leqslant \|h\| + \tfrac{1}{2}\eta,$$

(20) $$\big||H(t)| - H(t)\big| = \Big|\Big|\sum_{\nu=1}^{M} a_\nu H_\nu(t)\Big| - \sum_{\nu=1}^{M} a_\nu H_\nu(t)\Big|$$
$$\leqslant \Big|\sum_{\nu=1}^{M} a_\nu |H_\nu(t)| - \sum_{\nu=1}^{M} a_\nu H_\nu(t)\Big| = \Big|\sum_{\nu=1}^{M} a_\nu F_\nu(t)\Big| \leqslant \Big\|\sum_{\nu=1}^{M} a_\nu F_\nu\Big\| \leqslant \tfrac{1}{2}\eta < \eta \quad \text{for} \quad t \text{ in } T,$$

(21) if $\varrho(t, S) \geqslant \delta$ $(t \in T)$, then (by (15) and (16)) either

$$1 < \tfrac{1}{2}\eta\, \|H_0\|^{-1} \text{ and } |H(t)| \leqslant \sum_{\nu=1}^{M} a_\nu |H_\nu(t)| \leqslant \sum_{\nu=1}^{M} a_\nu (\|H_0\| + \tfrac{1}{4}\eta)$$
$$= \|H_0\| + \tfrac{1}{4}\eta \leqslant \eta/2 + \eta/4 < \eta, \text{ or } 1 \geqslant \tfrac{1}{2}\eta\, \|H_0\|^{-1} \text{ and}$$
$$|H(t)| \leqslant \sum_{\nu=1}^{M} a_\nu |H_\nu(t)| \leqslant \sum_{\nu=1}^{M} a_\nu \big(|H_0(t)| + \tfrac{1}{4}\eta\big) \cdot \tfrac{1}{2}\eta \cdot \|H_0\|^{-1} < \eta,$$

(22) if $\varrho(s, t) < \delta$ $(s \in S;\ t \in T)$, then (by (14) and (16))

$$h(s) - \operatorname{Re} H(t) \geqslant h(s) - |H(t)| = h(s) - \Big|\sum_{\nu=1}^{M} a_\nu H_\nu(t)\Big|$$
$$\geqslant \sum_{\nu=1}^{M} a_\nu \big(h(s) - |H_\nu(t)|\big) \geqslant \sum_{\nu=1}^{M} a_\nu \big(h(s) - |H_0(t)| - \tfrac{1}{4}\eta\big)$$
$$= h(s) - |H_0(t)| - \tfrac{1}{4}\eta > -\tfrac{1}{4}\eta - \tfrac{1}{2}\eta > -\eta.$$

Obviously properties (18)-(22) imply that H is the required extension of h satisfying (10)-(13), q. e. d.

LEMMA 6. *Let ε and ω be positive numbers. Let us assume* (B). *Let $\lambda = (\lambda_i)_{i=1}^{N_\lambda}$ and $\mu = (\mu_j)_{j=1}^{N_\mu}$ be peak partitions of unit in $C(S)$, let μ be ε-subordinated to λ and let $\pi_\mu^a \colon C(S) \to E_\mu$ be a projection ε-subordinating μ to λ. Let $L_\lambda \colon E_\lambda \to X$ be a linear operator of extension with $\|L_\lambda\| < 1 + \omega$. Then there is a linear operator of extension $L_\mu \colon E_\mu \to X$ such that*

(23) $$\|L_\mu \pi_\mu^a f - L_\lambda f\| \leqslant 2N_\lambda \cdot \varepsilon \cdot \|f\| \quad \textit{for} \quad f \textit{ in } E_\lambda,$$

(24) $$\|L_\mu\| < 1 + \omega + 2N_\lambda \varepsilon.$$

Proof. Let $q = (q_j)_{j=1}^{N_\mu}$. Let us set $J_1 = \{j_1, j_2, \ldots, j_{N_\lambda}\}$, where j_i satisfies (4) $(i = 1, 2, \ldots, N_\lambda)$, and $J_2 = \{j \leqslant N_\mu : j \notin J_1\}$.

First we choose extensions $L_\mu \pi_\mu^q \lambda_i$ of $\pi_\mu^q \lambda_i$ such that

$$\|L_\mu \pi_\mu^q \lambda_i - L_\lambda \lambda_i\| \leqslant 2\varepsilon \quad (i = 1, 2, \ldots, N_\lambda). \tag{25}$$

The existence of such extensions follows immediately from (5) and Lemma 4, because $\|\pi_\mu^q \lambda_i - \lambda_i\| \leqslant \varepsilon \|\lambda_i\| = \varepsilon$ $(i = 1, 2, \ldots, N_\lambda)$. Let η be an arbitrary positive number such that

$$0 < \eta < \frac{1+\omega - \|L_\lambda\|}{4N_\mu + 1} \tag{26}$$

and let δ_0 be chosen in such a way that

$$\text{if } \varrho(t_1, t_2) < \delta_0, \text{ then } \sum_{i=1}^{N_\lambda} |L_\mu \pi_\mu^q \lambda_i(t_1) - L_\mu \pi_\mu^q \lambda_i(t_2)| < \eta \quad \text{for } (t_1, t_2) \in T \times T. \tag{27}$$

According to Lemma 5 there exist δ with $0 < \delta < \delta_0$ and extensions $L_\mu \mu_j$ $(j \in J_2)$ such that

$$\|L_\mu \mu_j\| \leqslant \|\mu_j\| + \eta = 1 + \eta, \tag{28}$$

$$\big||L_\mu \mu_j(t)| - L_\mu \mu_j(t)\big| < \eta \quad \text{for} \quad t \text{ in } T, \tag{29}$$

$$\text{if } t \in T \text{ and } \varrho(t, S) \geqslant \delta, \text{ then } |L_\mu \mu_j(t)| < \eta, \tag{30}$$

$$\text{if } \varrho(t, s) < \delta, \text{ then } \mu_j(s) - \operatorname{Re} L_\mu \mu_j(t) > -\eta \text{ for } t \text{ in } T \text{ and for } s \text{ in } S. \tag{31}$$

Further we set

$$L_\mu \mu_{j_i} = L_\mu \pi_\mu^q \lambda_i - \sum_{j \in J_2} \lambda_i(q_j) L_\mu \mu_j \quad (i = 1, 2, \ldots, N_\lambda). \tag{32}$$

Finally for arbitrary complex numbers $c_1, c_2, \ldots, c_{N_\mu}$ we put

$$L_\mu \Big(\sum_{j=1}^{N_\mu} c_j \mu_j\Big) = \sum_{j=1}^{N_\mu} c_j L_\mu \mu_j. \tag{33}$$

Since $q_{j_k} \in V_\lambda^k \subset S \setminus U_\lambda^i$ for $i \neq k$,

$$\lambda_i(q_{j_k}) = \begin{cases} 1 & \text{for} \quad i = k, \\ 0 & \text{for} \quad i \neq k \end{cases} \quad (i, k = 1, 2, \ldots, N_\lambda).$$

Thus

$$\pi_\mu^q \lambda_i = \sum_{j=1}^{N_\mu} \lambda_i(q_j) \mu_j = \mu_{j_i} + \sum_{j \in J_2} \lambda_i(q_j) \mu_j \quad (i = 1, 2, \ldots, N_\lambda). \tag{34}$$

It follows from (32)-(34) that the definition of $L_\mu \pi_\mu^q \lambda_i$ $(i = 1, 2, \ldots, N_\lambda)$ which we get by putting in (33) $c_j = \lambda_i(q_j)$ for $j = 1, 2, \ldots, N_\mu$ coincides with the earlier choice of $L_\mu \pi_\mu^q \lambda_i$ (satisfying (25)).

Hence the construction given above defines a linear operator of extension $L_\mu : E_\mu \to X$.

We shall establish that the operator L_μ defined in this way satisfies conditions (23) and (24).

From (8), (25) and the linearity of operators L_λ and L_μ we get

$$\|L_\lambda f - L_\mu \pi_\mu^q f\| = \Big\|\sum_{i=1}^{N_\lambda} c_i L_\lambda \lambda_i - \sum_{i=1}^{N_\lambda} c_i L_\mu \pi_\mu^q \lambda_i\Big\| \tag{35}$$

$$\leqslant \sum_{i=1}^{N_\lambda} |c_i| \, \|L_\lambda \lambda_i - L_\mu \pi_\mu^q \lambda_i\| \leqslant 2N_\lambda \varepsilon \max_{1 \leqslant i \leqslant N_\lambda} |c_i| = 2N_\lambda \varepsilon \|f\| \text{ for } f = \sum_{i=1}^{N_\lambda} c_i \lambda_i.$$

Hence condition (23) is satisfied.

We are now going to prove (24). We begin with the estimation of the quantity $\sum_{j=1}^{N_\mu} |L\mu_j(t)|$ for t in T. We consider two cases

1° $\varrho(t, S) \geqslant \delta$. Then, by (32), we have

$$\sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| = \sum_{j \in J_2} |L_\mu \mu_j(t)| + \sum_{i=1}^{N_\lambda} \Big|\Big(L_\mu \pi_\mu^q \lambda_i - \sum_{j \in J_2} \lambda_i(q_j) L_\mu \mu_j\Big)(t)\Big| \tag{36}$$

$$\leqslant \sum_{j \in J_2} |L_\mu \mu_j(t)| + \sum_{i=1}^{N_\lambda} \sum_{j \in J_2} \lambda_i(q_j) |L_\mu \mu_j(t)| + \sum_{i=1}^{N_\lambda} |L_\mu \pi_\mu^q \lambda_i(t)|;$$

since (by (1)) $\sum_{i=1}^{N_\lambda} \lambda_i(q_j) = 1$ $(j \in J_2)$, we have

$$\sum_{i=1}^{N_\lambda} \sum_{j \in J_2} \lambda_i(q_j) |L_\mu \mu_j(t)| = \sum_{j \in J_2} \sum_{i=1}^{N_\lambda} \lambda_i(q_j) |L_\mu \mu_j(t)| = \sum_{j \in J_2} |L_\mu \mu_j(t)|. \tag{37}$$

It follows from (30) that

$$\sum_{j \in J_2} |L_\mu \mu_j(t)| \leqslant N_\mu \cdot \eta. \tag{38}$$

Now

$$\sum_{i=1}^{N_\lambda} |L_\mu \pi_\mu^q \lambda_i(t)| \leqslant \sum_{i=1}^{N_\lambda} |L_\mu \pi_\mu^q \lambda_i(t) - L_\lambda \lambda_i(t)| + \sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)|. \tag{39}$$

Using (25) we get

$$\sum_{i=1}^{N_\lambda} |(L_\mu \pi_\mu^q \lambda_i - L_\lambda \lambda_i)(t)| \leqslant 2N_\lambda \varepsilon. \tag{40}$$

By Lemma 3 we have

$$\sum_{i=1}^{N_\lambda} |L_\lambda \lambda_i(t)| \leqslant \|L_\lambda\| \quad \text{for} \quad t \text{ in } T. \tag{41}$$

Thus finally, by (35)-(41), we get

$$\sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| \leqslant 2N_\mu \eta + 2N_\lambda \varepsilon + \|L_\lambda\|. \tag{42}$$

2° There exists an s in S such that $\varrho(s,t) < \delta$. Then, by (32), we have

$$\sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| = \sum_{j\in J_2} |L_\mu \mu_j(t)| + \sum_{i=1}^{N_\lambda} \Big|\Big(L_\mu \pi_\mu^q \lambda_i - \sum_{j\in J_2} \lambda_i(q_j) L_\mu \mu_j\Big)(t)\Big|. \tag{43}$$

Using (27) and the fact that L_μ is an extension operator we get

$$\sum_{i=1}^{N_\lambda} |L_\mu \pi_\mu^q \lambda_i(t) - \pi_\mu^q \lambda_i(s)| = \sum_{i=1}^{N_\lambda} |L_\mu \pi_\mu^q \lambda_i(t) - L_\mu \pi_\mu^q \lambda_i(s)| < \eta. \tag{44}$$

Thus, by (34), we get

$$\sum_{i=1}^{N_\lambda} \Big|\Big(L_\mu \pi_\mu^q \lambda_i - \sum_{j\in J_2} \lambda_i(q_j) L_\mu \mu_j\Big)(t)\Big| < \eta + \sum_{i=1}^{N_\lambda} \Big|\pi_\mu^q \lambda_i(s) - \sum_{j\in J_2} \lambda_i(q_j) L_\mu \mu_j(t)\Big| \tag{45}$$
$$\leqslant \eta + \sum_{i=1}^{N_\lambda} \mu_{j_i}(s) + \sum_{i=1}^{N_\lambda} \Big|\sum_{j\in J_2} \lambda_i(q_j)\big(\mu_j(s) - L_\mu \mu_j(t)\big)\Big|.$$

Let us set

$$a_i = \sum_{j\in J_2} \lambda_i(q_j)\big(\mu_j(s) - L_\mu \mu_j(t)\big) \qquad (i = 1, 2, \ldots, N_\lambda). \tag{46}$$

It follows from (2) that

$$\operatorname{Im} a_i = \operatorname{Im}\Big(\sum_{j\in J_2} \lambda_i(q_j) - L_\mu \mu_j(t)\Big) = -\sum_{j\in J_2} \lambda_i(q_j) \operatorname{Im} L_\mu \mu_j(t). \tag{47}$$

Applying (1), (29) and the elementary inequality $|\operatorname{Im} z| \leqslant \big||z| - z\big|$ we get

$$\sum_{i=1}^{N_\lambda} |\operatorname{Im} a_i| = \sum_{i=1}^{N_\lambda} \Big|\sum_{j\in J_2} \lambda_i(q_j) \operatorname{Im} L_\mu \mu_j(t)\Big| \tag{48}$$
$$\leqslant \sum_{j\in J_2} \sum_{i=1}^{N_\lambda} \lambda_i(q_j) |\operatorname{Im} L_\mu \mu_j(t)| \leqslant N_\mu \eta.$$

Now we shall estimate the quantity $\sum_{i=1}^{N_\lambda} (|\operatorname{Re} a_i| - \operatorname{Re} a_i)$. Let us set

$$J_2^*(t) = \{j \in J_2 : \mu_j(s) - \operatorname{Re} L_\mu \mu_j(t) < 0\}.$$

We have

$$|\operatorname{Re} a_i| - \operatorname{Re} a_i \leqslant -2 \sum_{j\in J_2^*(t)} \lambda_i(q_j)\big(\mu_j(s) - \operatorname{Re} L_\mu \mu_j(t)\big) \qquad (i = 1, 2, \ldots, N_\lambda).$$

Hence from (31) we get

$$\sum_{i=1}^{N_\lambda} (|\operatorname{Re} a_i| - \operatorname{Re} a_i) \leqslant 2 \sum_{i=1}^{N_\lambda} \sum_{j\in J_2^*(t)} \lambda_i(q_j) \eta < 2N_\mu \eta.$$

Thus

$$\sum_{i=1}^{N_\lambda} |a_i| \leqslant \sum_{i=1}^{N_\lambda} (|\operatorname{Im} a_i| + |\operatorname{Re} a_i|) < N_\mu \eta + 2N_\mu \eta + \sum_{i=1}^{N_\lambda} \operatorname{Re} a_i. \tag{49}$$

It follows from (1), (2) and (46) that

$$\sum_{i=1}^{N_\lambda} \operatorname{Re} a_i = \operatorname{Re}\Big(\sum_{i=1}^{N_\lambda} a_i\Big) = \operatorname{Re}\Big(\sum_{i=1}^{N_\lambda} \sum_{j\in J_2} \lambda_i(q_j)\big(\mu_j(s) - L_\mu \mu_j(t)\big)\Big) \tag{50}$$
$$= \sum_{j\in J_2} \mu_j(s) - \sum_{j\in J_2} \operatorname{Re} L_\mu \mu_j(t).$$

Comparing (43) with (45), (49) and (50) and using the equality $\sum_{j=1}^{N_\mu} \mu_j(s) = 1$ we get

$$\sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| < \sum_{j\in J_2} |L_\mu \mu_j(t)| + \eta + \sum_{i=1}^{N_\lambda} \mu_{j_i}(s) + 3N_\mu \eta \tag{51}$$
$$+ \sum_{j\in J_2} \mu_j(s) - \sum_{j\in J_2} \operatorname{Re} L_\mu \mu_j(t)$$
$$= 1 + \sum_{j\in J_2} \big(|L_\mu \mu_j(t)| - \operatorname{Re} L_\mu \mu_j(t)\big) + (3N_\mu + 1)\eta.$$

By (29) and the elementary inequality $|z| - \operatorname{Re} z \leqslant \big||z| - z\big|$ we have

$$\sum_{j\in J_2} \big(|L_\mu \mu_j(t)| - \operatorname{Re} L_\mu \mu_j(t)\big) \leqslant \sum_{j\in J_2} \big||L_\mu \mu_j(t)| - L_\mu \mu_j(t)\big| < N_\mu \eta. \tag{52}$$

Finally (in the case 2°) comparing (51) with (52) we get

$$\sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| < (4N_\mu + 1)\eta + 1. \tag{53}$$

Hence in both cases, according to (26), (42) and (53) we obtain

$$\sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| < 1 + \omega + 2N_\lambda \varepsilon \quad \text{for } t \text{ in } T. \tag{54}$$

Thus, by Lemma 3, $\|L_\mu\| < 1 + \omega + 2N_\lambda \varepsilon$, q. e. d.

LEMMA 7. *Let us assume* (G). *Let* $\mu = (\mu_j)_{j=1}^{N_\mu}$ *be a peak partition of unit in* $C(S)$ *and let* $L_\mu : E_\mu \to X$ *be a linear operator of extension with* $\|L_\mu\| < 1 + \sigma$, *where* $0 < \sigma < \frac{1}{2}$. *Then there exists a linear operator of extension* $L_\mu' : E_\mu \to X$ *such that* $\|L_\mu'\| = 1$ *and* $\|L_\mu - L_\mu'\| < 16\sigma$.

Proof (4). Let us choose the integer n such that $2^{-n} > \sigma \geqslant 2^{-n-1}$. Let us set

$$\Lambda(t) = \sum_{j=1}^{N_\mu} |L_\mu \mu_j(t)| \quad \text{for} \quad t \text{ in } T,$$

$$T_m = \{t \in T : 1+2^{-m} > \Lambda(t) \geqslant 1+2^{-m-1}\} \quad (m = n, n+1, \ldots).$$

Note that $(\mathrm{cl}\, T_m) \cap S = \emptyset$, because $\Lambda(s) = 1$ for s in S. Therefore $\sup_{t \in T_m} |K(t)| < 1$, where $K \in X$ is a peak function for S with the properties as in (G). Let us choose an integer p_m such that $|K(t)|^{p_m} < 2^{-2m}$ for t in T_m $(m = n, n+1, \ldots)$. Let us put

$$L'_\mu f = \sum_{m=1}^{n-1} 2^{-m} L_\mu f + \sum_{m=n}^{\infty} 2^{-m} K^{p_m} L_\mu f \quad \text{for} \quad f \text{ in } E_\mu.$$

Obviously, by (G), L'_μ is a linear operator of extension from E_μ into X. To compute $\|L'_\mu\|$ we estimate the quantity

$$\Lambda'(t) = \sum_{j=1}^{N_\mu} |L'_\mu \mu_j(t)| \quad \text{for} \quad t \text{ in } T.$$

We have

$$\Lambda'(t) \leqslant \sum_{j=1}^{N_\mu} \Big(\sum_{m=1}^{n-1} 2^{-m} |L_\mu \mu_j(t)| + \sum_{m=n}^{\infty} 2^{-m} |L_\mu \mu_j(t)|\, |K(t)|^{p_m} \Big)$$

$$= \Lambda(t) \Big(\sum_{m=1}^{n-1} 2^{-m} + \sum_{m=n}^{\infty} 2^{-m} |K(t)|^{p_m} \Big) \quad (t \in T).$$

Let us consider two cases.

1° there is an integer $m_0 \geqslant n$ such that $t \in T_{m_0}$. Then we have

$$\Lambda(t) \Big(\sum_{m=1}^{n-1} 2^{-m} + \sum_{m=n}^{\infty} 2^{-m} |K(t)|^{p_m} \Big) \leqslant \Lambda(t) \Big(\sum_{m \neq m_0} 2^{-m} + 2^{-m_0} |K(t)|^{p_{m_0}} \Big)$$

$$\leqslant (1+2^{-m_0})(1-2^{-m_0}+2^{-3m_0}) = 1-2^{-2m_0}(1-2^{-m_0}-2^{-2m_0}) < 1.$$

2° $\Lambda(t) \leqslant 1$. Then obviously

$$\Lambda(t) \Big(\sum_{m=1}^{n-1} 2^{-m} + \sum_{m=n}^{\infty} 2^{-m} |K(t)|^{p_m} \Big) \leqslant \Lambda(t) \sum_{m=1}^{\infty} 2^{-m} \leqslant 1.$$

Since $1+2^{-n} > 1+\sigma > \|L_\mu\| = \sup_{t \in T} \Lambda(t)$ (by Lemma 3), for every t in T either 1° or 2° holds. Therefore, by Lemma 3, $\sup_{t \in T} \Lambda'(t) = \|L'_\mu\| \leqslant 1$. Thus $\|L'_\mu\| = 1$, because every linear operator of extension has the norm $\geqslant 1$.

(4) The idea of this proof is similar to those of Glicksberg ([14], Lemma 4.5) and of Bishop [4].

Now, let $f \in E_\mu$. We have

$$\|L'_\mu f - L_\mu f\| = \Big\| L'_\mu f - \sum_{m=1}^{\infty} 2^{-m} L_\mu f \Big\| = \Big\| \sum_{m=n}^{\infty} 2^{-m} (K^{p_m} - 1) L_\mu f \Big\|$$

$$\leqslant \|L_\mu\| \, \|f\| \sum_{m=n}^{\infty} 2^{-m+1} \leqslant 2^{-n+2}(1+2^{-n}) \|f\|.$$

Hence $\|L'_\mu - L_\mu\| \leqslant 2^{-n+2}(1+2^{-n}) \leqslant 8\sigma(1+2\sigma) < 16\sigma$, q. e. d.

3.3. Proof of the Main Theorem. Let $0 < \delta < \frac{1}{3}$. Using Lemmas 1 and 2 one may define by induction positive numbers ε_n, peak partitions of unit $\lambda^{(n)} = (\lambda_i^{(n)})_{i=1}^{N_n}$ in $C(S)$ and projections $\pi_n : C(S) \to E_{\lambda^{(n)}}$ such that

(55) $$\lambda^{(0)} = (e_S), \text{ where } e_S(s) = 1 \text{ for } s \text{ in } S,$$

(56) $$\lambda^{(n+1)} \text{ is } \varepsilon_n\text{-subordinated to } \lambda^{(n)} \ (n = 0, 1, \ldots),$$

(57) $$\sum_{n=0}^{\infty} N_n \varepsilon_n < \tfrac{1}{2}\delta,$$

(58) $$\pi_{n+1} \text{ is a projection } \varepsilon_n\text{-subordinating } \lambda^{(n+1)} \text{ to } \lambda^{(n)},$$

(59) $$\text{if } f \in E_{\lambda^{(n)}}, \text{ then } \|\pi_{n+m} f - f\| \leqslant 2^{-n-m} \|f\| \ (m, n = 0, 1, \ldots),$$

(60) $$d(\lambda^{(n)}) \to 0 \text{ as } n \to +\infty.$$

Let us assume only (B). We shall define by an induction process a sequence (L_n) of linear operators of extension such that

(61) $$L_n : E_{\lambda^{(n)}} \to X \quad (n = 0, 1, \ldots),$$

(62) $$\|L_{n+1} \pi_{n+1} f - L_n f\| \leqslant 2N_n \varepsilon_n \|f\| \quad \text{for} \quad f \in E_{\lambda^{(n)}} \quad (n = 0, 1, \ldots),$$

(63) $$\|L_n\| < 1 + \omega_0 + \sum_{\nu=0}^{n-1} 2N_\nu \varepsilon_\nu,$$

where $\omega_0 = \frac{1}{2}\delta - \sum_{\nu=0}^{\infty} N_\nu \varepsilon_\nu$ $(n = 0, 1, \ldots)$ (5). We define $L_0 e_S$ as such an extension of e_S to X that $\|L_0 e_S\| < 1 + \omega_0$ and we put $L_0 c e_S = c L_0 e_S$ for every complex number c. Let us suppose that L_n is defined for some $n \geqslant 0$. Then we define L_{n+1} as a linear operator of extension satisfying the assertion of Lemma 6 in the case where $\lambda = \lambda^{(n)}$, $\mu = \lambda^{(n+1)}$, $\varepsilon = \varepsilon_n$, $\pi_\mu^\lambda = \pi_{n+1}$, $L_\lambda = L_n$ and $\omega = \omega_0 + \sum_{\nu=0}^{n-1} 2N_\nu \varepsilon_\nu$. We omit the easy verification that sequence (L_n) defined in this way satisfies conditions (61)-(63).

(5) In the case where $n = 0$ we admit $\sum_{\nu=0}^{-1} N_\nu \varepsilon_\nu = 0$.

Let us consider the sequence of linear operators $(L_n\pi_n)$. Obviously $L_n\pi_n: C(S)\to X$ $(n = 0, 1, \ldots)$. We shall show that there exists a limit $\lim_n L_n\pi_n f$ for every f in $C(S)$. To prove it we shall apply the Banach-Steinhaus principle ([12], p. 55). It follows from (63) and the fact that $\|\pi_n\| = 1$ that $\sup_n \|L_n\pi_n\| < 1+\delta$. Hence it is sufficient to show that $\lim_n L_n\pi_n f$ exists for every f in E, where $E = \bigcup_{n=0}^{\infty} E_{\lambda(n)}$ is a dense subset in $C(S)$ (the density of E is an immediate consequence of Lemma 1 and property (60)). Let $f \in E_{\lambda(n)}$, n being fixed for the moment. Using (59), (62) and (63) we get

$$\|L_{m+n+1}\pi_{m+n+1}f - L_{m+n}\pi_{m+n}f\| \leqslant \|L_{m+n+1}\pi_{m+n+1}f - L_{m+n+1}\pi_{m+n+1}\pi_{m+n}f\| + \|L_{m+n+1}\pi_{m+n+1}\pi_{m+n}f - L_{m+n}\pi_{m+n}f\| \leqslant \|L_{m+n+1}\pi_{m+n+1}\|\,\|f - \pi_{m+n}f\| + 2N_{m+n}\varepsilon_{m+n}\|\pi_{m+n}f\| \leqslant (1+\delta)2^{-n-m}\|f\| + 2N_{m+n}\varepsilon_{m+n}\|f\|.$$

Since the series $\sum_{m=0}^{\infty}\big((1+\delta)2^{-n-m} + 2N_{m+n}\varepsilon_{m+n}\big)$ is absolutely convergent, $(L_{n+m}\pi_{n+m}f)_{m=1}^{\infty}$ is a Cauchy sequence. Thus there exists a limit $\lim_m L_{n+m}\pi_{n+m}f = \lim_m L_m\pi_m f$ for every f in $E_{\lambda(n)}$ $(n = 0, 1, \ldots)$. Therefore there exists a limit

$$\lim_n L_n\pi_n f = L_\delta f \quad \text{for every } f \text{ in } C(S).$$

Obviously $L_\delta: C(S)\to X$ is a linear operator with the norm $\|L_\delta\| \leqslant \sup_n \|L_n\pi_n\| < 1+\delta$. Since $\pi_n f \in E_{\lambda(n)}$ and $L_n: E_{\lambda(n)}\to X$ are linear operators of extension, $L_n\pi_n f(s) = \pi_n f(s)$ for every s in S and for every f in $C(S)$ $(n = 0, 1, \ldots)$. Thus $L_\delta f(s) = \lim_n L_n\pi_n f(s) = \lim_n \pi_n f(s)$. It follows immediately from Lemma 1 and property (60) that $\lim_n \pi_n f(s) = f(s)$ for every s in S and for every f in $C(S)$. Hence $L_\delta f(s) = f(s)$ for s in S and for f in $C(S)$. Thus L_δ is a linear operator of extension required in the first part of the Main Theorem.

Now, let us assume (B) and (G). Then we shall define by an induction process a sequence (L'_n) of linear operators of extension such that

(61′) $$L'_n: E_{\lambda(n)} \to X \quad (n = 0, 1, \ldots),$$

(62′) $$\|L'_{n+1}\pi_{n+1}f - L'_n f\| \leqslant 50N_n\varepsilon_n \quad (n = 0, 1, \ldots),$$

(63′) $$\|L'_n\| = 1.$$

Let $L'_0: E_{\lambda^0} \to X$ be an arbitrary linear operator of extension with $\|L'_0\| = 1$. The existence of such an operator follows immediately from Lemma 7 and the fact that there exists a linear operator of extension $L_0: E\to X$ with $\|L_0\| < 1+\frac{1}{2}$. Let us suppose that for some $n \geqslant 0$ the operator $L'_n: E_{\lambda(n)}\to X$ is defined. We define $L_{n+1} = L_\mu: E_{\lambda(n+1)}\to X$ as a linear operator of extension satisfying the assertion of Lemma 6 in the case where $\lambda = \lambda^{(n)}$, $\mu = \lambda^{(n+1)}$, $\varepsilon = \varepsilon_n$, $\pi^q_\mu = \pi_{n+1}$, $L_\lambda = L'_n$ and $\omega = N_n\varepsilon_n > \|L'_n\| - 1 = 0$. It follows from Lemma 6 that

(64) $$\|L_{n+1}\pi_{n+1}f - L'_n f\| \leqslant 2N_n\varepsilon_n\|f\| \quad \text{for} \quad f\in E_{\lambda(n)},$$

(65) $$\|L_{n+1}\| < 1+3N_n\varepsilon_n < 1+\tfrac{3}{2}\delta < 1+\tfrac{1}{2}.$$

Now, using Lemma 7, we define $L'_{n+1}: E_{\lambda(n+1)}\to X$ as such a linear operator of extension that $\|L'_{n+1}\| = 1$ and

(66) $$\|L_{n+1} - L'_{n+1}\| < 1+16\cdot 3N_n\varepsilon_n = 1+48N_n\varepsilon_n.$$

Obviously the sequence (L'_n) defined in this way satisfies the conditions (61′)-(63′). Condition (63′) for $n+1$ is an immediate consequence of (64) and (66).

To complete the proof we put

$$Lf = \lim_n L'_n\pi_n f \quad \text{for} \quad f \text{ in } C(S).$$

In exactly the same way as in the proof of the first part of the theorem (using formulas (61′)-(63′) instead of (61)-(63)) we show that $L': C(S)\to X$ is a linear operator of extension with $\|L'\| = 1$, q. e. d.

Added in proof. Recently we obtained the following improvement of the Main Theorem:

Let X be a closed subspace of the space $C(T)$ of all continuous complex-valued functions on a compact metric space T and let S be a closed subset of T. Then (B) (see p. 286) *implies the existence of a linear operator of extension L: $C(S)\to X$ with $\|L\| = 1$.*

For the proof see A. Pełczyński, *Supplement to my paper "On simultaneous extension of continuous functions"*, Studia Math. 25.1 (1965).

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On formal trigonometrical series

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Using derivatives of infinite order we have extended in [5] the methods of the elementary theory of distributions due to J. Mikusiński and R. Sikorski (see [3] and [4]). The elements so obtained, which we call *ultra-distributions*, may be regarded as Fourier transforms of distributions of finite order.

This note deals with Fourier series expansions of ultra-distributions. For simplicity we restrict ourselves to the case of one variable. We show that each periodic ultra-distribution has a Fourier series converging to that ultra-distribution. The Fourier coefficients are defined by the classical formulas, which we interprete similarly as for periodic distributions (see [3], § 20). Moreover, each formal trigonometrical series, i. e. a series with no restrictions on the coefficients, converges in the sense of ultra--distributions and is the Fourier series of its sum. We also prove an analogue of the last result for generalized trigonometrical series.

Throughout the paper we use the notation and basic properties of ultra-distributions given in [5]. We recall briefly the definition of the convergence. Let $\varphi_t(x)$ be an ultra-distribution of x for each value of the parameter t and $\varphi(x)$ another ultra-distribution. We say that

$$\lim_{t\to t_0}\varphi_t(x)=\varphi(x),$$

if there exists an entire function $A(x)$, an integer ν, and continuous functions $\Phi_t(x)$, $\Phi(x)$, which are $O(|x|^{2\nu})$ as $|x|\to\infty$ and satisfy the following conditions:

(L_1) $$A\left(\frac{1}{i}D\right)\Phi_t(x)=\varphi_t(x),\qquad A\left(\frac{1}{i}D\right)\Phi(x)=\varphi(x),$$

(L_2) For $t\to t_0$, $(1+x^2)^{-\nu}\Phi_t(x)$ converges to $(1+x^2)^{-\nu}\Phi(x)$ uniformly in R_1.

A series of ultra-distributions $\sum_{n=1}^{\infty}\varphi_n(x)$ *converges* to $\varphi(x)$ if the sequence