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On operators with a finite d-characteristic

by

D. PRZEWORSKA-ROLEWICZ and S. ROLEWICZ (Warszawa)

The theory of linear integral equations founded by Fredholm [1-4] was a starting point for the development of the theory of normed spaces and compact operators in these spaces (Riesz [1], Banach [1]). This theory created its own methods different from the methods of the determinant theory used in the original papers by Fredholm. Subsequently compact operators in different types of linear topological spaces were investigated by Hyers [1], Marinescu [1], Altman [1-2]. Most general results was obtained by Leray [1] in the case of locally convex spaces and Williamson [1] in the case of general linear topological spaces. There was a return to Fredholm's original idea of the determinant theory after second world war. The first determinant theory of operators in a Banach space was created by Ruston [1-4]. This theory was developed and modified by Grothendieck [1-3]. Another, more general theory was given by Leżański [1-2] and completed and modified by Sikorski [1-5]. Sikorski has remarked that many facts have a quite algebraic character and can be formulated and proved in the language of linear algebra.

The theory of linear integral singular equations, developed by Muskhelishvili [1] for one dimension, and by Michlin [1] for many dimensions, and others, arouse an interest of mathematicians in the theory of linear operators in Banach spaces which have finite nullity and deficiency. These operators are called *operators with a finite d-characteristic*. The basic properties of this theory was independently formulated by Gochberg and Krein [1] and soon after by Kato [1]. Other results concerning this subject are contained in the papers of Gochberg [1-4], Feldman, Gochberg and Markus [1] and Yood [1].

Buraczewski [1] transfered the determinant theory of Leżański and Sikorski to the case of operators with a finite d-characteristic.

One of the most important methods of solution of a linear singular integral equation is the method of regularization (Muskhelishvili [1], Michlin [1], [2]). Algebraic principles of the theory of regularization were given by Przeworska-Rolewicz [1-2].

The aim of this paper is an investigation of the theory of operators with a finite d-characteristic exclusively in the language of linear algebra without consideration topological properties. A part of the results is contained in the author's paper [1].

§ 1. Definition of operators with a finite d-characteristic. Theorem of the index of superposition. We are given two linear spaces X and Y and a linear operator A determined on a set $D_A \subset X$ and with values in Y. By E_A we denote the set of values of operator A, by Z_A we denote $\{x \in X: Ax = 0\}$. The number $\alpha_A = \dim Z_A$ is called the *nullity* of A. The number $\beta_A = \dim Y/E_A$, where by Y/E_A we denote the quotient space, is called the *deficiency* of the operator A. The pair of number (α_A, β_A) is called *d-characteristic* of the operator A. We say that the d-characteristic of the operator A is *finite*, or that the operator A has a *finite* d-characteristic, if numbers α_A and β_A are both finite. The *index* of the operator A is the number

$$\varkappa_A = \beta_A - \alpha_A. \tag{1.1}$$

The following important theorem holds for the index of superposition of two operators:

THEOREM 1.1. *Let A and B be linear operators with a finite d-characteristic, let B map X in Y, and let A map Y in Z. If A is determined on the whole space Y, then the superposition AB is an operator with a finite d-characteristic and*

$$\varkappa_{AB} = \varkappa_A + \varkappa_B. \tag{1.2}$$

Proof. Let $\mathfrak{C}_1 = E_B \cap Z_A$ and $n_1 = \dim \mathfrak{C}_1$. Obviously we can represent the set Z_A as a direct sum

$$Z_A = \mathfrak{C}_1 \oplus \mathfrak{C}_2 \tag{1.3}$$

where $\dim \mathfrak{C}_2 = \alpha_A - n_1$ and Y as a direct sum

$$Y = E_B \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_3 \tag{1.4}$$

where $\dim \mathfrak{C}_3 = n_3$.

Since the space $\mathfrak{C}_2 \oplus \mathfrak{C}_3$ is isomorphic to the quotient space Y/E_B, we have $\dim(\mathfrak{C}_2 + \mathfrak{C}_3) = \beta_B$, whence $\alpha_A - n_1 + n_3 = \beta_B$ and

$$\alpha_A - \beta_B = n_1 - n_3.$$

Formula (1.4) implies

$$E_A = E_{AB} \oplus A\mathfrak{C}_3, \tag{1.5}$$

but $\dim A\mathfrak{C}_3 = \dim \mathfrak{C}_3 = n_3$, and definitively

$$\varkappa_{AB} = \beta_{AB} - \alpha_{AB} = \beta_A + n_3 - (\alpha_B + n_1) = \beta_A - \alpha_A + \beta_B - \alpha_B = \varkappa_A + \varkappa_B,$$

q. e. d.

If a superposition of two operators AB has a finite d-characteristic, then obviously β_A and α_B are finite. This immediately implies a theorem in some measure inverse to theorem 1.1.

THEOREM 1.2. *Let an operator A map X in Y and an operator B map Y in X. If both superpositions AB and BA have a finite d-characteristic, then both operators A and B have a finite d-characteristic.*

§ 2. Finite dimensional operators. We are given two linear spaces X and Y. An operator K of the type

$$y = Kx = \sum_{i=1}^{n} \varphi_i(x) y_i$$

where $y_i \in Y$ and φ_i denote linear functionals determined on X, is called a *finite dimensional operator*.

We shall assume that $Y = X$ and we shall consider the operator $I+K$, where I denotes identity. By simple calculations we find that the operator $I+K$ has a finite d-characteristic and its index is equal to zero. This fact and theorem 1.1 imply

THEOREM 2.1. *Let A be an operator with a finite d-characteristic and let K be a finite dimensional operator. Let A and K transform the linear space X in the linear space Y. Then the operator $A+K$ has a finite d-characteristic and*

$$\varkappa_{A+K} = \varkappa_A.$$

Proof. We decompose the space X into a direct sum of the space Z_A and some space $\mathfrak{C}$: $X = Z_A \oplus \mathfrak{C}$. Obviously the operator A_1, which is a restriction of the operator A to the space $\mathfrak{C}$, is invertible. Let K_1 denote the restriction of the operator K to the space $\mathfrak{C}$. Then

$$A_1 + K_1 = (I + K_1 A_1^{-1}) A_1.$$

The operator $K_1 A_1^{-1}$ is a finite dimensional operator mapping Y in Y. Basing ourselves on theorem 1.1 we obtain

$$\varkappa_{A_1+K_1} = \varkappa_{I+K_1A_1^{-1}} + \varkappa_{A_1} = \beta_A. \tag{2.1}$$

On the other hand, the operator $A+K$ is an extension of the operator A_1+K_1, and we shall show that $\varkappa_{A+K} = \varkappa_{A_1+K_1} - \alpha_A$. We shall consider three cases:

1° $KZ_A \subset E_{A_1+K_1}$. Then $E_{A+K} = E_{A_1+K_1}$, whence $\beta_{A+K} = \beta_{A_1+K_1}$. But $\alpha_{A+K} = \alpha_{A_1+K_1} + \alpha_A$. Therefore $\varkappa_{A+K} = \varkappa_{A_1+K_1} - \alpha_A$.

2° For each $x \in Z_A$, $x \neq 0$, $Kx \notin E_{A_1+K_1}$. We write $r = \dim KZ_A$. Obviously $\beta_{A+K} = \beta_{A_1+K_1} - r$. On the other hand, $\alpha_{A+K} = \alpha_{A_1+K_1} - (\alpha_A - r)$, because $\alpha_A - r$ is equal to the nullity of the operator K restricted to the space Z_A. Therefore $\varkappa_{A+K} = \varkappa_{A_1+K_1} - \alpha_A$.

3° In the general case we decompose the space Z_A into a direct sum of two spaces $\mathfrak{C}_1$ and $\mathfrak{C}_2: Z_A = \mathfrak{C}_1 \oplus \mathfrak{C}_2$, where $K\mathfrak{C}_1 \subset E_{A_1+K_1}$ and for each $x \in \mathfrak{C}_2$, $x \neq 0$, $Kx \notin E_{A_1+K_1}$. At the begining we consider the restriction A_2+K_2 of the operator $A+K$ to the space $\mathfrak{C} \oplus \mathfrak{C}_2$. Basing ourselves on 2° we obtain $\varkappa_{A_2+K_2} = \varkappa_{A_1+K_1} - \dim \mathfrak{C}_2$. On the other hand, by 1°, we obtain $\varkappa_{A+K} = \varkappa_{A_2+K_2} - \dim \mathfrak{C}_1$. But $\dim \mathfrak{C}_1 + \dim \mathfrak{C}_2 = \alpha_A$. Therefore $\varkappa_{A+K} = \varkappa_{A_1+K_1} - \alpha_A$.

By (2.1) this implies the theorem.

§ 3. Perturbations of operators. We are given a class of operators $\mathfrak{A}$. An operator A is called an *$\mathfrak{A}$-perturbation of an operator B* if $A+B \in \mathfrak{A}$. An operator C is called an *$\mathfrak{A}$-perturbation of a class* $\mathfrak{R}$ of operators if C is an $\mathfrak{A}$-perturbation of an arbitrary operator B belonging to $\mathfrak{R}$. An $\mathfrak{A}$-perturbation of the class $\mathfrak{A}$ we shall call briefly $\mathfrak{A}$-perturbation.

In the preceding paragraph we proved that the finite dimensional operators are perturbations of the class of operators with a finite d-characteristic. Now we shall show that the inverse theorem is also true.

THEOREM 3.1. *If an operator K mapping a linear space X into itself is a perturbation of the class of all operators with a finite d-characteristic, then K is a finite dimensional operator.*

Proof. Suppose that the operator K is not a finite dimensional operator. Then there exists a sequence $\{y_n\}$ $(n = 1, 2, \ldots)$ of linearly independent elements belonging to the image of the space X by the operator K. Let x_n be such an element that $Kx_n = y_n$. Let X_0 denote the linear space generated by sequences $\{x_n\}$, $\{y_n\}$. We decompose the space X into a direct sum: $X = \mathfrak{C} \oplus X_0$. Now we determine the operator A in the following way:

A is an identity on $\mathfrak{C}$,

$$Ax_n = y_n \quad \text{for} \quad n = 1, 3, 5, \ldots$$

The elements x_n $(n = 2, 4, 6, \ldots)$ and those y_n which do not belong to the space generate by the sequence $\{x_n\}$ $(n = 1, 3, \ldots)$ we order in a sequence $\{z_n\}$. Similarly we order in a sequence $\{z'_n\}$ all elements y_n $(n = 2, 4, 6, \ldots)$ and those x_n which do not belong to the space generate by y_n $(n = 1, 3, 5, \ldots)$. We determine: $Az_n = z'_n$.

The operator A is invertible and maps the space X onto itself, whence it has a finite d-characteristic. On the other hand, the operator $A-K$ does not have a finite d-characteristic because $(A-K)x_n = 0$ for $n = 1, 3, 5, \ldots$ and $\alpha_{A-K} = +\infty$. Therefore the operator K is not a perturbation of the class of all operators with a finite d-characteristic.

Remark. We considered the case of operators transforming the space X into itself. In the case where the powers of the bases X and Y are equal we can obtain the same result by isomorphism of the spaces X and Y. In the case when the powers of the bases of the spaces X and Y are infinite and different, operators with finite d-characteristic mapping X in Y do not exist.

In the case where we do not consider all operators with a finite d-characteristic, the situation can be different, as follows from

THEOREM 3.2. *Let $\mathscr{X}_0$ be a ring of linear operators mapping the space X into itself. Let $\mathscr{X}_0$ contain all finite dimensional operators. Let $\mathscr{J}$ be an arbitrary right-side (or left-side) ideal such that $I+T$ has a finite d-characteristic for each $T \in \mathscr{J}$. Then operators belonging to $\mathscr{J}$ are perturbations of the class of all operators with a finite d-characteristic belonging to $\mathscr{X}_0$. If, moreover, $\varkappa_{I+T} = 0$ for each $T \in \mathscr{J}$, then this perturbation does not change the index.*

Proof. Let A be an arbitrary operator with a finite d-characteristic belonging to $\mathscr{X}_0$. Let the space X be decomposed into a direct sum $X = Z_A \oplus \mathfrak{C}$. This decomposition induces the following projective operators:

$$P_1 = \begin{cases} x & \text{for} \quad x \in \mathfrak{C}, \\ 0 & \text{for} \quad x \in Z_A; \end{cases} \qquad P_2 = \begin{cases} 0 & \text{for} \quad x \in \mathfrak{C}, \\ x & \text{for} \quad x \in Z_A. \end{cases}$$

Obviously the operator $A_1 = AP_1$ has the d-characteristic $(0, \beta_A)$. Since $\mathscr{X}_0$ contains finite dimensional operators, we can extend the operator A_1^{-1} (determined on E_A) onto the whole space, without changing the set of values. This extension we denote by $\tilde{A}_1^{-1}$. Obviously $\tilde{A}_1^{-1}A = P_1$.

Let $T \in \mathscr{J}$. Let $T_1 = TP_1$. Then

$$A_1 + T_1 = (I + TP_1\tilde{A}_1^{-1})A_1$$

but $T \in \mathscr{J}$, whence $C = TP_1\tilde{A}_1^{-1} \in \mathscr{J}$; therefore A_1+T_1 and obviously $A+T$, has a finite d-characteristic. If, moreover, $\varkappa_{I+T} = 0$ for all $T \in \mathscr{J}$, then

$$\varkappa_{A_1+T_1} = \varkappa_{I+TP_1\tilde{A}_1^{-1}} + \varkappa_{A_1} = \varkappa_{A_1} = \beta_A,$$

whence

$$\varkappa_{A+TP_1} = \beta_A - \alpha_A = \varkappa_A.$$

On the other hand, TP_2 is a finite dimensional operator, whence

$$\varkappa_{A+T} = \varkappa_{A+TP_1+TP_2} = \varkappa_{A+TP_1} = \varkappa_A.$$

For left-side ideals the proof is similar. It is enough to remark that the operator $P_1 = A\tilde{A}_1^{-1}$ is a projective operator mapping X on E_A and the operator $P_2 = I - P_1$ is a finite dimensional operator. On the other hand,

$$A + P_1T = A(I + \tilde{A}_1^{-1}T)$$

and by theorem 1.1 and the properties of the ideal we find that the operator $A+P_1T$ has a finite d-characteristic; and if, moreover, $\varkappa_{I+T}=0$ for each $T\in\mathcal{J}$, then

$$\varkappa_{A+P_1T}=\varkappa_A. \tag{3.1}$$

Since P_2 is a finite dimensional operator, then P_2T is an unchanging index perturbation of class of operators with a finite d-characteristic. Therefore $A+P_1T+P_2T$ is an operator with a finite d-characteristic and

$$\varkappa_{A+P_1T}=\varkappa_{A+T}. \tag{3.2}$$

The formulas (3.1) and (3.2) imply $\varkappa_{A+T}=\varkappa_A$, q. e. d.

We do not know if we have a right-side (left-side) ideal $\mathcal{J}$, such that for each $T\in\mathcal{J}$ the operator $I+T$ has a finite d-characteristic, then $\varkappa_{I+T}=0$ for all $T\in\mathcal{J}$?

§ 4. Conjugate operators, d_H-characteristic and Φ_H-operators.

Let X, Y be linear spaces. Let X', Y' denote their conjugate spaces, i. e. the spaces of all linear functionals determined respectively on the spaces X and Y. Let $H\subset Y'$ be a total space, i. e. if $\eta y=0$ for all $\eta\in H$, then $y=0$. A linear operator A transforming the space X into the space Y induces the operator ηA mapping H in $\Xi\subset X'$ determined in the following way:

$$(\eta A)x=\eta(Ax).$$

This operator is called *conjugate operator* to A and we shall denote it by A'. Then $A'\eta\overset{\mathrm{df}}{=}\eta A$.

If $H=Y'$, then the nullity of the operator A' is equal to β_A.

We shall consider the deficiency of the operator A', i. e. the dimension of the space Ξ/HA. Obviously the deficiency depends of the space Ξ, but it is easy to check that if $H=Y'$ and $\Xi=X'$, then

$$\beta_{A'}=\dim X'/YA'=\alpha_A.$$

Indeed, let $E_{A'}=\{\xi:\xi=\eta A,\ \eta\in Y'\}$ and $Z_A^0=\{x:\xi Ax=0,\ \xi\in X'\}$. Obviously $Z_A^0=Z_A$. Let f be an arbitrary element of X'. This functional induces the functional on the space Z_A, whence

$$\beta_{A'}\leqslant\dim Z_A^0=\dim Z_A\leqslant\alpha_A.$$

On the other hand, each linear functional determined on the space Z_A can be extended on to the space X; therefore $\alpha_A\leqslant\beta_{A'}$ and definitely $\beta_{A'}=\alpha_A$. We have proved

PROPOSITION 4.1. *Let X and Y be linear spaces. Let X' and Y' be their conjugate spaces. Let an operator A map X into Y and the conjugate operator A' map Y' into X'. Then the nullity of A is equal to the deficiency of A' and the deficiency of A is equal to the nullity of A' and*

$$\varkappa_{A'}=-\varkappa_A.$$

If $H\neq Y'$ or $\Xi\neq X'$, the proposition 4.1 is not true, even in the case of $X=Y$, $\Xi=H$. An appropriate example is given in the author's paper [1].

Now we replace the index of operator A by another number (which be called *H-index*) possessing the desired property. We determine the *d_H-characteristic* as the pair of numbers (α_A,β_A^H), where $\beta_A^H=\alpha_{A'}$. We determine the H-index by the formula:

$$\varkappa_A^H=\beta_A^H-\alpha_A.$$

Obviously $\varkappa_A^H=-\varkappa_{A'}^X$.

The example given in the author's paper [1] shows that the d_H-characteristic need not be equal to d-characteristic, even in the case of $X=Y$, $\Xi=H$. Moreover, it shows that the theorem similar to theorem 1.1 is not true.

If the d_H-characteristic of an operator A is equal to the d-characteristic, then such an operator is called *Φ_H-operator.*

PROPOSITION 4.2. *Let $X=Y$ and $\Xi=H$. If an operator A is the sum of the identity operator I and a finite dimensional operator K mapping Ξ into itself, $A=I+K$, then A is a Φ_Ξ-operator.*

Proof. Let $Kx=\sum_{i=1}^{n}x_i\xi_ix$. By a simple calculation we can prove that $\alpha_{I+K}=\beta_{I+K}=n-k$, where k is the rank of the matrix (ξ_ix_j). If we consider the conjugate operator, i. e. the operator $I+K'$, where $K'\xi=\sum_{i=1}^{n}\xi x_i\xi_i$ then obviously $\alpha_{I+K'}=\beta_{I+K'}=n-k'$, where k' is the rank of the matrix (ξ_jx_i). But $k=k'$, whence β_{I+K}^Ξ, equal to $\alpha_{I+K'}$ by definition, is equal to $\alpha_{I+K}=\beta_{I+K}$, q. e. d.

In some cases it is possible to prove that the d_Ξ-characteristic is equal to the d_{Ξ_0}-characteristic, where Ξ_0 is a subspace of Ξ. This follows from

THEOREM 4.1. *Let X be a linear space. Let Ξ be the total space of linear functionals determined on X. Let T map X into itself. Let X_0 be an arbitrary subspace of X containing TX and let Ξ_0 be an arbitrary subspace of the space Ξ containing ΞT. If the operator $A=I+T$ has a finite d-characteristic, then the restriction of operator A to the space X_0 has a finite d_{Ξ_0}-characteristic which is equal to the d_Ξ-characteristic of operator A.*

The proof immediately follows from the fact that all solutions of the equation $(I+T)x=0$ considered in the space X must belong to X_0 and, similarly, all solutions of the equation $(I+T')\xi=0$ considered in the space Ξ must belong to the space Ξ_0.

We say that the subspace $X_0\subset X$ is *described by the family of functionals Ξ_0* when $x\in X_0$ if and only if $\xi_0x=0$ for all $\xi_0\in\Xi_0$. If Ξ_0 is a finite dimensional set describing a subspace $X_0 \subset X$, then obviously each subspace X_1 containing X_0 can be described by a finite dimensional set of functionals $\Xi_1 \subset \Xi_0$.

An operator A with a finite d-characteristic is a Φ_H-operator if and only if the set of its values E_A can be described by a finite system of functionals.

THEOREM 4.2. *Let X, Y, S be linear spaces. Let Ξ, H, Σ be the total spaces of functionals determined on spaces X, Y, S respectively. Let B be a Φ_H-operator mapping the linear space X into Y. Let A be a Φ_Σ-operator mapping Y into S. Let $HA \subset \Sigma$. Then the superposition AB is a Φ_Σ-operator and*

$$\varkappa_{AB}^{\Sigma} = \varkappa_A^{\Sigma} + \varkappa_B^{H}. \tag{4.1}$$

Proof. Let Y be decomposed in a direct sum of type (1.4), i. e.

$$Y = E_B \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_3.$$

Since B is a Φ_H-operator, then the space E_B can be described by a system of r functionals, where $r = \dim(\mathfrak{C}_2 \oplus \mathfrak{C}_3)$. Hence the space $E_B \oplus \mathfrak{C}_2$ can be described by a finite system of functionals ξ_i $(i = 1, 2, \ldots, n_3$, where $n_3 = \dim \mathfrak{C}_3)$. Since A is a Φ_Σ-operator, E_A can be described by a finite system of functionals η_i $(i = 1, 2, \ldots, \beta_A)$. Basing ourselves on (1.5)

$$E_A = E_{AB} \oplus A\mathfrak{C}_3$$

and on the fact that $HA \subset \Sigma$, we find that E_{AB} can be described by the system of functionals:

$$\eta_1, \ldots, \eta_{\beta_A}, \quad \xi_1 A, \ldots, \xi_{n_3} A \in \Sigma.$$

Therefore AB is a Φ_Σ-operator and $\varkappa_{AB}^{\Sigma} = \varkappa_{AB}$ and, by theorem 1.1, we obtain the formula (4.1), q. e. d.

The following theorem is in some measure inverse to theorem 4.2:

THEOREM 4.3. *Let an operator A map the space X into the space Y and let an operator B map Y into X. If the superpositions AB and BA are, respectively, a Φ_H-operator and a Φ_Ξ-operator, then the operators A and B are, respectively, a Φ_H-operator and a Φ_Ξ-operator.*

Proof. Theorem 1.2 implies that the operators A and B have a finite d-characteristic. From the assumption it follows that the space E_{AB} can be described by a finite system of functionals. Hence, basing ourselves on formula (1.5)

$$E_A = E_{AB} \oplus A\mathfrak{C}_3,$$

we can described E_A by a finite system of functionals. This implies that $\beta_A = \beta_A^H$ and A is a Φ_H-operator. For the operator B the proof is similar, q. e. d.

§ 5. Regularization. General properties. Let $\mathscr{X}_0$ be a ring of linear operators mapping the linear space X into itself. Let $\mathscr{J} \subset \mathscr{X}_0$ be a two-side ideal. We shall assume that $\mathscr{J}$ is a proper ideal, i. e. $\mathscr{J} \neq \mathscr{X}_0$.

For example, all finite dimensional operators contained in $\mathscr{X}_0$ state a two-side ideal. We shall denote this ideal by $\mathscr{K}$.

If for an operator A there is such an operator $R_A \in \mathscr{X}_0$ that

$$R_A A = I + T \quad (\text{or } AR_A = I + T), \quad \text{where} \quad T \in \mathscr{J},$$

then this operator R_A will be called a *left-side* (or *right-side*) *regularizer* of the operator A to the ideal $\mathscr{J}$. If the regularizer is simultaneously left-side and right-side, then it will be called a *simple regularizer*.

Obviously a (left-side or right-side) regularizer cannot belong to $\mathscr{J}$. Indeed, if $R_A \in \mathscr{J}$, then $R_A A = I + T \in \mathscr{J}$ and $T \in \mathscr{J}$ hence $I \in \mathscr{J}$ and we obtain a contradiction because $\mathscr{J}$ is proper.

PROPOSITION 5.1. *If an operator $A \in \mathscr{X}_0$ possesses left-side regularizer R_1 and a right-side regularizer R_2 to the ideal $\mathscr{J}$, then $R_1 - R_2 \in \mathscr{J}$.*

Proof. Indeed, $R_1 A = I + T_1$, $AR_2 = I + T_2$, where $T_1, T_2 \in \mathscr{J}$. Therefore $R_1(AR_2) = R_1(I + T_2) = R_1 + R_1 T_2$, $(R_1 A) R_2 = (I + T_1) R_2 = R_2 + T_1 R_2$ and subtracting these equalities we obtain $R_1 - R_2 = T_1 R_2 - R_1 T_2 \in \mathscr{J}$, q. e. d.

PROPOSITION 5.2. *If an operator A possesses a left-side regularizer R_1 and a right-side regularizer R_2 to ideal $\mathscr{J}$, then each of them is simple.*

Proof. Proposition 5.1 implies that $R_1 - R_2 = T \in \mathscr{J}$; therefore $R_1 = R_2 + T$ and

$$AR_1 = A(R_2 + T) = AR_2 + AT = I + T_2 + AT,$$

but $T_2 + AT \in \mathscr{J}$, whence R_1 is also a right-side regularizer; therefore it is simple. Similarly we can show that R_2 is a left-side regularizer.

PROPOSITION 5.3. *A simple regularizer to ideal $\mathscr{J}$ is uniquely determined with respect to a component belonging to $\mathscr{J}$.*

Proof. Suppose that an operator A has two different simple regularizers to the ideal $\mathscr{J}$. They will be denoted by R_1 and R_2. Obviously R_1 is a right-side regularizer and R_2 is a left-side regularizer. The proposition 5.1 implies that $R_1 - R_2 \in \mathscr{J}$, q. e. d.

PROPOSITION 5.4. *If an operator $A \in \mathscr{X}_0$ is the sum $A = B + T$ of an operator B invertible on the whole space X and $T \in \mathscr{J} \subset \mathscr{X}_0$, then the operator A possesses a simple regularizer $R_A = B^{-1}$ to the ideal $\mathscr{J}$. Inversely, if an operator A possesses an invertible simple regularizer R_A to the ideal $\mathscr{J}$, then $A = R_A^{-1} + T$, where $T \in \mathscr{J}$.*

Proof. Indeed, if $A = B + T$ and the operator B is invertible, then

$$B^{-1}A = I + B^{-1}T, \quad AB^{-1} = I + TB^{-1}$$

and $B^{-1}\epsilon\mathcal{J}$, $TB^{-1}\epsilon\mathcal{J}$; therefore B^{-1} is a simple regularizer to ideal $\mathcal{J}$. Inversely, if a regularizer R_A is invertible, then, basing ourselves on the fact that

$$R_A A = I+T, \quad \text{where} \quad T\epsilon\mathcal{J},$$

we obtain $A = R_A^{-1}+R_A^{-1}T$, where $R_A^{-1}T\epsilon\mathcal{J}$, q. e. d.

PROPOSITION 5.5. *If operators A and B possess left-side (or right-side) regularizers R_A and R_B to ideal $\mathcal{J}$, then a left-side (resp. right-side) regularizer of the superposition AB to the ideal $\mathcal{J}$ exists and $R_{AB} = R_B R_A$.*

Proof. Let

$$R_A A = I+T_1, \quad R_B B = I+T_2, \quad \text{where} \quad T_1, T_2\epsilon\mathcal{J}.$$

Then

$$R_B R_A AB = R_B(I+T_1)B = R_B B+R_B T_1 B = I+T_2+R_B T_1 B$$

but $T_2+R_B T_1 B\epsilon\mathcal{J}$, whence $R_B R_A$ is a left-side regularizer of the operator AB to the ideal $\mathcal{J}$. We can prove the same for right-side ideals, q. e. d.

Let Ξ be the total space of linear functionals ξ determined on a linear space X. Let the conjugate operators to the operators belonging to $\mathcal{X}_0$ transform Ξ into itself. The operators conjugate to the operators $T\epsilon\mathcal{X}_0$ constitute a ring, which will be denoted by $\Xi\mathcal{X}_0$. The operators conjugate to the operators $T\epsilon\mathcal{J}\subset\mathcal{X}_0$ constitute an ideal which will be denoted by $\Xi\mathcal{J}$.

PROPOSITION 5.6. *If R_A is a left-side (or right-side) regularizer of an operator A to the ideal $\mathcal{J}$, then R_A' is a right-side (resp. left-side) regularizer of the conjugate operator A' to the ideal $\Xi\mathcal{J}$.*

Proof. We have $A'R_A' = (R_A A)' = (I+T)' = I+T'$, where $T'\epsilon\Xi\mathcal{J}$, q. e. d.

PROPOSITION 5.7. *Let the ideal $\mathcal{J}$ be such that $I+T$ has a finite d-characteristic for each $T\epsilon\mathcal{J}$. If an operator A mapping X into itself possesses a simple regularizer to the ideal $\mathcal{J}$, then the operator A has a finite d-characteristic.*

This follows immediately from the theorem 1.2.

An ideal $\mathcal{J}$ of operators is called *Ξ-quasifredholm ideal*, if for each $T\epsilon\mathcal{J}$ the operator $I+T$ is a Φ_Ξ-operator.

PROPOSITION 5.8. *If an operator A possesses a simple regularizer R_A to a Ξ-quasifredholm ideal $\mathcal{J}$, then the operator A is a Φ_Ξ-operator.*

Proof. Obviously $R_A A$ and AR_A are Φ_Ξ-operators; then theorem 5.3 implies that A and R_A are Φ_Ξ-operators.

§ 6. Regularization of operators with a finite d-characteristic to the ideal of finite dimensional operators. Applying proposition 5.8 we find that if an operator A possesses a simple regularizer to the ideal $\mathcal{K}$ of finite dimensional operators, then it has a finite d-characteristic. The following theorems solve the inverse problem:

THEOREM 6.1. *If the nullity of an operator A is finite, then this operator possesses a left-side regularizer to the ideal $\mathcal{K}$ of finite dimensional operators.*

Proof. Let X decompose into a direct sum: $X = Z_A\oplus\mathfrak{C}$. We determine the operator K in the following way:

$$Kx = \begin{cases} x & \text{for} \quad x\epsilon Z_A, \\ 0 & \text{for} \quad x\epsilon\mathfrak{C}. \end{cases}$$

Obviously K is a finite dimensional projective operator. We write

$$y = (I-K)x. \tag{6.1}$$

From the definition of the operator K, $y\epsilon\mathfrak{C}$ and $z = x-y\epsilon Z_A$. The restriction A_1 of the operator A to the space $\mathfrak{C}$ is invertible. We denote the inverse operator by A_1^{-1}, and by $\tilde{A}_1^{-1}$ we denote an arbitrary extension of this operator onto the whole space X. Since $y\epsilon\mathfrak{C}$, then

$$y = \tilde{A}_1^{-1}Ay = \tilde{A}_1^{-1}(Ay+Az) = \tilde{A}_1^{-1}A(y+z) = \tilde{A}_1^{-1}Ax. \tag{6.2}$$

Therefore

$$(I-K)x = (I-K)\tilde{A}_1^{-1}Ax,$$

whence the operator $R_A = (I-K)\tilde{A}_1^{-1}$ is a left-side regularizer or the operator A to the ideal $\mathcal{K}$ of finite dimensional operators, q. e. d.

THEOREM 6.2. *If a linear operator A has a finite d-characteristic, then it possesses a simple regularizer to the ideal $\mathcal{K}$ of finite dimensional operators.*

Proof. In theorem 6.1 we showed that the operator $R_A = (I-K)\tilde{A}_1^{-1}$ is a left-side regularizer of the operator A to the ideal $\mathcal{K}$. Obviously the operators $\tilde{A}_1^{-1}$ and $(I-K)$ have a finite d-characteristic, whence the operator R_A as its superposition has a finite d-characteristic. From theorem 6.1 the operator R_A possesses a left-side regularizer R_1 to the ideal $\mathcal{K}$. whence $R_1R_A = I+K_1$, whence $K_1\epsilon\mathcal{K}$ but since $R_A A = I-K$, we have

$$R_1(R_A A) = R_1(I-K) = R_1-R_1K, \qquad (R_1R_A)A = (I+K_1)A = A+K_1A$$

and subtracting these equalities by sides we obtain $R_1-A = R_1K+ +K_1A = K_2\epsilon\mathcal{K}$. Therefore

$$AR_A = (R_1-K_2)R_A = R_1R_A-K_2R_A = I+K_1-K_2R_A,$$

where $K_1-K_2R_A\epsilon\mathcal{K}$. This implies that the operator R_A is a simple regularizer to the ideal $\mathcal{K}$ of the operator A, q. e. d.

COROLLARY. $\varkappa_{R_A} = -\varkappa_A$.

Indeed, $\varkappa_{R_A}+\varkappa_A = \varkappa_{R_AA} = \varkappa_{I-K} = 0$.

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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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On a generalization of regularly increasing functions

by

W. MATUSZEWSKA (Poznań)

1. In this section we shall denote by $f, g, h, \ldots$ real functions defined and non-decreasing in $(-\infty, \infty)$. The following notation will be used:

$$\bar{\varrho}_f(\mu) = \limsup_{u\to\infty}\big(f(u+\mu)-f(u)\big),$$

$$\underline{\varrho}_f(\mu) = \liminf_{u\to\infty}\big(f(u+\mu)-f(u)\big).$$

We denote by C_0 the space whose elements are functions $\mu(\cdot)$ continuous in $(-\infty, \infty)$ and converging to 0 with $u\to\infty$ and to a finite limit with $u\to-\infty$. Equipped with the usual metric defined by $d\big(\mu_1(\cdot), \mu_2(\cdot)\big) = \|\mu_1(\cdot)-\mu_2(\cdot)\|$, where $\|\mu(\cdot)\| = \sup_{-\infty\leqslant t<\infty}|\mu(t)|$ for $\mu(\cdot)\in C_0$, C_0 is a complete metric space. We write $\mu(\cdot)\in C_0^+$ if $\mu(\cdot)\in C_0$ and $\mu(u)>0$ everywhere.

The aim of section 1 is to present some lemmas the use of which simplifies the proofs of the theorems given further in section 3 and 4.

1.1. *The following equalities are satisfied for any function f:*

$$(*)\quad \lim_{\mu\to 0+}\frac{\bar{\varrho}_f(\mu)}{\mu} = \sup_{\mu>0}\frac{\bar{\varrho}_f(\mu)}{\mu}, \qquad (**)\quad \lim_{\mu\to\infty}\frac{\bar{\varrho}_f(\mu)}{\mu} = \inf_{\mu>0}\frac{\bar{\varrho}_f(\mu)}{\mu};$$

$$(+)\quad \lim_{\mu\to 0+}\frac{\underline{\varrho}_f(\mu)}{\mu} = \inf_{\mu>0}\frac{\underline{\varrho}_f(\mu)}{\mu}, \qquad (++)\quad \lim_{\mu\to\infty}\frac{\underline{\varrho}_f(\mu)}{\mu} = \sup_{\mu>0}\frac{\underline{\varrho}_f(\mu)}{\mu}.$$

The proofs of (**), (++) can be found in [2], the proofs of (*), (+) run on the same lines.

1.2. *If for any $\mu(\cdot)$ in C_0^+ there exists a limit*

$$(+)\qquad \lim_{u\to\infty}\big[f\big(u+\mu(u)\big)-f(u)\big] = g\big(\mu(\cdot)\big),$$

then

(a) *$g\big(\mu(\cdot)\big)=0$ for any $\mu(\cdot)\in C_0$;*

(b) *for any $\varepsilon>0$ there exist a $\delta>0$ and u_0 such that the inequality*

$$|f(u+\mu)-f(u)|<\varepsilon$$

holds for $|\mu|\leqslant\delta$ and $u\geqslant u_0$.