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A topology for Mikusiński operators*

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1. Introduction

The class of continuous complex-valued functions of a non-negative real variable forms a commutative algebra without zero divisors where the product is defined as the finite convolution and sums and scalar products are defined in the usual way. Mikusiński [2] calls the quotient field of this algebra the *operator field*. In this operator field he introduces concepts for limits, continuity, differentiation and integration, and applies the resulting theory to the study of differential equations.

However, his theory suffers from two defects which he has noted. First, Urbanik [6] has shown that the definition of operational convergence given by Mikusiński is topologically inadequate. Second, Mikusiński's operational calculus applies only to functions defined for non-negative reals. This creates an unpleasing situation with regard to left shifts or translates of a function. The right translate of a function is always a function, whereas this is not always true for the left translate of a function.

It is our purpose to develop an algebra of operators in which these defects are eliminated. In order to overcome the problem associated with left translates of a function we will start with the continuous complex-valued functions of a real variable, each of which is zero to the left of some point which depends upon the function itself. In fact, Mikusiński [2], p. 124, suggests that this should be done himself. This class will form a commutative algebra without zero divisors where product is taken to be convolution and sums and scalar products are defined in the usual way. We select from the quotient field of this algebra a subalgebra which contains the operators needed for application.

Furthermore, we will topologize this subalgebra in a way which yields satisfactory concepts for limits, continuity and differentiation; satisfactory in that the concepts used by Mikusiński when applied to this subalgebra are compatible with our topological concepts.

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2. Preliminaries

2.1. We denote by C the set of all complex-valued continuous functions of a real variable with the property that for every c in C there is a real number σ such that $c(t) = 0$ for all $t < \sigma$. The supremum of these σ is denoted by $\sigma(c)$, if c is not the zero function. In the case of the zero function we use $\sigma(0) = +\infty$. Addition in C and multiplication of an element in C by a complex number is defined in the usual way. We define the product of two elements a and b in C to be

$$(a*b)(t) = \int_{-\infty}^{+\infty} a(x)b(t-x)\,dx.$$

It is readily seen that C is closed under these operations. Furthermore, for real numbers $r \leqslant \sigma(a)$ and $s \leqslant \sigma(b)$ it is clear that

$$(a*b)(t) = \int_{r}^{t-s} a(x)b(t-x)\,dx.$$

We denote by C_0 the subset of C consisting of those elements c for which $\sigma(c) \geqslant 0$. Clearly, C_0 is closed under the algebraic operations given above. Let a and b belong to C. Suppose r and s are any two numbers such that $r \leqslant \sigma(a)$ and $s \leqslant \sigma(b)$. Then $a_1(t) = a(t+r)$ and $b_1(t) = b(t+s)$ belong to C_0 and $(a*b)(t) = (a_1*b_1)(t-r-s)$.

2.2. LEMMA. *The real valued function $\sigma(c)$ has the following properties:*

(i) *If c belongs to C and r is a real number, then $c_1(t) = c(t+r)$ is in C and $\sigma(c_1) = \sigma(c)-r$.*

(ii) *If c is a function whose derivative c' belongs to C, then $\sigma(c') = \sigma(c)$.*

(iii) *If c belongs to C and α is a non-zero complex number, then $\sigma(\alpha c) = \sigma(c)$.*

(iv) *If a and b belong to C, then $\sigma(a+b) \geqslant \min(\sigma(a), \sigma(b))$.*

(v) *If a and b belong to C, then $\sigma(a*b) = \sigma(a)+\sigma(b)$.*

Proof. Only the proof of (v) is non-trivial. If either a or b is the zero function then (v) is obviously true. If neither is the zero function we may choose $r = \sigma(a)$, $s = \sigma(b)$ and set $a_1(t) = a(t+r)$, $b_1(t) = b(t+s)$. We have observed that $(a*b)(t) = (a_1*b_1)(t-r-s)$. Thus $\sigma(a*b) \geqslant \sigma(a)+\sigma(b)$ since $\sigma(a_1*b_1) \geqslant 0$. On the other hand, assume that $(a_1*b_1)(x) = 0$ for $0 \leqslant x \leqslant t-r-s$ where $t > r+s$. By a theorem of Titchmarsh ([5], p. 324) there are non-negative numbers α and β such that $\alpha+\beta = t-r-s$ and $a_1(x) = 0$ on $0 \leqslant x \leqslant \alpha$, $b_1(x) = 0$ on $0 \leqslant x \leqslant \beta$. But, $\sigma(a_1) = \sigma(b_1) = 0$ by (i) above, so it follows that $\alpha = \beta = 0$ and $t = r+s$ contrary to assumption. Thus $(a_1*b_1)(t-r-s)$ is not identically zero in any neighborhood of zero, — that is, $(a*b)(t)$ is not identically zero in any neighborhood of $\sigma(a)+\sigma(b)$ — so that $\sigma(a*b) \leqslant \sigma(a)+\sigma(b)$ from which the desired conclusion follows.

2.3. We shall let L denote the set of all complex-valued functions of a real variable which satisfy (i) if f belongs to L, then there is a real number σ such that $f(t) = 0$ for all $t \leqslant \sigma$ and (ii) if f belongs to L, then f is Lebesgue integrable on every interval $-T \leqslant t \leqslant T$. The algebraic operations defined on C can also be defined on L. Furthermore, L is closed under these operations. We shall let D denote the subclass of C consisting of infinitely differentiable functions. D is a subalgebra of C. We shall have occasion to make use of the fact that $D*L \subset D$ and $C*L \subset C$. Mikusiński and Ryll-Nardzewski [3] have proved these results for functions defined on the non-negative reals. Their results carry over immediately to our case with use of the observation made at the end of part 2.1.

3. General theory

3.1. We will now investigate the problem of introducing a topology on a subalgebra of the quotient field of an integral domain when the integral domain is already topologized. This will serve as the theoretical basis for our work with Mikusiński operators. We shall let A denote a complex commutative algebra without zero divisors which has a topology generated by a family of seminorms $\{p(\cdot\,; n) : n \in N \text{ an index set}\}$ such that $p(a; n) = 0$ for every n if and only if $a = 0$. We shall call this the *A-topology* for A. Let Q denote the quotient field of A and let S denote a subalgebra of Q. Let $\omega\colon A \to Q$ be the natural embedding of A in Q. We can provide ωA with a topology homeomorphic to the A-topology be merely setting $p(\omega a; n) = p(a; n)$. We shall call this the *ωA-topology* for ωA. We shall be particularly interested in the case when ωA is contained in S.

3.2. Definition. Let $\omega A \subset S$. Let F denote a family of linear maps from S into ωA such that

(i) F distinguishes points,

(ii) if a and b belong to S and f belongs to F, then $f(a*b) = a*f(b)$. Here $*$ denotes the multiplication operation in Q. We shall also write $f_a(b) = a*f(b)$,

(iii) f_a belongs to F for every a in S and every f in F. We say, in this case, F is *multiplicative on S*.

3.3. Let $p(x; f, n) = p(f(x); n)$ where x belongs to S, f belongs to F, and n belongs to N. It is easy to verify that each $p(\cdot\,; f, n)$ is a seminorm on S. This collection of seminorms generates a topology for S which we shall call the *S-topology* for S. It is well-known that S, with this topology, is a locally convex linear topological space. Since F distinguishes points, the S-topology will be Hausdorff. Furthermore, the S-topology is the smallest topology for which each f in F is continuous. S has a local subbase at zero consisting of sets of the form $V(f,n,\varepsilon) = \{x \in S : p(x;f,n) < \varepsilon\}$. These results can all be found in [4].

If $\{a_k\}$ is a net ([1], p. 65) in S, then $\lim a_k = a$ in the S-topology if and only if $\lim f(a_k) = f(a)$ in the ωA-topology for every f in F. The necessity follows immediately from the continuity of each f. Since $V(f,n,\varepsilon) = f^{-1}(V(n,\varepsilon))$ where $V(n,\varepsilon) = \{x \in \omega A : p(x;n) < \varepsilon\}$, the sufficiency is also clear.

3.4. THEOREM. *Let F be multiplicative on S; then products are separately continuous in the S-topology.*

Proof. Since S is a linear topological space, it suffices to show that for each neighborhood V of zero in the S-topology and every a in S, there is a neighborhood U of zero in the S-topology such that $a * U \subset V$. Let V be given. Then there is a basic neighborhood $V(f_1, n_1, \varepsilon_1) \cap \ldots \cap V(f_k, n_k, \varepsilon_k) \subset V$. Let $U = V((f_1)_a, n_1, \varepsilon_1) \cap \ldots \cap V((f_k)_a, n_k, \varepsilon_k)$. If u is in U, then $p(a*u; f_j, n_j) = p(a*f_j(u); n_j) = p((f_j)_a(u); n_j) = p(u; (f_j)_a, n_j) < \varepsilon_j$. Therefore, $a*u$ is in V.

4. The topology

4.0. We shall now use the theory developed in section 3 in order to topologize a certain subalgebra S of Q, the quotient field of C. With the aid of the remark at the end of 2.1 it is not difficult to show that C is a complex commutative algebra without zero divisors. We give C the topology of compact convergence through the family of seminorms $p(c;n) = \sup\{|c(t)| : -\infty < t \leqslant n,\ c \text{ belongs to } C\}$. Clearly, $p(c;n) = 0$ for all n if and only if $c = 0$.

4.1. Definition. (i) $S = \{a \in Q : a * \omega D \subset \omega D\}$. Here ωD is the natural embedding of D (infinitely differentiable functions in C) in Q.

(ii) Let ωd belong to ωD. $\omega d: S \to \omega C$ such that $\omega d(a) = \omega d * a$. Thus, we interpret ωD as a family of maps from S into ωC as well as a subalgebra of ωC.

4.2. It can be shown that S is a subalgebra of Q such that ωC is contained in S. Furthermore, the complex field is isomorphically embedded in S. It can also be shown that all the elements in L (see 2.3) have a representation in S. The Heaviside unit step function l behaves as an integral operator in S just as it does in Mikusiński's theory. We denote the inverse of l by s (as does Mikusiński) and since the elements of ωD are infinitely differentiable, s is in S. Of course, it is the differentiation operator. Let

$$H_a(t) = \begin{cases} 0 & \text{for} \quad t < a, \\ 1 & \text{for} \quad t \geqslant a. \end{cases}$$

We shall let H_a denote the representation of $H_a(t)$ in S. Then, as in Mikusiński's theory $s * H_a = h^a$ is the translation operator. Now, however, if a is negative we merely translate functions to the left. That is, the left translate of a function is now always a function.

4.3. It can be easily verified that ωD is multiplicative on S and therefore S has a locally convex, Hausdorff topology in which products are separately continuous where the seminorms on S are defined by $p(a; \omega d, n) = p(\omega d * a; n)$. A useful criterion for establishing convergence in the S-topology was given in 3.3; i. e., if $\{a_k\}$ is a net in S, then $\lim a_k = a$ in the S-topology if and only if $\lim \omega d * a_k = \omega d * a$ in the ωC-topology for every ωd in ωD.

4.4. Definition. Let P denote a subvector space of C. Convergence in S is *P-compatible with convergence in C* (with the C-topology) if ω restricted to P is continuous from P into S. If P is all of C we shall merely say *convergence in S is compatible with convergence in C*.

4.5. We shall now give an example to show that convergence in S is not compatible with convergence in C. In the next theorem we will give a collection of subspaces of C for which convergence in S is compatible.

Let

$$c_k(t) = \begin{cases} 0 & \text{for} \quad t \leqslant -k, \\ (1/k)\exp(-1/(t+k)) & \text{for} \quad t > -k. \end{cases}$$

Clearly, $\{c_k\}$ is in C and $\lim c_k = 0$ in the C-topology. We know (see 4.3) that $\lim \omega c_k = 0$ in the S-topology if and only if $\lim \omega c_k * \omega d = 0$ in the ωC-topology for every ωd in ωD. Because ωC is homeomorphic to C, this is equivalent to requiring $\lim c_k * d = 0$ in the C-topology for every d in D. Let

$$T(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ e^{-1/t} & \text{for} \quad t > 0. \end{cases}$$

Now T belongs to D and we will show $\lim c_k * T \neq 0$ in the C-topology. From the monotone character of $T(t)$ and each $c_k(t)$ it is clear that

$$p(c_k * T; 2) \geqslant (1/k) \int_{-k}^{2} \exp(-1/(x+k)) \exp(-1/(2-x))\, dx$$

$$\geqslant (1/k) \int_{1-k}^{1} e^{-2}\, dx = e^{-2} \quad \text{for all } k \geqslant 1.$$

Thus, $\lim c_k * T \neq 0$ in the C-topology.

4.6. THEOREM. *Let P denote the set of all c in C for which $\sigma(c) \geqslant \tau$. Convergence in S is P-compatible with convergence in C.*

Proof. It is clear that P is a subvector space of C. Let $V(\omega d; n, \varepsilon)$ be given (see 3.3). If $n \leqslant \tau+\sigma(d)$, then ωc belongs to $V(\omega d, n, \varepsilon)$ for all c in P for $p(\omega c; \omega d, n) = 0$ since $\sigma(c*d) = \tau+\sigma(d) \geqslant n$. Suppose $n > \tau+\sigma(d)$. Let $n^* = \max(n, n-\sigma(d), n-\tau)$. Let $M = p(d; n^*)$ and let

$$U = \left\{c \in P: p(c; n^*) < \frac{\varepsilon}{M(n-\sigma(d)-\tau)}\right\}.$$

Then ωU is contained in V and ω restricted to P is thus continuous.

5. Relationship between M-convergence and convergence in the S-topology

5.1. Definition. Let $\{a_k\} \subset Q$ and let a belong to Q. Suppose there is a non-zero b in ωC such that:

(i) $\{b*a_k\} \subset \omega C$ and $b*a$ belongs to ωC,

(ii) $\sigma(b*a_k) \geqslant \tau$ for all k,

(iii) $\lim b*a_k = b*a$ in the ωC-topology.

Then we say $\lim a_k = a$ *in the sense of Mikusiński*, and we shall call b a *regularization factor* for the sequence.

This is a slight modification of Mikusiński's definition of convergence ([2], p. 144) so that his definition fits our set-up and terminology. Mikusiński does not require the regularization factor to be in ωC, but this can be shown to be no restriction on the definition. Furthermore, he does not require (ii) but for functions defined on the positive reals this is automatically satisfied. The example in 4.5 shows that such a restriction is necessary.

In the next theorem the connection between M-convergence and convergence in the S-topology will be shown.

If $\{a_k\}$ and $\{b_k\}$ are sequences in C such that (i) $\sigma(a_k) \geqslant \tau$, $\sigma(b_k) \geqslant \tau$ for some real τ, and (ii) $\lim a_k = a$, $\lim b_k = b$ in the C-topology, then $\lim a_k*b_k = a*b$ in the C-topology. Mikusiński [2] proves this result when $\sigma(a_k) \geqslant 0$ and $\sigma(b_k) \geqslant 0$. Using the remark at the end of 2.1 it is easy to prove the result for any τ.

5.2. Theorem. *Let $\{a_k\}$ and $\{b_k\}$ be two sequences in S which converge in the sense of Mikusiński to a and b in S. Let g and h be their respective regularization factors and assume g^{-1} and h^{-1} are in S. Then,*

(i) $\lim a_k = a$ *in the S-topology,*

(ii) $\lim(a_k \pm b_k) = a \pm b$ *in the S-topology,*

(iii) $\lim a_k*b_k = a*b$ *in the S-topology,*

(iv) $\lim \alpha_k*a_k = \alpha*a$ *in the S-topology where $\{\alpha_k\}$ is the isomorphic image in S of a sequence of complexes which converge to α in the usual topology.*

Proof. (i) In 4.6 we showed that convergence in S is P-compatible with convergence in C, so $\lim g*a_k = g*a$ in the S-topology. Products are separately continuous in S and g^{-1} belongs to S, so $\lim a_k = a$ in the S-topology.

(ii) S is a linear topological space, so the result holds since the individual sequences converge in the S-topology by (i).

(iii) $\{(g*a_k)*(h*b_k)\}$ converges to $(g*a)*(h*b)$ in the ωC-topology. $g*h$ is a regularization factor for the sequence $\{a_k*b_k\}$ and this sequence satisfies the conditions for M-convergence, so by (1) above $\lim a_k*b_k = a*b$ in the S-topology.

(iv) S is a linear topological space, so products by scalars are continuous.

6. Relationship between M-continuity and continuity of operational functions in the S-topology

6.1. Let Z denote the class of functions whose domain is a subset of the reals and whose range is in S. If f belongs to Z, then $f(\alpha)*d$ can be interpreted as a complex-valued function of two variables. We want to emphasize this interpretation. Therefore, when considering $f(\alpha)*d$ as a function of two variables we will write $f_d(\alpha, t)$. We shall have need to consider other functions of two variables which we shall denote by $f(\alpha, t)$. Such functions can often be interpreted as elements of Z, i. e. for each α, $f(\alpha; \cdot)$ is an element of C. Finally, we shall drop the cumbersome distinction between ωC and C since they are isomorphic and homeomorphic.

6.2. Theorem. *Suppose f belongs to Z. Then*

(i) $\lim f(\alpha) = L$ *as $\alpha \to \alpha_0$ in the S-topology if and only if* $\lim f(\alpha)*d = L*d$ *as $\alpha \to \alpha_0$ in the C-topology for every d in D.*

(ii) *If* $\lim f(\alpha) = L$ *as $\alpha \to \alpha_0$ in the S-topology, then* $\lim b*f(\alpha) = b*L$ *as $\alpha \to \alpha_0$ in the S-topology for every b in S.*

(iii) *f is continuous at $\alpha = \alpha_0$ if and only if, for each d in D,* $\lim f_d(\alpha, t) = f_d(\alpha_0, t)$ *as $\alpha \to \alpha_0$ uniformly in t on every interval $-\infty < t \leqslant n$.*

(iv) *If f is continuous on an interval I, then $b*f$ belongs to Z and is continuous on I for every b in S.*

Proof. (i) This follows from the remarks made at the end of 4.3.

(ii) This is an almost immediate consequence of the fact that products are separately continuous in the S-topology (see 3.4).

(iii) f is continuous at $\alpha = \alpha_0$ if and only if $\lim f(\alpha) = f(\alpha_0)$ as $\alpha \to \alpha_0$ in the S-topology, or equivalently, if and only if $\lim f(\alpha)*d = f(\alpha_0)*d$ in the C-topology for every d in D by (i) above. This happens if and only if for every $\varepsilon > 0$, d in D and integer n, there is a $\delta > 0$ such that $|a-a_0| < \delta$ implies $p(f(a)*d-f(a_0)*d; n) < \varepsilon$, or equivalently, $|f_d(a,t)-f_d(a_0,t)| < \varepsilon$ for $-\infty < t \leqslant n$ which is the desired result.

(iv) This is immediate from (ii) above.

6.3. LEMMA. *Let $f(a,t)$ be continuous in a and t. Suppose $f(a,\cdot)$ belongs to C for each a in a closed interval I and also suppose $\sigma(f(a,\cdot)) \geqslant \tau$ for some real τ and every a in I. Then for each g in C and a_0 in I it is true that* $\lim(g*f(a,\cdot))(t) = (g*f(a_0,\cdot))(t)$ *as $a \to a_0$ uniformly in t on every interval $-\infty < t \leqslant n$.*

Proof. Let $\varepsilon > 0$ and n be given. Without loss of generality assume $\sigma(g) > \tau$. If $n-\tau < \tau$, then the result is trivially true, so assume $n-\tau > \tau$. We have

$$|(g*f(a,\cdot))(t)-(g*f(a_0,\cdot))(t)| \leqslant \int_{\tau}^{t-\tau} |g(t-x)|\,|f(a,x)-f(a_0,x)|\,dx.$$

But, $f(a,x)$ is uniformly continuous on $I\times[\tau, n-\tau]$ and $g(t)$ is bounded on $[\tau, n-\tau]$, so the integral on the right can be made arbitrarily small simply by requiring a to be sufficiently close to a_0.

6.4. Definition. Let f belong to Z. f is *continuous in the sense of Mikusiński* if

(i) there is a continuous function $f_1(a,t)$ such that $f_1(a,\cdot)$ belongs to C for each a,

(ii) $\sigma(f(a,\cdot)) \geqslant \tau$ for some real τ and every a in an interval I,

(iii) there is a b in S such that $f(a) = b*f_1(a,\cdot)$ for every a in I, where $f_1(a,\cdot)$ is considered as an element of Z.

Note that (ii) is already inherent in Mikusiński's definition ([2], p. 180) since his functions are only defined on the non-negative reals. We can now easily establish the connection between continuity in Mikusiński's sense and continuity in the S-topology.

6.5. THEOREM. (i) *If f is continuous in the sense of Mikusiński, then f is continuous in the S-topology.*

(ii) *If $f(a,t)$ satisfies the conditions listed in Lemma* 6.3, *then $f(a,\cdot)$ is continuous in the S-topology.*

Proof. (i) Suppose f is continuous in the sense of Mikusiński. Then there is a b in S and $f_1(a,t)$ with the properties listed in 6.4. Then $\lim f_d(a,t) = \lim(d*b*f_1(a,\cdot))(t)$ as $a\to a_0$. But, $b*d$ belongs to D, say $b*d = d_1$, and by Lemma 6.3, $\lim(d_1*f_1(a,\cdot))(t) = (d_1*f_1(a_0,\cdot))(t)$ as $a\to a_0$ uniformly in t on $-\infty < t \leqslant n$. Thus, f is continuous in the S-topology, for $\lim f_d(a,t) = (f_1(a_0,\cdot)*d_1)(t) = (b*d*f(a_0,\cdot))(t) = f_d(a_0,t)$ as $a\to a_0$ uniformly in t on $-\infty < t\leqslant n$.

(ii) If $f(a,t)$ satisfies the conditions listed in Lemma 6.3, then $f(a,\cdot)$ is continuous in the sense of Mikusiński (choose $b = 1$), hence is continuous in the S-topology.

7. Relationship between M-differentiability and differentiability of operational functions in the S-topology

7.1. Definition. Let f belong to Z. Then

$$\frac{d}{da}f(a_0) = f'(a_0) = \lim_{a\to a_0}\frac{f(a)-f(a_0)}{a-a_0}$$

when this limit exists.

7.2. THEOREM. *Let f belong to Z. Suppose $f'(a_0)$ exists. Then*

$$\frac{\partial}{\partial a}f_d(a,t)_{a=a_0}$$

exists for all d in D.

Proof. We have

$$f'(a_0)*d = d*\lim_{a\to a_0}\frac{f(a)-f(a_0)}{a-a_0}$$

where d belongs to D. Since the limit exists by hypothesis and since products are separately continuous in S, we have

$$d*\lim_{a\to a_0}\frac{f(a)-f(a_0)}{a-a_0} = \lim_{a\to a_0}\frac{d*f(a)-d*f(a_0)}{a-a_0} = \lim_{a\to a_0}\frac{f_d(a,t)-f_d(a_0,t)}{a-a_0} = \frac{\partial}{\partial a}f_d(a,t)_{a=a_0}.$$

7.3. LEMMA. *Suppose $\frac{\partial}{\partial a}f(a,t)$ is continuous in a and t, where a belongs to an interval I and, for each a, $\frac{\partial}{\partial a}f(a,\cdot)$ belongs to C. Suppose*

$$\sigma\left(\frac{\partial}{\partial a}f(a,\cdot)\right)\geqslant\tau$$

for some real number τ and every a in I. Then $f(a,\cdot)$, considered as an element of Z, has a derivative with respect to a in the S-topology. This derivative $\frac{d}{da}f(a,\cdot)$ is equal to the function $\frac{\partial}{\partial a}f(a,\cdot)$ considered as an element of Z.

Proof. We must show

$$\lim_{\alpha\to\alpha_0}\frac{f(\alpha,\cdot)-f(\alpha_0,\cdot)}{\alpha-\alpha_0}=\frac{\partial}{\partial\alpha}f(\alpha,\cdot)_{\alpha=\alpha_0}$$

in the S-topology. This is true if and only if for every d in D we have

$$\lim\frac{[f(\alpha,\cdot)-f(\alpha_0,\cdot)]}{\alpha-\alpha_0}*d=\frac{\partial}{\partial\alpha}f(\alpha,\cdot)_{\alpha=\alpha_0}*d$$

in the C-topology. Let $\alpha-\alpha_0=h$. Then

$$\begin{aligned}\lim_{\alpha\to\alpha_0}\left[\frac{f(\alpha,\cdot)-f(\alpha_0,\cdot)}{\alpha-\alpha_0}\right]*d&=\lim_{\alpha\to\alpha_0}\frac{f(\alpha,\cdot)*d-f(\alpha_0,\cdot)*d}{\alpha-\alpha_0}\\&=\lim_{h\to 0}\int_{-\infty}^{+\infty}\left[\frac{f(\alpha_0+h,x)-f(\alpha_0,x)}{h}\right]d(t-x)\,dx\\&=\int_{-\infty}^{+\infty}\frac{\partial}{\partial\alpha}f(\alpha,x)_{\alpha=\alpha_0}d(t-x)\,dx\\&=\frac{\partial}{\partial\alpha}f(\alpha,\cdot)_{\alpha=\alpha_0}*d.\end{aligned}$$

7.4. Definition. Let f belong to Z. f is *differentiable in the sense of Mikusiński* if there is a continuous function $f_1(\alpha,t)$ such that $f_1(\alpha,\cdot)$ belongs to C and $\sigma(f_1(\alpha,\cdot))\geqslant\tau$ for some real τ and every α in an interval I; if $\frac{\partial}{\partial\alpha}f_1(\alpha,t)$ is continuous and if there is a b in C such that $f(\alpha)=b*f_1(\alpha,\cdot)$. We write $f'(\alpha)=b*\frac{\partial}{\partial\alpha}f_1(\alpha,\cdot)$.

Again note the slight modification of Mikusiński's definition ([2], p. 183) so that it fits our set-up.

7.5. THEOREM. *If f is differentiable in the sense of Mikusiński, then f has a derivative in the S-topology, and the values of the two derivatives are equal.*

Proof. Suppose f is differentiable in the sense of Mikusiński. Then we find a b in S and $f_1(\alpha,t)$ such that $f(\alpha)=b*f_1(\alpha,\cdot)$. Now in the S-topology

$$f'(\alpha_0)=\lim_{\alpha\to\alpha_0}\frac{f(\alpha)-f(\alpha_0)}{\alpha-\alpha_0}$$

if this limit exists. But, we can write

$$\lim_{\alpha\to\alpha_0}\frac{f(\alpha)-f(\alpha_0)}{\alpha-\alpha_0}=\lim_{\alpha\to\alpha_0}b*\left[\frac{f_1(\alpha,\cdot)-f_1(\alpha_0,\cdot)}{\alpha-\alpha_0}\right].$$

Products are separately continuous in the S-topology and with the aid of Lemma 7.3 we have

$$\lim_{\alpha\to\alpha_0}\frac{f(\alpha)-f(\alpha_0)}{\alpha-\alpha_0}=b*\frac{\partial}{\partial\alpha}f_1(\alpha,\cdot)_{\alpha=\alpha_0}$$

and the derivative exists in the S-topology.

References

[1] J. L. Kelley, *General topology*, New York 1955.

[2] J. G. Mikusiński, *Operational calculus*, 5th Ed., International Series of Monographs on Pure and Applied Mathematics 8, New York 1959.

[3] J. G. Mikusiński and C. Ryll-Nardzewski, *Sur le produit de composition*, Studia Mathematica 12 (1951), p. 52-57.

[4] A. E. Taylor, *Introduction to functional analysis*, New York 1958.

[5] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, 2nd Ed., Oxford-London 1948.

[6] K. Urbanik, *Sur la structure non-topologique du corps des opérateurs*, Studia Mathematica 14 (1953), p. 243-246.

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