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On higher gradients of harmonic functions

by

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Chapter I

1. Let $U(x) = U(x_1, x_2, \ldots, x_n)$ be a real-valued harmonic function defined in a domain D of the n-dimensional Euclidean space $(n \geqslant 2)$. Consider the norm $W(x)$ of the gradient of $U(x)$,

$$W(x) = |\text{grad}\, U| = \left\{\sum_{j=1}^{n}\left(\frac{\partial U}{\partial x_j}\right)^2\right\}^{1/2}.$$

It is a classical fact that $W(x)$ is subharmonic in D, and therefore $\{W(x)\}^p$ is also subharmonic for any $p \geqslant 1$. E. M. Stein and G. Weiss [3] established a remarkable fact that $\{W(x)\}^p$ is subharmonic in D for some values of p less than 1, more precisely, subharmonic for any

$$p \geqslant \frac{n-2}{n-1}. \tag{1.1.1}$$

The example $U(x) = \left(\sum x_j^2\right)^{-(n-2)/2}$ shows that the result is false for p less than $(n-2)/(n-1)$. The case $n = 2$ is, of course, classical if we interpret the result as the subharmonicity of $\log W$.

In this chapter we extend the Stein-Weiss result to higher gradients.

2. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be any multi-index of weight m, that is, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers and $m = |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. We write $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!$ and

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\ldots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Given any harmonic function $U(x)$ we consider its gradient of order m, that is, the set of all distinct derivatives of order m (arranged in any fixed way)

$$\text{grad}_m U(x) = \{D^\alpha U\}_{|\alpha|=m}$$

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and its norm

(1.2.1) $$W(x) = |\mathrm{grad}_m U| = \Big\{\sum_{|a|=m} (D^a U)^2 (a!)^{-1}\Big\}^{1/2}.$$

We have then the following result which for $m=1$ reduces to the one stated above:

THEOREM 1. *If $U(x) = U(x_1, x_2, \ldots, x_n)$ is harmonic, the function $\{W(x)\}^p = |\mathrm{grad}_m U|^p$ is subharmonic for*

(1.2.2) $$p \geqslant \frac{n-2}{m+n-2}.$$

We note that for fixed n the right-hand side here is arbitrarily small if m is large enough. If $n=2$ the result should again be interpreted as the subharmonicity of $\log W(x)$ (and also easily follows from classical facts).

3. The rest of the chapter will be devoted to the proof of Theorem 1. In this chapter (but not in the other two) we shall also use another notation for the derivatives of order m. If $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ is a multi-index *of m components* such that $1 \leqslant \beta_i \leqslant n$ for $i = 1, 2, \ldots, m$, we will write

$$U_\beta = \frac{\partial}{\partial x_{\beta_1}} \frac{\partial}{\partial x_{\beta_2}} \cdots \frac{\partial}{\partial x_{\beta_m}} U$$

and the set $\{U_\beta\}$ of such derivatives designate by a single letter u. The numbers m and n are kept fixed throughout. Thus u is a vector function with m^n components and the norm $|u|$ of u will be

$$\Big\{\sum U_\beta^2\Big\}^{1/2}.$$

In this sum each component $D^a U$ of $\mathrm{grad}_m U$ occurs exactly $m!/a!$ times so that

$$|u| = (m!)^{1/2} |\mathrm{grad}_m U|.$$

By u_i we shall denote the derivatives $\partial u/\partial x_i$ of the vector function u. Let

$$f = \sum U_\beta^2 = |u|^2,$$

where U is harmonic. We are interested in functions $\varphi(t)$, $t \geqslant 0$, increasing and such that $\varphi(f)$ is subharmonic. We may restrict ourselves to functions φ that are concave; hence $\varphi' \geqslant 0$, $\varphi'' \leqslant 0$. We begin by computing the Laplacian of $\varphi(f)$.

We have

$$\Delta\varphi(f) = \varphi''(f)|\mathrm{grad} f|^2 + \varphi'(f)\Delta f.$$

Clearly,

$$|\mathrm{grad} f|^2 = 4\sum_i (u\cdot u_i)^2,$$

where the dot designates scalar product, and

$$\Delta f = 2\sum_\beta [U_\beta \Delta U_\beta + |\mathrm{grad}\, U_\beta|^2] = 2\sum_{\beta, i}\Big(\frac{\partial U_\beta}{\partial x_i}\Big)^2 = 2\sum_i |u_i|^2.$$

Let now

(1.3.1) $$M = \max \frac{\sum_i (u\cdot u_i)^2}{|u|^2 \sum_i |u_i|^2}.$$

Since $\varphi'' \leqslant 0$, we have

(1.3.2) $$\begin{aligned}\Delta\varphi(f) &= \varphi''(f) 4\sum_i (u\cdot u_i)^2 + \varphi'(f) 2\sum_i |u_i|^2 \\ &\geqslant \varphi''(f) 4M|u|^2 \sum |u_i|^2 + 2\varphi'(f)\sum |u_i|^2.\end{aligned}$$

It follows that if

(1.3.3) $$2Mt\varphi''(t) + \varphi'(t) \geqslant 0,$$

the function $\varphi(f)$ is subharmonic. In particular, we have subharmonicity if

$$2Mt\varphi''(t) + \varphi'(t) = 0,$$

which can be written $\varphi'(t) = Ct^{-1/2M}$. Here $C > 0$ since $\varphi' \geqslant 0$. Taking the second constant of integration 0, we find $\varphi(t) = C't^{1-1/2M}$, and the main problem now is finding the value of M. If we show that

(1.3.4) $$M = \frac{m+n-2}{2m+n-2},$$

the function $\varphi(f) = f^{(n-2)/2(m+n-2)} = |u|^{(n-2)/(m+n-2)}$ will be subharmonic, and Theorem 1 established. If $n=2$, then $M = \frac{1}{2}$ and the preceding argument leads to $\varphi(t) = \log t$.

4. In the lemma that follows the index β has the meaning explained in Section 3.

LEMMA 1. *Let $U(x) = U(x_1, x_2, \ldots, x_n)$ be a solid harmonic of degree m. Then*

$$|u|^2 = \sum_\beta U_\beta^2 = C \int_{|x|\leqslant 1} U^2(x)\,dx, \tag{1.4.1}$$

where the constant C depends on m and n only. More generally, if U and U' are any two solid harmonics of degree m, then

$$\sum_\beta U_\beta U'_\beta = C \int_{|x|\leqslant 1} U(x)\,U'(x)\,dx. \tag{1.4.2}$$

Proof. It is enough to prove (1.4.1). By homogeneity, we have

$$\int_{|x|\leqslant 1} U^2(x)\,dx = C \int_{|x|=1} U\frac{\partial U}{\partial \nu}\,d\sigma,$$

C denoting a constant depending on m and n only. On the other hand, Green's formula gives

$$\int_{|x|=1} U\frac{\partial U}{\partial \nu}\,d\sigma = \int_{|x|\leqslant 1} \sum_i \left(\frac{\partial U}{\partial x_i}\right)^2 dx,$$

so that

$$\int_{|x|\leqslant 1} U^2(x)\,dx = C \int_{|x|\leqslant 1} \sum_i \left(\frac{\partial U}{\partial x_i}\right)^2 dx,$$

with the same C. Successive application of this gives (1.4.1).

5. We now pass to the calculation of M.

Let $U(x)$ be a harmonic function and x_0 a point of its definition. Expand U in spherical harmonics at x_0. If V and W are the terms of the development of degrees m and $m+1$ respectively, then at the point x_0 the derivatives of order m of U are the same as the derivatives of order m of V, and the derivatives of order $m+1$ of U are the same as those of W. On account of (1.4.2) we have in (1.3.1)

$$u\cdot u_i = C \int_{|x|\leqslant 1} V\frac{\partial W}{\partial x_i}\,dx,$$

$$|u|^2 = C \int_{|x|\leqslant 1} V^2 dx,$$

$$\sum |u_i|^2 = C \int_{|x|\leqslant 1} \sum \left(\frac{\partial W}{\partial x_i}\right)^2 dx.$$

Hence M is simply the maximum of

$$I = \sum_{i=1}^n \left[\int_{|x|\leqslant 1} V\frac{\partial W}{\partial x_i}\,dx\right]^2 = \sum_{i=1}^n \left(V\cdot\frac{\partial W}{\partial x_i}\right)^2, \tag{1.5.1}$$

where V and W are all possible solid harmonics of degrees m and $m+1$ respectively, satisfying the conditions

$$\int_{|x|\leqslant 1} V^2 dx = \int_{|x|\leqslant 1} \sum \left(\frac{\partial W}{\partial x_i}\right)^2 dx = 1,$$

the dot product in (1.5.1) being the integral over $|x|\leqslant 1$ of product.

Let us now fix W and maximize with respect to V. Then

$$\tfrac{1}{2}\delta I = \left[\sum_{i=1}^n \left(V\cdot\frac{\partial W}{\partial x_i}\right)\frac{\partial W}{\partial x_i}\right]\cdot \delta V = 0,$$

provided

$$\delta(V\cdot V) = 2(V\cdot\delta V) = 0.$$

But this implies that

$$\sum_i \left(V\cdot\frac{\partial W}{\partial x_i}\right)\frac{\partial W}{\partial x_i} = \lambda V.$$

The largest value of λ is the maximum. Write $\xi_i = (V\cdot\partial W/\partial x_i)$ and multiply the last equation by $\partial W/\partial x_j$. We obtain

$$\sum_i \xi_i\left(\frac{\partial W}{\partial x_i}\cdot\frac{\partial W}{\partial x_j}\right) = \lambda\xi_j,$$

and if we now multiply by ξ_j and sum,

$$\sum_{i,j} \xi_i\xi_j\left(\frac{\partial W}{\partial x_i}\cdot\frac{\partial W}{\partial x_j}\right) = \lambda\sum_{i,j}\xi_j^2.$$

Now if we assume that the $\partial W/\partial x_i$ are linearly independent, then the quantities $t\xi_i$ are arbitrary. It follows that λ is simply the maximum of

$$\sum \xi_i\xi_j\left(\frac{\partial W}{\partial x_i}\cdot\frac{\partial W}{\partial x_j}\right) = \left(\sum \xi_i\frac{\partial W}{\partial x_i}\right)^2$$

with the condition that $\Sigma\xi_i^2 = 1$. Let us denote by ξ the unit vector with components ξ_i. Then

$$\lambda = \max_\xi \left(\frac{\partial W}{\partial \xi}\right)^2 = \max_\xi \left(\frac{\partial W}{\partial \xi}\cdot\frac{\partial W}{\partial \xi}\right),$$

and

$$M = \max_{\xi, W} \left(\frac{\partial W}{\partial \xi}\right)^2, \quad \sum \left(\frac{\partial W}{\partial x_i}\right)^2 = 1.$$

(We use here systematically the notation: $(F)^2 = (F \cdot F) =$ the integral of F^2 over $|x| \leqslant 1$.)

We have excluded sofar the W for which the derivatives $\partial W/\partial x_i$ are linearly dependent. But since any such W can be approximated in norm ($\|W\|^2 = (W)^2$) by a W with linearly independent derivatives, the maximum of I remains the same if one imposes on W the last condition.

Finally, since the space of W is rotation invariant we may replace ξ by any unit vector, so that

$$M = \max \left(\frac{\partial W}{\partial x_1}\right)^2 \quad \text{with} \quad \sum \left(\frac{\partial W}{\partial x_i}\right)^2 = 1.$$

Now

$$\sum \left(\frac{\partial W}{\partial x_i}\right)^2 = \int_{|x|=1} W \frac{\partial W}{\partial \nu} d\sigma = (m+1) \int_{|x|=1} W^2 d\sigma,$$

and

$$(W)^2 = \int_0^1 \varrho^{n-1+2m+2} d\varrho \left[\int_{|x|=1} W^2 d\sigma\right] = \frac{1}{n+2m+2} \int_{|x|=1} W^2 d\sigma.$$

It follows that

$$(W)^2 = (n+2m+2)^{-1}(m+1)^{-1}$$

or

$$M = \max \left(\frac{\partial W}{\partial x_1}\right)^2 \quad \text{with} \quad (W)^2 = (n+2m+2)^{-1}(m+1)^{-1},$$

and finally we arrive at the formula

$$M = \max_W \frac{(\partial W/\partial x_1)^2}{(m+1)(n+2m+2)(W)^2}. \tag{1.5.2}$$

6. Consider now a solid harmonic W of degree $m+1$. We have (see [1], p. 239)

$$W = \sum_{\mu=0}^{m+1} r^{m+1-\mu} C_{m+1-\mu}^{\mu+(n-2)/2}\left(\frac{x_1}{r}\right) H_\mu(x_2, \ldots, x_n), \tag{1.6.1}$$

where H_μ is a solid harmonic of degree μ in $x_2, \ldots, x_n$, C_k^ν is an ultraspherical polynomial defined by the equation

$$(1-2xt+t^2)^{-\nu} = \sum_{k=0}^{\infty} C_k^\nu(x) t^k,$$

and $r^2 = x_1^2 + x_2^2 + \ldots + x_n^2$.

We want to maximize $(\partial W/\partial x_1)^2/(W)^2$. We observe that the terms of the sum in (1.6.1) are orthogonal over the unit sphere $x_1^2+x_2^2+\ldots+x_n^2 = 1$, for if we fix x_1 and r the functions resulting from the terms are orthogonal over the sphere $x_2^2+\ldots+x_n^2 = r^2 - x_1^2$. If we differentiate with respect to x_1 we obtain a similar expression for $\partial W/\partial x_1$ whose terms will likewise be orthogonal. It follows from this that in order to maximize it is enough to assume that the right-hand side of (1.6.1) consists of a single term. For if we denote by W_μ the μ-th term in the sum (1.6.1), then

$$\frac{(\partial W/\partial x_1)^2}{(W)^2} = \frac{\sum_\mu (\partial W_\mu/\partial x_1)^2}{\sum_\mu (W_\mu)^2} \leqslant \max_\mu \frac{(\partial W_\mu/\partial x_1)^2}{(W_\mu)^2},$$

and if the maximum on the right is attained for $\mu = \mu_0$ and a suitable W_{μ_0}, then the left-hand side attains its maximum for W consisting of a single term W_{μ_0}.

We now fix μ and calculate $(\partial W_\mu/\partial x_1)^2$ and $(W_\mu)^2$. We have

$$(W_\mu)^2 = \int_{|x|\leqslant 1} r^{2m+2-2\mu}\left[C_{m+1-\mu}^{\mu+(n-2)/2}\left(\frac{x_1}{r}\right)\right]^2 H_\mu(x_2, \ldots, x_n)^2 dx.$$

We may assume that H_μ^2 is normalized; say, its integral over the surface of the unit sphere in the space $x_2, \ldots, x_n$ is equal to 1. Then

$$(W_\mu)^2 = \int_{-1}^{+1} dx_1 \int_0^{\sqrt{1-x_1^2}} r^{2m+2-\mu}\left[C_{m+1-\mu}^{\mu+(n-2)/2}\left(\frac{x_1}{r}\right)\right]^2 \varrho^{2\mu+n-2} d\varrho,$$

where $\varrho^2 + x_1^2 = r^2$, or setting $x_1 = r\cos\theta$, $\varrho = r\sin\theta$,

$$(W_\mu)^2 = \int_0^\pi d\theta \int_0^1 r^{2m+n+1}[C_{m+1-\mu}^{\mu+(n-2)/2}(\cos\theta)]^2 (\sin\theta)^{2\mu+n-2} dr$$

$$= \frac{1}{2m+n+2}\int_0^\pi [C_{m+1-\mu}^{\mu+(n-2)/2}(\cos\theta)]^2(\sin\theta)^{2\mu+n-2} d\theta.$$

Finally, substituting $\cos\theta = x$, we obtain

$$(1.6.2)\qquad (W_\mu)^2 = \frac{1}{2m+n+2}\int_{-1}^{+1}[C_{m+1-\mu}^{\mu+(n-2)/2}(x)]^2(1-x^2)^{\mu+(n-3)/2}\,dx.$$

7. From the definition of W_μ,

$$\frac{\partial W_\mu}{\partial x_1} = \left\{(m+1-\mu)\frac{x_1}{r}C_{m+1-\mu}^{\mu+(n-2)/2}(x_1/r) + \left[C_{m+1-\mu}^{\mu+(n-2)/2}\left(\frac{x_1}{r}\right)\right]'\left(1-\frac{x_1^2}{r^2}\right)\right\}r^{m-\mu}H_\mu.$$

But (see [1], p. 175, formula (15))

$$(1-x^2)[C_k^\nu(x)]' + kxC_k^\nu(x) = (k+2\nu-1)C_{k-1}^\nu(x),$$

which shows that

$$\frac{\partial W_\mu}{\partial x_1} = (m+\mu+n-2)r^{m-\mu}C_{m-\mu}^{\mu+(n-2)/2}\left(\frac{x_1}{r}\right)H_\mu(x_2, \ldots, x_n).$$

Comparing this with (1.6.2) we obtain

$$(1.7.1)\qquad \left(\frac{\partial W_\mu}{\partial x_1}\right)^2 = \frac{(m+\mu+n-2)^2}{2m+n}\int_{-1}^{+1}[C_{m-\mu}^{\mu+(n-2)/2}(x)]^2(1-x^2)^{\mu+(n-3)/2}\,dx.$$

From the classical formula (see [1], p. 236, formula (26))

$$J_k^\nu = \int_{-1}^{+1}[C_k^\nu(x)]^2(1-x^2)^{\nu-1/2}dx = \frac{2^{1-2\nu}\Gamma(k+2\nu)\pi}{k!(k+\nu)\Gamma(\nu)^2}$$

we deduce that

$$\frac{J_{k-1}^\nu}{J_k^\nu} = \frac{k(k+\nu)}{(k-1+\nu)(k-1+2\nu)}.$$

From this, (1.6.2), (1.7.1), (1.5.2) and also the observation that in computing $(\partial W/\partial x_1)^2/(W)^2$ we may restrict ourselves to the W's of the form W_μ, we obtain after elementary computation that

$$M = \max_{0\leqslant\mu\leqslant m+1}\frac{(m+\mu+n-2)(m+1-\mu)}{(m+1)(2m+n+2)}$$
$$= \max_{0\leqslant\mu\leqslant m+1}\frac{(m+1)^2-\mu^2+(n-3)[(m+1)-\mu]}{(m+1)(2m+n+2)}.$$

If $n\geqslant 3$ the maximum is clearly attained when $\mu=0$; if $n=2$ it is attained when $\mu=0$ or $\mu=1$. In all cases, therefore,

$$M = \frac{m+n-2}{2m+n-2}.$$

This proves the formula (1.3.4) and so establishes Theorem 1.

8. Theorem 1 asserts that, for any harmonic U and $u = \operatorname{grad}_m U$ the function $\psi_0(|u|)$ is subharmonic, where

$$\psi_0(t) = t^{p_0},\qquad p_0 = \frac{n-2}{m+n-2}\qquad (t\geqslant 0).$$

One may ask whether this is a best possible result. Of course, if $\omega(t)$ is convex and increasing, then the subharmonicity of $\psi_0(|u|)$ implies that of $\omega[\psi_0(|u|)]$, and a positive answer to our question is given by the following

THEOREM 2. *Suppose that $\psi(t)$ is continuous for $t\geqslant 0$ and $\psi(|u|)$, $u=\operatorname{grad}_m U$, is subharmonic for any harmonic U. Then $\psi(t)=\omega(t^{p_0})$, where $\omega(t)$ is increasing and convex and $p_0=(n-2)/(m+n-2)$. If $n=2$, we replace here t^{p_0} by* $\log t$.

Write $|u|^2 = f$, $\varphi(t) = \psi(t^{1/2})$. We have to show that $\psi(t)$ is a convex function of t^{p_0}, i. e., $\varphi(t)$ is a convex function of $t^{p_0/2}$. Suppose first that ψ is continuously twice differentiable for $t>0$. Take a fixed point x_0 and any U such that $|u|>0$, $\sum u_i^2>0$ at x_0, and denote by M_U the ratio under the max sign in (1.3.1). Since $\Delta\varphi(f)\geqslant 0$ at x_0, the first equation (1.3.2) shows that $2M_U f\varphi''(f)+\varphi'(f)\geqslant 0$. Taking the upper bound of M_U and observing that by replacing U by λU we may give f any preassigned positive value without changing M_U, we see that we have $2Mt\varphi''(t)+\varphi'(t)\geqslant 0$ for all $t>0$, a fact which as can be easily verified expresses the convexity of $\varphi(t)$ with respect to $t^{p_0/2}$.

To dispose of the hypothesis that ψ is twice differentiable for $t>0$ we use the method of regularization. Let $\{\chi_n(t)\}$ be a sequence of functions defined for $t\geqslant 0$, non-negative, in C'', satisfying the condition $\int_0^\infty\chi_n(t)\,dt = 1$ and having support shrinking to the point 1 as $n\to\infty$. Let

$$\psi_n(t) = \int_0^\infty\psi(st)\chi_n(s)\,ds.$$

The functions $\psi_n(t)$ are in C'' for $t>0$. Moreover, as easily seen, if $\psi(|u|)$ is subharmonic for all U, so is $\psi_n(|u|)$. Hence $\psi_n(t)$ is a convex function of t^{p_0} for $t>0$. But $\psi_n(t)$ tends to $\psi(t)$ for $t>0$. It follows that $\psi(t)$ is a convex function of t^{p_0} for $t>0$, and so also for $t\geqslant 0$ since it is continuous for $t=0$.

To show that $\psi(t)$ is increasing (i. e., non-decreasing) for $t\geqslant 0$, observe that if it were not so, then we would have $\psi(0)>\psi(t)$ for all t positive and sufficiently small. Take any U such that $|u|=0$ at x_0. Then $\psi(|u|)$ would have a strict maximum at x_0 which is incompatible with the hypothesis that $\psi(|u|)$ is subharmonic.

Remark. Further extensions of theorem 1 have been obtained by E. M. Stein and G. Weiss in a paper not yet published.

Chapter II

1. It is a familiar fact that a system of n functions of n variables is the gradient of a harmonic function if and only if both the divergence and the curl of the system vanish. In this chapter we investigate the problem when is a given system of functions a gradient of order m of a harmonic function. We shall need the results in Chapter III. Some of the arguments below are borrowed from [2].

We recall the notation. We consider functions of a variable $x = (x_1, x_2, \dots, x_n)$ and we write $|x| = (\sum x_j^2)^{1/2}$. By a we denote *multi-indices* $(a_1, a_2, \dots, a_n)$, where the a_j are non-negative integers, and by the *weight* of a we mean the number $|a| = \sum a_j$. We write

$$a! = a_1! a_2! \dots a_n!, \qquad x^a = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}, \qquad \left(\frac{\partial}{\partial x}\right)^a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{a_n}.$$

If $P(x)$ is a polynomial $\sum a_a x^a$, we mean by $P(\partial/\partial x)$ the operator $\sum a_a (\partial/\partial x)^a$.

By Π_m we shall denote the linear space of all homogeneous polynomials of degree m in x. By h_m we shall mean the subclass of Π_m consisting of all harmonic polynomials of degree m.

If P and Q are in Π_m, we set

$$(P, Q) = P\left(\frac{\partial}{\partial x}\right) Q.$$

It is easy to see that (P, Q) *is an inner product on* Π_m. For suppose that $|a| = |\beta| = m$. Then $(\partial/\partial x)^a x^\beta = 0$ if $\beta \neq a$, and $(\partial/\partial x)^a x^a = a!$. Thus if $P = \sum_{|a|=m} a_a x^a$, $Q = \sum_{|\beta|=m} b_\beta x^\beta$, then

$$(P, Q) = P\left(\frac{\partial}{\partial x}\right) Q = \sum a_a b_a a! = (Q, P). \tag{2.1.1}$$

2. Lemma 1. *Suppose that $Q \in \Pi_m$. Then $(Q, P) = 0$ for all $P \in h_m$ if and only if Q is divisible by $x_1^2 + x_2^2 + \dots + x_n^2$.*

Let $\Delta = \sum \partial^2/\partial x_j^2$. If Q is divisible by $x_1^2 + \dots + x_n^2$, then $Q = R \cdot (x_1^2 + \dots + x_n^2)$ and

$$(Q, P) = R\left(\frac{\partial}{\partial x}\right) \Delta P = 0 \quad \text{for all} \quad P \in h_m.$$

Suppose, conversely, that $(Q, P) = 0$ for all $P \in h_m$. Consider the mapping $\varphi: P \to \Delta P$ of Π_m into Π_{m-2}; we claim that the mapping is "onto". In fact, if $R \in \Pi_{m-2}$ and R is orthogonal to all polynomials of the form ΔP, $P \in \Pi_m$, then $R(\partial/\partial x)\Delta P = 0$ for all $P \in \Pi_m$. Setting $P(x) = (x_1^2 + \dots + x_n^2) R(x)$ we obtain

$$R\left(\frac{\partial}{\partial x}\right)\left[\frac{\partial^2}{\partial x_1} + \dots + \frac{\partial^2}{\partial x_n}\right] R(x)(x_1^2 + \dots + x_n^2) = 0,$$

which, in view of the fact that the operation (P, Q) is an inner product (see (2.1.1)) implies that $R(x)(x_1^2 + \dots + x_n^2) = 0$, i. e. $R = 0$. Thus the mapping φ is actually "onto".

The kernel of the mapping φ is precisely h_m. Hence $\dim h_m = \dim \Pi_m - \dim \Pi_{m-2}$, and the orthogonal complement $h_m^\perp$ has dimension

$$\dim \Pi_m - \dim h_m = \dim \Pi_{m-2}.$$

Consider now the mapping $\psi: \Pi_{m-2} \to \Pi_m$ given by $\psi(Q) = (x_1^2 + \dots + x_n^2) Q(x)$, $Q \in \Pi_{m-2}$. The mapping is one-one and so the image $\psi(\Pi_{m-2})$ of Π_{m-2} has dimension $\dim \Pi_{m-2}$. Furthermore, $\psi(\Pi_{m-2}) \in h_m^\perp$, for f $P \in h_m$, then

$$(\psi(Q), P) = ((x_1^2 + \dots + x_n^2) Q, P) = Q\left(\frac{\partial}{\partial x}\right) \Delta P(x) = 0.$$

Consequently, since $\dim \psi(\Pi_{m-2}) = \dim h_m^\perp$, we have $h_m^\perp = \psi(\Pi_{m-2})$, that is, every $P \in h_m^\perp$ is of the form $(x_1^2 + \dots + x_n^2) Q$, $Q \in \Pi_{m-2}$, and the lemma is established.

3. Theorem 1. *Let $\{P_a\}$ be a set of homogeneous polynomials of degree k, where a runs through all multi-indices of weight m. Then $P_a = (\partial/\partial x)^a P$, where $P \in h_{m+k}$ if and only if $\sum Q_a(\partial/\partial x) P = 0$ for all sets of polynomials Q_a of degree k such that $\sum x^a Q_a(x)$ is divisible by $x_1^2 + \dots + x_n^2$.*

The necessity of the condition is clear. To prove the sufficiency, consider the set of polynomials $R_a = (\partial/\partial x)^a P$, where $P \in h_{m+k}$. They form a linear subspace of the space of the vectors $\{S_a\}$, $S_a \in \Pi_k$. In the space of vectors $S = \{S_a\}$ we have an inner product $(S_1, S_2) = \sum (S_{1a}, S_{2a})$. If $\{Q_a\}$ is a vector orthogonal to all R_a, $R_a = (\partial/\partial x)^a P$, then we have

$$\sum \left(Q_a, \left(\frac{\partial}{\partial x}\right)^a P\right) = 0$$

for all $P \in h_{m+k}$. But

$$\sum \left(Q_a, \left(\frac{\partial}{\partial x}\right)^a P\right) = \sum (x^a Q_a, P) = \left(\sum x^a Q_a, P\right) = 0$$

for all $P \in h_{m+k}$. According to Lemma 1 this implies that $\sum x^a Q_a$ is divisible by $x_1^2 + \dots + x_n^2$ and thus, by hypothesis, $\sum (Q_a, P_a) = \sum Q_a(\partial/\partial x) P_a(x) = 0$.

Consequently, if Q_a is orthogonal to the space of vectors $\{R_a\}$, it is also orthogonal to the vector $\{P_a\}$, i. e., $\{P_a\}$ is among the $\{R_a\}$.

4. THEOREM 2. *Let $\{u_a\}$ be a set of C^∞ functions in the sphere $|x| < R$ where a runs through all multi-indices of weight m. Then $u_a = (\partial/\partial x)^a u$, where u is a harmonic function if and only if $\sum Q_a(\partial/\partial x) u_a = 0$ whenever Q_a are homogeneous polynomials of the same degree such that $\sum x^a Q_a(x)$ is divisible by $x_1^2+\ldots+x_n^2$.*

Remark. The condition that the u_a are in C^∞ can be dropped, but then $Q_a(\partial/\partial x)$ must be taken in the sense of distributions.

Proof. The necessity of the condition is obvious as before.

In the proof of sufficiency, observe, first of all that the u_a are necessarily harmonic. For set $Q_a(x) = x_1^2+\ldots+x_n^2$ if $a = \beta$ and $Q_a = 0$ if $a \neq \beta$. Then our hypotheses imply that $\Delta u_\beta = 0$.

Let now $u_a = \sum a_\nu^a P_\nu(x)$ be the expansion of u_a into normalized spherical harmonics. We observe that a series $\sum a_\nu P_\nu(x)$ of normalized spherical harmonics converges for $|x| < R$ if and only if $\sum |a_\nu| \varrho^\nu < \infty$ for all $\varrho < R$. Consequently, we have $\sum |a_\nu^a| \varrho^\nu < \infty$ for $\varrho < R$.

Suppose now that $Q_a(x)$ are homogeneous polynomials of the same degree such that $\sum x^a Q_a(x)$ is divisible by $x_1^2+x_2^2+\ldots+x_n^2$. Then

$$\sum_a Q_a\left(\frac{\partial}{\partial x}\right) u_a = \sum_\nu \sum_a a_\nu^a Q_a\left(\frac{\partial}{\partial x}\right) P_\nu^a(x) = 0.$$

Since the inner sum on the right represents a harmonic polynomial of degree $\nu - |a|$, the vanishing of the series implies the vanishing of each of the terms. Thus we have

$$\sum_a Q_a\left(\frac{\partial}{\partial x}\right)[a_\nu^a P_\nu^a(x)] = 0$$

whenever $\sum x^a Q_a(x)$ is divisible by $x_1^2+\ldots+x_n^2$. By the preceding theorem there exist harmonic polynomials, which we will denote by $b_\nu P_\nu$ such that P_ν is a normalized spherical harmonic and

$$a_\nu^a P_\nu^a(x) = \left(\frac{\partial}{\partial x}\right)^a b_{\nu+m} P_{\nu+m} \qquad (m = |a|).$$

These polynomials $b_\nu P_\nu$ are uniquely determined for $\nu \geqslant m$. For if also $a_\nu^a P_\nu^a = (\partial/\partial x)^a b'_{\nu+m} P'_{\nu+m}$, then

$$\left(\frac{\partial}{\partial x}\right)^a [b'_{\nu+m} P'_{\nu+m} - b_{\nu+m} P_{\nu+m}] = 0$$

for all a, $|a| = m$, which would imply that $b'_{\nu+m} P'_{\nu+m} - b_{\nu+m} P_{\nu+m}$ is

a polynomial of degree $\leqslant m-1$. If we show that the series $\sum b_\nu P_\nu(x)$ converges for $|x| < R$, then denoting its sum by $u(x)$ we shall have

$$\left(\frac{\partial}{\partial x}\right)^a u(x) = \sum_\nu \left(\frac{\partial}{\partial x}\right)^a b_\nu P_\nu(x) = \sum_\nu a_\nu^a P_\nu^a(x) = u_a(x).$$

Now, for a normalized spherical harmonic P_ν of degree ν we have

$$|P_\nu(x)| \leqslant C \nu^{n-2} |x|^\nu,$$

where C depends on the dimension n only. If $\partial/\partial\varrho$ denotes differentiation in the direction of the unit vector $(\mu_1, \mu_2, \ldots, \mu_n)$, we have

$$\frac{\partial}{\partial \varrho} = \sum_{j=1}^m \mu_j \frac{\partial}{\partial x_j}$$

and

$$\left|\left(\frac{\partial}{\partial \varrho}\right)^m b_{\nu+m} P_{\nu+m}(x)\right| = \left|\left[\sum \mu_j \frac{\partial}{\partial x_j}\right]^m b_{\nu+m} P_{\nu+m}(x)\right| \leqslant C \sum |a_\nu^a| \, |P_\nu^a(x)|,$$

where C is a sufficiently large constant, and this is majorized by $C \sum_a |a_\nu^a| \nu^{n-2} |x|^\nu$.

Integrating along the ray we obtain

$$|b_{\nu+m} P_{\nu+m}(x)| \leqslant C |x|^{\nu+m} \sum_a |a_\nu^a| \nu^{n-2}.$$

Since $\sum |a_\nu^a| \varrho^\nu < \infty$ for $\varrho < R$, it follows that $\sum |b_{\nu+m} P_{\nu+m}(x)| < \infty$ for $|x| < R$ and Theorem 2 is established.

Chapter III

1. Let $f(x) = f(x_1, x_2, \ldots, x_n) \in L^p = L^p(E_n)$. We consider its Poisson integral

$$P_t f = \frac{2}{\omega_{n+1}} \int_{E_n} \frac{t}{[(x-z)^2 + t^2]^{(n+1)/2}} f(z)\, dz, \qquad t > 0.$$

Young's inequality implies that $P_t f \in L^q$, $p \leqslant q \leqslant \infty$, for each $t > 0$. If $t = t_1 + t_2$, then $P_t f = P_{t_1} P_{t_2} f$.

We consider Riesz transforms $R_j f$, $j = 1, 2, \ldots, n$, of f. There are a number of definitions of Riesz transforms. Using Fourier transforms we may define $R_j f$ by the equation

$$(R_j f)\hat{} = -i \frac{x_j}{|x|} \hat{f}.$$

This definition is legitimate for $f \in L^2$, and it then turns out that for any f in C^∞ and having finite support we have $\|R_j f\|_p \leqslant A_p \|f\|_p$, where $1 < p < \infty$, and A_p depends on p only. Thus R_j can be extended by continuity to all of $L_p(E_n)$. This extension defines $R_j f$ only almost everywhere, but there is another definition of $R_j f$, given by means of singular integrals, which shows that $R_j f$ can be defined everywhere, and is pointwise continuous if f is continuously differentiable (and in L^p). In the arguments below, where we consider Riesz transforms of Poisson integrals the transforms are assumed to be continuous.

If $a = (a_1, a_2, \ldots, a_n)$ is a multi-index and the a_j are non-negative integers, we set $R^a = R_1^{a_1} R_2^{a_2} \ldots R_n^{a_n}$. If $f \in L^p$, $1 < p < \infty$, then $R^a f$ is defined and $\|R^a f\|_p \leqslant A_p^{|a|} \|f\|_p$.

Suppose that $f \in L^2$. Then by differentiating under the integral sign one sees that $P_t f$, as a function of x and t, is in C^∞. Furthermore, all derivatives of $P_t f$ are in $L^2(E_n)$. The Fourier transform of the Poisson kernel is $e^{-|x|t}$. Consequently, we have $(P_t f)\hat{\ } = \hat{f}(x) e^{-|x|t}$.

Let again $f \in L^2$. Taking Fourier transforms we see that $R^a P_t f = P_t R^a f$ and consequently $R^a P_t f$ and all its derivatives are in C^∞ and $L^2(E_n)$. If D is a monomial differential operator in x and t, then, since $DP_t f$ is in L^2 and $R^a P_t f$ is in C^∞, both $R^a D P_t f$ and $D R^a P_t f$ are well defined and by taking their Fourier transforms we see that

$$R^a D P_t f = D R^a P_t f. \tag{3.1.1}$$

Finally, by again taking Fourier transforms, we see that

$$\frac{\partial}{\partial x_j} R^a D P_t f = R_j \frac{\partial}{\partial t} R^a D P_t f, \tag{3.1.2}$$

i. e., the operators $\partial/\partial x_j$ and $R_j \partial/\partial t$ coincide on all functions $R^a D P_t f$, $f \in L^2$.

2. THEOREM 1. *Let*

$$\beta = (a_1, a_2, \ldots, a_n, k) = (a, k)$$

be the multi-indices of weight m,

$$m = |\beta| = |a| + k,$$

and $f_\beta(x, t)$ *a system of functions of* x *and* t *given by*

$$f_\beta(x, t) = R^a P_t f, \quad \beta = (a, k), \tag{3.2.1}$$

where f *is real-valued and in* $L^p(E_n)$, $1 \leqslant p < \infty$. *Then the* $f_\beta(x, t)$ *are harmonic functions and*

$$\Big\{\sum_\beta f_\beta^2(x, t)(\beta!)^{-1}\Big\}^{1/2} = \Big\{\sum_{k=0}^{m} \frac{1}{k!} \sum_{|a| = m-k} (R^a P_t f)^2 (a!)^{-1}\Big\}^{1/2} \tag{3.2.2}$$

is subharmonic for

$$l \geqslant \frac{n-1}{m+n-1}. \tag{3.2.3}$$

Proof. We assume first that f is bounded and has bounded support; hence $f \in L^2$. The function $R^a P_t f$ is in C^∞, and in view of (3.1.1) and the fact that $P_t f$ is harmonic, the functions $f_\beta(x, t)$ are harmonic.

To prove the subharmonicity of $\big(\Sigma f_\beta^2 (\beta!)^{-1}\big)^{l/2}$ we apply Theorem 1 of Chapter I and Theorem 2 of Chapter II. It is enough to show that if $Q_\beta(x, t)$ are homogeneous polynomials of the same degree N, such that

$$\sum x^a t^k Q_\beta(x, t), \quad \beta = (a, k), \quad |\beta| = m,$$

is divisible by $x_1^2 + \ldots + x_n^2 + t^2$, then $\sum Q_\beta(\partial/\partial x, \partial/\partial t) f = 0$. Now, since $f_\beta = R^a P_t f$ and the operators $\partial/\partial x_j$ and $R_j(\partial/\partial t)$ coincide for all functions of the form $R^a D P_t f$, $f \in L^2$ (see (3.1.2)), we have

$$\sum_\beta Q_\beta\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) f_\beta = \sum_\beta Q_\beta\Big(R\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) R^a P_t f$$
$$= \sum_\beta R^a Q_\beta(R, 1)\Big(\frac{\partial}{\partial t}\Big)^N P_t f.$$

Since

$$\sum_\beta x^a t^k Q_\beta(x, t) = (x_1^2 + \ldots + x_n^2 + t^2) L(x, t),$$

we have

$$\sum_\beta R^a Q_\beta(R, 1) = L(R, 1)(R_1^2 + R_2^2 + \ldots + R_n^2 + I) = 0,$$

in view of the identity $\sum R_j^2 = -I$, which is an immediate consequence of the definition of the R_j. Thus $\sum Q_\beta(\partial/\partial x, \partial/\partial t) f_\beta = 0$, as we wished to show.

Suppose now that $1 < p < \infty$, $f \in L^p$, and let f_n be bounded, of finite support and tend to f in L^p. Then $P_t f_n$ converges to $P_t f$ in L^p for each $t > 0$, and thus $f_\beta^{(n)} = R^a P_t f_n$ converges to $R^a P_t f$ in $L^p(E_n)$. On the other hand,

$$f_\beta^{(n)} = R^a P_t f_n = P_t R^a f_n,$$

and since $R^a f_n$ converges to $R^a f$ in $L^p(E_n)$, it follows that $f_\beta^{(n)} = P_t R^a f_n$ converges uniformly for $t \geqslant \varepsilon > 0$ to $P_t R^a f = R^a P_t f = f_\beta$. Thus the $f_\beta(x, t)$ are harmonic and $\{\sum f_\beta^2(x, t)(\beta!)^{-1}\}^{l/2}$ subharmonic for $t > 0$ and satisfying (3.2.2).

It remains to consider the case $p = 1$ of Theorem 1. Observe that if $t > \varepsilon$, then $P_t f = P_{t-\varepsilon}(P_\varepsilon f)$, and that $P_\varepsilon f \in L_p$ for all $p \geqslant 1$. This reduces the case to the previous cases.

3. If p is strictly greater than 1, then the functions (3.2.1) are the Poisson integrals of the functions $R^\alpha f$. This is in general not true if $p = 1$ even though $R^\alpha f$ can be defined in that case. It is however not integrable, even locally, so that $P_t R^\alpha f$ has no meaning.

4. The significance of the theorem of this chapter is as follows. If f is in L^p, $p \geqslant 1$, then $|P_t f|^r$ is subharmonic for $r \geqslant 1$. The Stein-Weiss result quoted in Chapter I asserts that if we adjoin to $P_t f$ its Riesz transforms, we obtain a harmonic vector $P_t f, R_1 P_t f, \ldots, R_n P_t f$ whose norm is subharmonic when raised to the power $(n-1)/n$. By the theorem of this chapter, if we keep adding to the last system higher and higher Riesz transforms we obtain harmonic vectors whose norms remain subharmonic when raised to smaller and smaller powers.

Perhaps a change in notation will make this a little clearer. In defining the norm of $\mathrm{grad}_m U$ we considered only distinct derivatives of order m. If, however, we define $u = \mathrm{grad}_m U$ successively as the first gradient of the $(m-1)$-st (which is in a way more natural, as the argument of Chapter I shows) and set

$$R_\gamma f = R_{\gamma_1}, R_{\gamma_2}, \ldots, R_{\gamma_m} f$$

for any multi-index $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ *of m components*, where now $1 \leqslant \gamma_j \leqslant n$ for all j, then the factorials in (3.2.2) can be dropped and our theorem asserts that the function

$$\Big\{\sum_{k=0}^{m} \sum_{\gamma_1,\gamma_2,\ldots,\gamma_k=1}^{n} (R_{\gamma_1} R_{\gamma_2} \ldots R_{\gamma_k} P_t f)^2\Big\}^{l/2}$$

is subharmonic for $l \geqslant (n-1)/(m+n-1)$.

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