On the other hand, $\mathcal{S}$ maps $X(B)\oplus B$ onto $\Lambda(B, X)$ and consequently $\mathcal{S}\mathcal{I}$, which is the identity, maps $[\Lambda(B_0, X_0), \Lambda(B_1, X_1)]_s$ continuously into $\Lambda(B, X)$. Thus $[\Lambda(B_0, X_0), \Lambda(B_1, X_1)]_s$ is continuously embedded in $\Lambda(B, X)$.

Now, $\mathcal{S}$ maps $X_i(B_i)\oplus B_i$ continuously into $\Lambda(B_i, X_i)$ $(i = 0, 1)$ and therefore it maps $[X_0(B_0)\oplus B_0, X_1(B_1)\oplus B_1]_s = X(B)\oplus B$ into $[\Lambda(B_0), X_0, \Lambda(B_1, X_1)]_s$. But the image of $X(B)\oplus B$ under $\mathcal{S}$ is $\Lambda(B, X)$. Consequently

$$\Lambda(B, X) \subset [\Lambda(B_0, X_0), \Lambda(B_1, X_1)]_s.$$

We already proved the reverse inclusion and its continuity, and thus the open mapping theorem yields the desired conclusion.

In the case where $X(B) = [X_0(B_0), X_1(B_1)]^s$ the result sought is obtained by using 7 instead of 4 in the preceding argument.

Bibliography

[1] A. P. Calderón, *Intermediate spaces and interpolation*, Studia Math., Special Series 1 (1963), p. 31-34.

[2] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Pub. 31 (1957).

[3] S. G. Krein, *On an interpolation theorem in the theory of operators*, Doklady Akad. Nauk 130 (1960), p. 491-494.

[4] — *On the notion of normal scale of spaces*, ibidem 132 (1960), p. 510-513.

[5] J. L. Lions, *Une construction d'espaces d'interpolation*, Comptes Rendus Acad. Sc. Paris 83 (1960), p. 1853-1855.

[6] G. G. Lorentz, *Some new functional spaces*, Annals of Math. 51 (1950), p. 37-55.

[7] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. 83 (1956), p. 482-492.

[8] — and G. Weiss, *Interpolation of operators with change of measures*, ibidem 87 (1958), p. 159-172.

[9] M. H. Taibleson, Doctoral Dissertation, University of Chicago, 1962.

[10] A. Zygmund, *Trigonometrical series*, (*I*), (*II*), London 1959.

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A ring of analytic functions

by

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This paper is devoted to an investigation of a topological ring of analytic functions. Specifically, this ring, denoted by R, is the set of functions analytic on the unit disc with the usual addition and scalar multiplication, the Hadamard product for its ring multiplication, and the compact-open topology. The ring R is identified algebraically with a subring $\hat{R}$ of the ring of continuous functions on the non-negative integers X. The operations in $\hat{R}$ are the usual pointwise operations, and the structure of $\hat{R}$ is determined by considering its isomorph $\hat{R}$.

In Section 2 we are concerned with the problems of identifiying the maximal ideal space of R and describing the maximal ideals intrinsically. We first show, using theorems on general rings of continuous functions, that the maximal ideals are in one-to-one correspondence with the points of the Stone-Čech compactification βX of X. We next give an intrinsic description of the maximal ideals, using the properties of the power series expansions of analytic functions. Using this description we strengthen the previous theorem appreciably and show that the maximal ideal space with the hull-kernel topology is homeomorphic to βX. Finally, the Hadamard product is used to give a simple characterization of the dual space of the topological linear space of analytic functions on the unit disc. This dual space is isomorphic to the set of functions in R whose radius of convergence exceeds one, which is exactly the intersection of the maximal ideals corresponding to points of $\beta X - X$ (the dense maximal ideals of R).

In Section 3 we continue the investigation of the maximal ideals by studying the structure of their associated residue class rings. The complex number field C is isomorphically embedded in R/M, where M is a maximal ideal of R. If M corresponds to a point of X, then R/M and the isomorph C^* of C are identical; whereas, if M corresponds to a point of $\beta X - X$, then R/M is a transcendental extension of C^* having transcendence degree $\mathfrak{c}$, the cardinality of the continuum. Moreover, we show,

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in the second case, that R/M is algebraically closed. Using theorems on transcendental extensions and algebraically closed fields, we show that, in either case, R/M and C are isomorphic fields. The two classes of maximal ideals are distinguished by the fact that their residue class rings admit radically different types of complex-valued isomorphisms.

In Section 4 we are concerned primarily with the structure of the closed ideals of R. The basic tool used is the rotational completeness theorem for analytic functions, which we prove using the methods and results of harmonic analysis. We show that the closure of every principal ideal is principal, give a necessary and sufficient condition that a principal ideal be closed, and show that every closed ideal is a principal ideal generated by an idempotent element of R. Using these theorems we indicate connections with the general theory of dual rings, of which R is an example. In the last portion of the section we investigate the prime and primary ideals and show that each prime (primary) ideal is contained in a unique maximal ideal and that closed prime (primary) ideals are maximal.

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1. Preliminaries. In this paper D will denote the interior of the unit disc in the complex plane C and R will denote the collection of all complex-valued functions which are analytic on D. The family R can also be regarded as the set of all power series with complex coefficients having radius of convergence greater than or equal to one. Addition and scalar multiplication in R are the usual pointwise operations and ring multiplication is given by the Hadamard product: if f and g are elements of R with power series $f(x) = \sum_{p=0}^{\infty} f_p x^p$ and $g(x) = \sum_{p=0}^{\infty} g_p x^p$, then $fg(x) = \sum_{p=0}^{\infty} f_p g_p x^p$. With these operations R is a commutative ring with identity e, where $e(x) = (1-x)^{-1} = \sum_{p=0}^{\infty} x^p$. If we give R the compact-open topology, or equivalently, the topology of uniform convergence on compact subsets of D, then R is a topological ring in the sense of Naimark ([2], p. 168). Moreover, multiplication is a continuous function on $R\times R$ into R.

The ring R has an algebraic realization which we will use extensively to study R. We denote by X the space of non-negative integers with the discrete topology, and for each $f\in R$, $f(x) = \sum_{p=0}^{\infty} f_p x^p$, we define $\hat{f}: X\to C$ by $\hat{f}(p) = f_p$. The map $f\to\hat{f}$ is an isomorphism of R onto $\hat{R}$, the family of all complex valued functions f on X satisfying $\limsup_p |f(p)|^{1/p} \leqslant 1$. The ring $\hat{R}$ is a subring of $C(X)$, the ring of all complex-valued continuous functions on X, and an over-ring of $B(X)$, the ring of all bounded complex-valued continuous functions on X. We will determine the structure of R by considering its isomorph $\hat{R}$. In the sequel R and $\hat{R}$ will be identified and the words "closed" and "dense" will be used to describe sets in $\hat{R}$ whose isomorphs in R have these topological properties. The symbol "ˆ" will be dropped since no confusion will result from this omission.

The scalar field C is embedded algebraically and topologically in R *via* the function η defined by $\eta(a) = ae$ for $a\in C$.

2. Maximal ideals. The dual space of R. Let M be a maximal ideal in R and let φ be the natural homomorphism from R onto the residue class ring R/M. Since M is maximal, R/M is a field which contains $\varphi\eta(C)$ as a subfield. For each $E\subset X$ the function k_E, the characteristic function of E, is an idempotent element of R, and $\varphi(k_E) = \varphi(e)$ or $\varphi(\theta)$, where θ is the zero element of R. In particular, if $k_n = k_{\{n\}}$, then k_n is idempotent, and, moreover, $n\neq m$ implies $k_n k_m = \theta$. Hence, there exists at most one $n\in X$ such that $\varphi(k_n) = \varphi(e)$, and $\varphi(k_m) = \varphi(\theta)$ for all other $m\in X$. By linearity and homogeneity we may divide the homomorphisms into two classes, determined by their action on the ideal of polynomials I_0 (the finitely non-zero elements of R).

(I) $\varphi(f) = \varphi(\theta)$ for all $f\in I_0$,

(II) there exists a unique $n\in X$ such that $\varphi(f) = f(n)\varphi(e)$ for each $f\in I_0$.

We will say that a maximal ideal is of *type* (I) or (II) according as its corresponding homeomorphism is of type (I) or (II). If M is of type (I), then $M\supset I_0$ and is dense in R. For each $n\in X$ the set $M^n = \{f\in R: f(n) = 0\}$ is a maximal ideal in R, and is, moreover, closed.

THEOREM 2.1. *A maximal ideal M in R is closed if, and only if, it is of type* (II). *If M is closed, then there exists a unique $n\in X$ such that $M = M^n$.*

Proof. In view of the above remarks the necessity is obvious. To prove the sufficiency we shall show that if M is of type (II), then $M = M^n$ for a unique $n\in X$. If φ is the natural homomorphism of R onto R/M, then there exists a unique $n\in X$ and that $\varphi(f) = f(n)\varphi(e)$ for each $f\in I_0$. Let $k_n' = e-k_n$. Then $f\in M$ implies $fk_n'\in M$ and $f-fk_n'\in M$. But $f-fk_n' = f(n)k_n$. Hence, $\varphi(f(n)k_n) = \varphi(\theta)$ or $f(n)\varphi(k_n) = \varphi(\theta)$. However, $\varphi(k_n) = \varphi(e)$. Therefore, $f(n) = 0$ and $f\in M^n$. Thus $M\subset M^n$, and from the maximality of M we have $M = M^n$.

We note here that the set of non-zero continuous multiplicative functionals on R to C, a subset of R^* (the dual space of R), is a countable discrete space in the weak*-topology of R^*. Since the space of such functionals is homeomorphic to the space of closed maximal ideals of R we infer that M_0, the closed maximal ideal space of R, is homeomorphic to the non-negative integers. Thus, as in the cases of normed rings and rings with continuous inverse, the natural place to study this ring is in its closed maximal ideal space.

Before completing the identification of the maximal ideals of R we must digress to the theory of rings of continuous functions. The non-negative integers with the discrete topology is a completely regular Hausdorff space, and, hence, admits a Stone-Čech compactification, which will be denoted by βX. $C(\beta X)$ denotes the continuous functions on βX. We shall now state without proof several facts which are important in the present study (for proofs, see [1]); 1) X is dense in βX, which is a compact Hausdorff space; 2) $B(X)$ is isomorphic and isometric to $C(\beta X)$; 3) to each point p of βX there corresponds exactly one maximal ideal M_1^p in $B(X)$ and $M_1^p = \{f \in B(X): f^\beta(p) = 0\}$, where f^β is the continuous extension of f to βX; 4) to each point p of βX there corresponds exactly one maximal ideal M_2^p in $C(X)$; 5) for each $p \in \beta X$, $M_2^p \cap B(X) \subset M_1^p$; 6) each prime ideal in $C(X)$ ($B(X)$) is contained in a unique maximal ideal in $C(X)$ ($B(X)$).

Lemma 2.2.1. *If f is an element of R, then f is a unit in R (i. e., is invertible) if, and only if, 1) $p \in X$ implies $f(p) \neq 0$ and 2) $\lim_p |f(p)|^{1/p}$ exists and equals one.*

Proof. If 1) and 2) hold, then f and f^{-1} have radius of convergence one, where $f^{-1}(p) = (f(p))^{-1}$. Hence f and f^{-1} are both in R and f is a unit in R. Conversely, if f is a unit in R, then $f^{-1} \in R$ and $f(p) \neq 0$ for all $p \in X$. Since $f^{-1} \in R$, $\limsup_p |f(p)|^{-1/p} \leqslant 1$ and $\liminf_p |f(p)|^{1/p} \geqslant 1$. However, $\limsup_p |f(p)|^{1/p} \leqslant 1$. Hence $\lim_p |f(p)|^{1/p} = 1$.

We shall now define two lattice-like operations in R.

For each $f \in R$ we define $f \vee e$ by the following formula:

$$(f \vee e)(p) = \begin{cases} f(p) & \text{if} \quad |f(p)| \geqslant 1, \\ 1 & \text{if} \quad |f(p)| < 1. \end{cases}$$

Similarly, we define

$$(f \wedge e)(p) = \begin{cases} f(p) & \text{if} \quad |f(p)| < 1, \\ 1 & \text{if} \quad |f(p)| \geqslant 1. \end{cases}$$

Lemma 2.2.2. *If $f \in R$, then 1) $f \vee e$ and $f \wedge e$ are elements of R, 2) $f \vee e$ is a unit in R, and 3) $f = (f \wedge e)(f \vee e)$.*

Proof. The proofs of 1) and 2) follow immediately from the definitions of $f \wedge e$ and $f \vee e$ and from Lemma 2.2.1. The proof of 3) follows by considering the cases $|f(p)| < 1$ and $|f(p)| \geqslant 1$.

Lemma 2.2.3. *If M and N are distinct maximal ideals in R, then there exist $f \in M \cap B(X)$ and $g \in N \cap B(X)$ such that $f + g = e$.*

Proof. If $M \neq N$, then there exist $f' \in M - N$, $h \in R$, and $g \in N$ such that $hf' + g = e$. But M is an ideal in R; hence, $hf' \in M$. If we define f by $f = hf' \wedge e$, then $f \in M \cap B(X)$, since f is bounded and $hf' \vee e$ is a unit in R. Let $A = \{p \in X: |f(p)| < 1\} = \{p \in X: |hf'(p)| < 1\}$. Then $k_A \in B(X) \subset R$ and $g = g' k_A \in N$. From the definitions of f and g we have $(f+g)(p) = 1$, or $f + g = e$. We must now show that g is bounded. For $p \in X - A$, $g(p) = 0$, and for $p \in A$, $|hf'(p)| < 1$. Therefore, $|g(p)| = |g'(p)| = |1 - hf'(p)| \leqslant 1 + |hf'(p)| < 2$ for $p \in A$. Consequently, $g \in N \cap B(X)$.

Theorem 2.2. *There exists a one-to-one correspondence between the maximal ideals of R and the points of the Stone-Čech compactification of the non-negative integers. Moreover, if M_1^p, M^p, and M_2^p are the maximal ideals of $B(X)$, R, and $C(X)$, respectively, which correspond to a point p of βX, then $M_1^p \cap B(X) \subset M^p \cap B(X) \subset M_2^p$. Finally, if $p \in X$, then $M_2^p \cap R = M^p$ and $M^p \cap B(X) = M_1^p$.*

Proof. Let M be a maximal ideal in R. Then $M \cap B(X)$ is a prime ideal in $B(X)$ and is contained in a unique maximal ideal M_1^p of $B(X)$, where p is an element of βX. We define the function σ from the set $\mathscr{M}$ of all maximal ideals of R to the set $\mathscr{M}_B$ of maximal ideals of $B(X)$ by $\sigma(M) =$ the unique maximal ideal M_1^p in $\mathscr{M}_B$ which contains $M \cap B(X)$. If M and N are distinct maximal ideals in R, then $M \cap B(X)$ and $N \cap B(X)$ cannot be contained in the same maximal ideal of $B(X)$, because of Lemma 2.2.3. Hence σ is well-defined and one-to-one. Now, if $p \in \beta X$ and M_2^p is the maximal ideal in $C(X)$ which corresponds to p, then $M_2^p \cap R$ is an ideal in R, and, hence, is contained in a maximal ideal M of R. Thus $M_2^p \cap B(X) \subset M \cap B(X) \subset M_1^q$ for some $q \in \beta X$. However, $M_2^p \cap B(X) \subset M_1^p$ and $M_2^p \cap B(X)$ is a prime ideal in $B(X)$. Therefore, $q = p$, and σ is onto. By composing σ with the function $\tau: \mathscr{M}_B \to \beta X$ defined by $\tau(M_1^p) = p$ we have a one-to-one map from $\mathscr{M}$ onto βX. This completes the proof of the first part of the theorem and also gives us the second statement. $M_2^p \cap B(X) \subset M^p \cap B(X) \subset M_1^p$ for each $p \in \beta X$. Finally, if $p \in X$, then $M_2^p = \{f \in C(X): f(p) = 0\}$, $M^p = \{f \in R: f(p) = 0\}$, and $M_1^p = \{f \in B(X): f(p) = 0\}$, from which it follows that $M_2^p \cap R = M^p$ and $M^p \cap B(X) = M_1^p$.

Theorem 2.2 gives an identification of the maximal ideals of R. We shall now give a more specific description of the maximal ideals M^p, $p \in \beta X - X$, and, using that description, we shall give a characterization of $\mathscr{M}$. We have, as a corollary to Theorem 2.2, that these ideals are exactly the dense maximal ideals of R.

Lemma 2.3.1. *If $p \in \beta X - X$ and $\{p_a: a \in \mathfrak{A}\}$ is a net in X converging to p, then for each $n \in X$ there exists an a_0 such that $p_a > n$ for $a \geqslant a_0$.*

Proof. Fix $n \in X$. Since $p \notin \{0, 1, \ldots, n\}$ and βX is a Hausdorff space, there exists an open set U containing p such that $U \cap \{0, 1, \ldots, n\} = \emptyset$.

Now, $p_\alpha \to p$ implies that there exists $\alpha_0 \in \mathfrak{A}$ such that $p_\alpha \in U$ for $\alpha \geq \alpha_0$. Thus $p_\alpha \notin \{0, 1, \ldots, n\}$, and $p_\alpha > n$.

We define for $f \in R$, $\bar{f}(p) = |f(p)|^{1/p}$ for $p \neq 0$ and $f(0) = 1$.

LEMMA 2.3.2. *If $f \in R$, then $\bar f \in B(X)$ and $\bar f^\beta(p) \leq 1$ for each $p \in \beta X - X$, where $\bar f^\beta$ is the unique continuous extension of $\bar f$ to βX.*

Proof. It is clear from the definition of R that $f \in R$ implies $\bar f \in B(X)$. If $p \in \beta X - X$, then there exists a net $\{p_\alpha : \alpha \in \mathfrak{A}\}$ in X such that $p_\alpha \to p$, and $\bar f^\beta(p) = \lim_\alpha \bar f^\beta(p_\alpha) = \lim_\alpha \bar f(p_\alpha)$. Fix $\varepsilon > 0$. Then there exists a positive integer N_ε such that $\sup\{\bar f(n): n > N_\varepsilon\} < 1 + 2^{-1}\varepsilon$, and by Lemma 2.3.1 there exists $\alpha_1 \in \mathfrak{A}$ such that $p_\alpha > N_\varepsilon$ for $\alpha \geq \alpha_1$. Moreover, there exists $\alpha_2 \in \mathfrak{A}$ such that $|\bar f(p_\alpha) - \bar f^\beta(p)| < 2^{-1}\varepsilon$ for $\alpha \geq \alpha_2$. Since $\mathfrak{A}$ is a directed set, there exists $\alpha_0 \in \mathfrak{A}$ such that $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. Then $\alpha \geq \alpha_0$ implies $\bar f^\beta(p) < \bar f(p_\alpha) + 2^{-1}\varepsilon$ and $\bar f(p_\alpha) < 1 + 2^{-1}\varepsilon$. Hence, $\bar f^\beta(p) < 1 + \varepsilon$ for arbitrary $\varepsilon > 0$, and $\bar f^\beta(p) \leq 1$.

LEMMA 2.3.3. *If $p \in \beta X - X$ and $f \in M^p$, then $\bar f^\beta(p) < 1$.*

Proof. Suppose there exists $p \in \beta X - X$ and $f \in M^p$ such that $\bar f^\beta(p) = 1$. If $\{p_\alpha : \alpha \in \mathfrak{A}\}$ is a net in X converging to p, it follows that $\bar f(p_\alpha) \to 1$, and there exists $\alpha_0 \in \mathfrak{A}$ such that $\bar f(p_\alpha) > 2^{-1}$ for $\alpha \geq \alpha_0$. We define g from X to the reals by $g(n) = (f(n))^{-1}$ if $n = p_\alpha$ for some $\alpha \geq \alpha_0$ and $g(n) = 1$ otherwise. Then $\limsup_n |g(n)|^{1/n} = 1$, since $\lim_\alpha \bar f(p_\alpha) = 1$. Hence, $g \in R$ and $fg \in M^p$. But $fg \in M^p$ implies that $(fg \wedge e) \in M^p$ and $(fg \wedge e)$ is bounded. Therefore $(fg \wedge e) \in M^p \cap B(X) \subset M_1^p$, and $(fg \wedge e)^\beta(p) = 0$. Hence, $(fg)^\beta(p_\alpha) \to 0$. However, for $\alpha \geq \alpha_0$, $(fg)(p_\alpha) = 1$. This is a contradiction, and it follows that $p \in \beta X - X$ and $f \in M^p$ implies $\bar f^\beta(p) < 1$.

LEMMA 2.3.4. *If each of f and g is in R and p is an element of $\beta X - X$ such that $\bar f^\beta(p) < 1$, then $\overline{fg}^\beta(p) < 1$.*

Proof. Suppose f and g are elements of R and $p \in \beta X - X$ such that $\bar f^\beta(p) < 1$. Now, for each $n \in X$, $\bar f(n)\bar g(n) = \overline{fg}(n)$, and by the continuity of the extensions $\bar f^\beta$, $\bar g^\beta$, and $\overline{fg}^\beta$, we have $\bar f^\beta(q)\bar g^\beta(q) = \overline{fg}^\beta(q)$ for each $q \in \beta X$. Therefore, $\overline{fg}^\beta(p) = \bar f^\beta(p)\bar g^\beta(p)$, $\bar f^\beta(p) < 1$, and $\bar g^\beta(p) \leq 1$. Hence, $\overline{fg}^\beta(p) < 1$.

LEMMA 2.3.5. *If f and g are elements of R and if p is an element of $\beta X - X$ such that $\bar f^\beta(p) < 1$ and $\bar g^\beta(p) < 1$, then $(\overline{f+g})^\beta(p) < 1$.*

Proof. If $n \in X$, then $(\overline{f+g})(n) = |f(n)+g(n)|^{1/n} \leq 2^{1/n}(\bar f(n) \vee \bar g(n)) = 2^{1/n}(\bar f \vee \bar g)(n)$, where $\vee$ denotes the usual maximum of real-valued functions. Thus, if $\{p_\alpha : \alpha \in \mathfrak{A}\}$ is a net in X converging to p, then $(\overline{f+g})^\beta(p) \leq \lim_\alpha 2^{1/p_\alpha}[\bar f(p_\alpha) \vee \bar g(p_\alpha)] < 1$.

THEOREM 2.3. *If $p \in \beta X - X$, then $M^p = \{f \in R: \bar f^\beta(p) < 1\}$.*

Proof. Let $J^p = \{f \in R: \bar f^\beta(p) < 1\}$. Then $J^p \neq R$, and from Lemma

2.3.3 it follows that $M^p \subset J^p$. We infer from Lemmas 2.3.4 and 2.3.5 that J^p is an ideal in R. Hence $M^p = J^p$.

COROLLARY 2.3.1. *If we define R_0 to be the set of all $f \in R$ such that $\limsup_p |f(p)|^{1/p} < 1$, then $R_0 = \bigcap \{M^p : p \in \beta X - X\}$.*

Proof. If $f \in R_0$, then $\limsup_p |f(p)|^{1/p} < 1$, and if $\{p_\alpha : \alpha \in \mathfrak{A}\}$ is any net in X converging to $p \in \beta X - X$, then $\lim_\alpha \bar f(p_\alpha) < 1$, and $f \in M^p$. Hence $R_0 \subset M^p$ for each $p \in \beta X - X$. For the other containment it will suffice to show that if $f \in R - R_0$, then there exists $p \in \beta X - X$ such that $f \in R - M^p$. Suppose that $f \in R - R_0$. Then $\limsup_n |f(n)|^{1/n} = 1$ and there exists a subsequence $\{n_k : k = 1, 2, \ldots\}$ of $\{n\}$ such that $\lim_k \bar f(n_k) = 1$. Now $\{n_k\}$ is a net in βX and, therefore, clusters to some point $p \in \beta X$. Moreover, since the subsequence $\{n_k\}$ can be chosen such that $n_1 < n_2 < \ldots$, it can be assumed that $p \in \beta X - X$. Then $\bar f^\beta(p) = 1$ and $f \in R - M^p$.

COROLLARY 2.3.2. *If $p \in \beta X - X$, then $M_2^p \cap B(X) \subsetneqq M^p \cap B(X) \subsetneqq M_1^p$.*

Proof. If $p \in \beta X - X$, then the function k defined by $k(0) = 1$ and $k(n) = n^{-1}$ for $n \neq 0$ is an element of $M_1^p - (M^p \cap B(X))$. Similarly, the function l defined by $l(n) = (n!)^{-1}$ is an element of $(M^p \cap B(X)) - (M_2^p \cap B(X))$ for each $p \in \beta X - X$. This follows from Corollary 2.3.1 and the fact that l is a unit in $C(X)$.

We can use Theorem 2.3 to strengthen Theorem 2.2 appreciably. However, we shall also need several facts from the theory of Stone-Čech compactifications and from the theory of structure spaces of commutative rings.

If Y is an arbitrary completely regular Hausdorff space and βY is the Stone-Čech compactification of Y, then the points of βY can be regarded as the indices of the ultrafilters in the lattice $Z(Y)$ of zero sets of $C(Y)$. In this case, if $p \in Y$, then p is the index of the ultrafilter $\mathscr{A}^p$ of all zero sets which contain the point p. A base for the closed sets in βY is given by all sets of the form $C(A) = \{p \in \beta Y : A \in \mathscr{A}^p\}$, where $\mathscr{A}^p$ is an ultrafilter in $Z(Y)$ and A is an element of $Z(Y)$. Moreover, the closure in βY of A, denoted by $\mathrm{cl}_{\beta Y}(A)$, is exactly $C(A)$ (for a more detailed discussion, see Chapter 6 of [1]).

If A is an arbitrary commutative ring with identity and $\mathscr{M}$ is the set of all maximal ideals of A, then we can give $\mathscr{M}$ a topology (called the *hull-kernel topology*) in the following manner. For each $a \in A$ we define $E(a) = \{M \in \mathscr{M} : a \in M\}$. The collection $\{E(a) : a \in A\}$ is a base for the closed sets in the hull-kernel topology on $\mathscr{M}$ (cf. [1], p. 111, or [2], p. 221-223).

In our case the space X is discrete; hence, $Z(X)$ is the collection of all subsets of X. Also the base for the closed sets in $\mathscr{M}$ $(= \mathscr{M}_R)$ can be chosen much smaller than in the general case. The sets $E(k_A)$, where $A \subset X$, form a base for the closed sets in the hull-kernel topology. To show this we must demonstrate that for each $f \in R$ there exists $A \subset X$ such that $E(f) \subset E(k_A)$. This is equivalent to showing that for each $f \in R$ there exists $A \subset X$ such that $f \in M^p$ implies $k_A \in M^p$. To this end we fix $f \in R$, and let $B = \{n \in X : \bar{f}(n) < 1\}$ and $A = X - B$. We note that $\bar{k}_A = k_A$. Suppose $f \in M^p$. It will suffice to show that if $\{p_a : a \in \mathfrak{A}\}$ is a net in X which converges to p, then $\{k_A(p_a)\}$ converges to zero, since then $k_A^\beta(p) = 0$ and $k_A \in M^p$. Now, if $\{p_a : a \in \mathfrak{A}\}$ is a net in X converging to p, then there exists $a_0 \in \mathfrak{A}$ such that $\bar{f}(p_a) < 1$ for $a > a_0$. Then, for $a > a_0$, $p_a \in B$ and $k_A(p_a) = 0$. Hence, $k_A^\beta(p) = 0$ and $k_A \in M^p$.

THEOREM 2.4. *The maximal ideal space $\mathscr{M}$ of R with the hull-kernel topology is homeomorphic to βX, the Stone-Čech compactification of X.*

Proof. We have, from Theorem 2.2, that the function $p \to M^p$ is one-to-one. We will show that this mapping is a homeomorphism. It will suffice to show the following: if $A \subset X$, then $p \in C(A)$ if, and only if, $M^p \in E(k'_A)$, where $k'_A = k_{A'}$ $(A' = K - A)$. If this is the case, then the function $p \to M^p$ takes a basis for closed sets in βX onto a basis for closed sets in $\mathscr{M}$; hence, it is a homeomorphism.

Let $A \subset X$, and suppose that $p \in C(A)$. Now $C(A) = \mathrm{cl}_{\beta X}(A)$, and there exists a net $\{p_a : a \in \mathfrak{A}\}$ in A which converges to p. Then $k'_A(p_a)$ is identically zero and $k'^\beta_A(p) = 0$. Hence, $k'_A \in M^p$. Conversely, if $k'_A \in M^p$, then $k'^\beta_A(p) = 0$, and if $\{p_a : a \in \mathfrak{A}\}$ is a net in X which converges to p, then $k'_A(p_a)$ must be eventually zero. Hence, $\{p_a : a \in \mathfrak{A}\}$ is eventually contained in A, and $p \in \mathrm{cl}_{\beta X}(A) = C(A)$.

From Theorem 2.4 we obtain the following topological information about the maximal ideal space of R: it is a compact totally disconnected Hausdorff space which contains a homeomorphic copy of X, namely, $\{M^p : p \in X\}$, as a dense discrete subspace.

We shall now give a simple characterization of the dual space of the topological linear space R in terms of its ring structure. Here we shall consider R as a space of analytic functions on D.

LEMMA 2.5.1. *If $\{C_n : n = 0, 1, \ldots\}$ is a sequence of complex numbers satisfying the condition $\limsup_n |C_n|^{1/n} \leqslant 1$, and if for each sequence $\{a_n : n = 0, 1, \ldots\}$ of complex numbers satisfying the same condition, the series $\sum_{n=0}^{\infty} a_n C_n$ converges, then $\limsup_n |C_n|^{1/n} < 1$.*

Proof. Suppose $\{C_n : n = 0, 1, \ldots\}$ is a sequence of complex numbers satisfying $\limsup_n |C_n|^{1/n} = 1$. Then there exists a subsequence $\{n_k : k = 1, 2, \ldots\}$ of $\{n : n = 0, 1, \ldots\}$ such that $|C_{n_k}|^{1/n_k}$ converges to one and $C_{n_k} \neq 0$ for each k. We define the sequence $\{a_n : n = 0, 1, \ldots\}$ by $a_n = C_{n_k}^{-1}$ if $n = n_k$ for some k, and $a_n = 0$ if $n \neq n_k$, all k. Then $\limsup_n |a_n|^{1/n} \leqslant 1$ and $\sum_{n=0}^{\infty} a_n C_n$ fails to converge.

THEOREM 2.5. *The dual space R^* of R is isomorphic to the ideal $R_0 \cap \{M^p : p \in \beta X - X\}$.*

Proof. First, we shall fix $g \in R_0$ and show that g defines a unique functional L_g in R^*. We note that if $f \in R$, then fg (juxtaposition denotes, as above, the Hadamard product) is an element of R_0 and fg is defined at 1. We define $L_g : R \to C$ by $L_g(f) = fg(1)$. The function L_g is linear, homogeneous, and continuous and is, therefore, an element of R^*. Moreover, if $g = \sum_{p=0}^{\infty} C_p w_p$, where $w_p(x) = x^p$, and if $f = \sum_{p=0}^{\infty} a_p w_p$, then $L_g(f) = \sum_{p=0}^{\infty} a_p C_p$. The mapping $g \to L_g$ is clearly a well-defined homomorphism of R_0 into R^*. We will now show that this mapping is one-to-one and onto. If $g_1 \neq g_2$, then there exists a non-negative integer n such that $C_n^1 \neq C_n^2$, where $g_1 = \sum_{p=0}^{\infty} C_p^1 w_p$ and $g_2 = \sum_{p=0}^{\infty} C_p^2 w_p$. But $L_{g_1}(w_n) = (g_1 w_n)(1) = C_n^1$ and $L_{g_2}(w_n) = (g_2 w_n)(1) = C_n^2$. Hence, $g_1 \neq g_2$ implies $L_{g_1} \neq L_{g_2}$, and the mapping $g \to L_g$ is one-to-one. Now we fix an element L of R^* and consider the power series $\sum_{p=0}^{\infty} L(w_p) z^p$. We will show that this power series has radius of convergence greater than one, and, hence, defines an element of R_0. If $f \in R$, $f = \sum_{p=0}^{\infty} a_p w_p$, then $L(f) = L\left(\sum_{p=0}^{\infty} a_p w_p\right) = \sum_{p=0}^{\infty} a_p L(w_p)$ by the continuity of L. Hence, if we define f by the series $\sum_{p=0}^{\infty} \exp(-i\theta_p) w_p$, where $L(w_p) = r_p \exp(i\theta_p)$; then f is an element of R and $L(f) = \sum_{p=0}^{\infty} r_p = \sum_{p=0}^{\infty} |L(w_p)|$. Therefore, $\sum_{p=0}^{\infty} L(w_p)$ converges absolutely, and if we define $g = \sum_{p=0}^{\infty} L(w_p) w_p$, then $g \in R$. Now, by Lemma 2.5.1, $g \in R_0$ and $L_g = L$. Hence, the mapping $g \to L_g$ is an isomorphism of R_0 into R^*, where the correspondence is given by $L_g(f) = fg(1)$.

3. Residue class rings.

In the preceding section we gave a description of the space of maximal ideals of R and a description of the ideals themselves. We continue here our study of these ideals by considering the structure of their residue class rings.

We will first concern ourselves with the maximal ideals of type (II). We denote by φ_p the multiplicative functional corresponding to M^p, $p \in X$ $(\bar{\varphi}_p(f) = f(p))$. We will show that if M^p is of type (II) $(p \in X)$, then $\varphi_p \eta(C) = R/M^p$. To this end we define $\sigma : R/M^p \to C$ by $\sigma(\varphi_p(f)) = \bar{\varphi}_p(f)$ for each $f \in R$. The function σ is an isomorphism of R/M^p onto C, and $a \in C$ implies $\sigma(\varphi_p \eta(a)) = a$. Hence, each class in R/M^p is determined by a complex number. Thus, for $p \in X$, there exists an isomorphism $\sigma_p: R/M^p \to C$ which is C-homogeneous ($\sigma_p \varphi_p \eta$ = identity).

If M^p is a maximal ideal of type (I) ($p \in \beta X - X$), then the structure of R/M^p is more complex than in the case just considered, and we need here several definitions and results from the theory of field extensions. A detailed discussion is found in Chapter II of [3]:

If k is a subfield of K, then $k[Z]$ will denote the polynomial ring in the indeterminate Z over k, $k[Z_1, \ldots, Z_n]$ will denote the polynomial ring in the n indeterminates $Z_1, \ldots, Z_n$ over k, and if L is any set of elements of K, $k(L)$ will denote the smallest subfield of K which contains both k and L. An element ξ in K is said to be *transcendental* over k if it satisfies no polynomial in $k[Z]$ except the zero polynomial. A finite set $\{\xi_1, \ldots, \xi_n\}$ of elements of K is said to be *algebraically independent* if the only polynomial in $k[Z_1, \ldots, Z_n]$ satisfied by $\{\xi_1, \ldots, \xi_n\}$ is the zero polynomial. A subset L of K is called a *transcendence set over* k if every finite subset of L is algebraically independent over k. By a Zorn's lemma argument it can be shown that every transcendence set in K is contained in a maximal transcendence set, called a *transcendence basis* of K over k. Moreover, any two transcendence bases of K over k have the same cardinality, which is called the *transcendence* degree of K over k. Finally, if L is a transcendence basis of K over k, then K is an algebraic extension of $k(L)$.

A field K is said to be *algebraically closed* if every polynomial in $K[Z]$ has a root in K, or equivalently, if the only irreducible polynomials in $K[Z]$ are of degree one. A field K is called an *algebraic closure* of a subfield k if 1) K is algebraic over k and 2) K is algebraically closed. If K is an algebraically closed field and K and K' are isomorphic, then K' is also algebraically closed. The following theorem will be stated without proof, a proof may be found on pages 107, 108 of [3]:

THEOREM 3.1. *Let K' be an algebraically closed field and K be an algebraic extension of a field k. If φ is an isomorphism of k into K', then φ can be extended to an isomorphism of K into K'.*

THEOREM 3.2. *If $p \in \beta X - X$, then $\varphi_p \eta(C) \subsetneq R/M^p$.*

Proof. Let f by the element of R defined by $f(0) = 1$ and $f(n) = n^{-1}$ for $n \neq 0$. Then f is a unit in R and $f \in M_1^p$ for each $p \in \beta X - X$, where M_1^p is the maximal ideal of $B(X)$ which corresponds to p. If f is equivalent to a complex multiple γe of e modulo M^p, then $\varphi_p(f - \gamma e) = \varphi_p(\theta)$ and $f - \gamma e \in M^p$. But $\gamma e \in B(X)$. Therefore, $f - \gamma e \in M^p \cap B(X) \subset M_1^p = \{g \in B(X): g^\beta(p) = 0\}$. Now, if $\{p_a : a \in \mathfrak{A}\}$ is any net in X converging to p, then, by Lemma 2.3.1, for each $n \in X$ there exists $a_0 \in \mathfrak{A}$ such that $p_a > n$ whenever $a \geqslant a_0$. Hence $\lim_a f(p_a) = 0$, and $\lim_a (f - \gamma e)(p_a) = 0$ implies $\gamma = 0$ or $f \in M^p$, which is a contradiction. Therefore, we see that $\varphi_p(f) \in R/M^p - \varphi_p \eta(C)$.

The complex number field is algebraically closed, and it follows that $\varphi_p \eta(C)$ must also be algebraically closed. Hence, if $p \in \beta X - X$, then R/M^p is a transcendental extension of $\varphi_p \eta(C)$, i. e., each element of $R/M^p - \varphi_p \eta(C)$ is transcendental over $\varphi_p \eta(C)$. The letter f used below denotes the function defined in Theorem 3.2.

LEMMA 3.3.1. *If r is a positive real number and if f^r is defined by $f^r(n) = (f(n))^r$, then $f^r \in R$ and $\varphi_p(f^r) \in R/M^p - \varphi_p \eta(C)$. Moreover, if $r_1 \neq r_2$, then $\varphi_p(f^{r_1}) \neq \varphi_p(f^{r_2})$.*

Proof. If r is a positive real number, then $\limsup_n |f^r(n)|^{1/n} = \limsup_n |f(n)|^{r/n} \leqslant 1$, and $f^r \in R$. Suppose $\varphi_p(f^r) = \varphi_p(\gamma e)$ for some $\gamma \in C$. Then $f^r - \gamma e \in M^p \cap B(X) \subset M_1^p$, and $(f^r - \gamma e)^\beta(p) = 0$. But $f^{r\beta}(p) = 0$, and, hence, $\gamma = 0$. This implies that $f^r \in M^p$ and, consequently, $f \in M^p$, a contradiction. Hence, $\varphi_p(r^r) \in R/M^p - \varphi_p \eta(C)$. Now suppose that r_1 and r_2 are distinct positive real numbers such that $\varphi_p(f^{r_1}) = \varphi_p(f^{r_2})$. Without loss of generality we may assume that $r_1 < r_2$ and, hence, that $r_2 = r_1 + r$, where r is a positive real number. From our original assumption it follows that $f^{r_2} - f^{r_1} \in M^p$ or $f^{r_1} f^r - f^{r_1} \in M^p$. Thus $f^{r_1}(f^r - e) \in M^p$, where f^{r_1} is a unit in R. Hence $f^r - e \in M^p$, a contradiction.

We have now established the existence of an uncountable collection $\{\varphi_p(f^r): r \text{ real, positive number}\}$ of distinct elements of R/M^p, each of which is transcendental over $\varphi_p \eta(C)$. In the following lemma and theorem we will show that from this collection an uncountable transcendence set can be extracted.

Let T be a transcendence basis of C over the field of rational numbers composed of positive real numbers. Then T has the cardinality of the continuum, and if $s_1, \ldots, s_n$ is a collection of rational numbers and if $r_1, \ldots, r_n$ is a collection of elements of T such that $\sum_{k=1}^n s_k r_k = 0$, then $s_1 = s_2 = \ldots = s_n = 0$.

LEMMA 3.3.2. *The collection $S(f, T) = \{\varphi_p(f^r): r \in T\}$ is a transcendence set in R/M^p over $\varphi_p \eta(C)$.*

Proof. Suppose $P(Z_1, \ldots, Z_n)$ is a polynomial in $\varphi_p \eta(C)[Z_1, \ldots, Z_n]$ satisfied by $\{\varphi_p(f^{r_1}), \ldots, \varphi_p(f^{r_n})\}$, a finite collection of elements from $S(f, T)$. We shall show that $P(Z_1, \ldots, Z_n)$ is the zero polynomial. Now $P(Z_1, \ldots, Z_n) = \sum_{i=0}^m a'_{i_1 \ldots i_n} Z_1^{i_1} \ldots Z_n^{i_n}$, where $i \neq j$ implies that there exists $l \in \{1, \ldots, n\}$ such that $i_l \neq j_l$, and where $a'_{i_1 \ldots i_n} \in \varphi_p \eta(C)$. Moreover,

$$P(\varphi_p(f^{r_1}), \ldots, \varphi_p(f^{r_n})) = \sum_{i=0}^m a'_{i_1 \ldots i_n} \varphi_p(f^{r_1})^{i_1} \ldots \varphi_p(f^{r_n})^{i_n},$$

and $\varphi_p(f^{r_j})^{i_j} = \varphi_p(f^{i_j r_j})$. Let $a_i' = a'_{i_1\ldots i_n}$ and let $k_i' = i_1r_1+\ldots+i_nr_n$. Then $P(\varphi_p(f^{r_1}), \ldots, \varphi_p(f^{r_n}))$ may be written as follows: $\sum_{i=0}^{m} a_i'\varphi_p(f^{k_i'})$, since $\varphi_p(f^{i_1r_1})\ldots\varphi_p(f^{i_nr_n}) = \varphi_p(f^{i_1r_1}f^{i_2r_2}\ldots f^{i_nr_n}) = \varphi_p(f^{i_1r_1+\ldots+i_nr_n}) = \varphi_p(f^{k_i'})$. From the algebraic independence of $\{r_1, \ldots, r_n\}$ it follows that $i \neq j$ implies that $k_i' \neq k_j'$. Now, since the k_i''s are distinct they can be rearranged in order of increasing magnitude. We denote the superscript of smallest magnitude by k_0, the next by k_1, and finally we denote the superscript of largest magnitude by k_m. Similarly, we rearrange the coefficients so that a_i is the coefficients of $\varphi_p(f^{k_i})$ in the new ordering. The polynomial $P(\varphi_p(f^{r_1}), \ldots, \varphi_p(f^{r_n}))$ now has the form: $\sum_{i=0}^{m} a_i\varphi_p(f^{k_i})$, where $k_0 < k_1 < \ldots < k_m$ and $a_i \in \varphi_p\eta(C)$ for each $i \in \{0, \ldots, m\}$. Moreover, each a_i is the image under φ_p of a complex multiple γ_i of e; hence, the polynomial has the form $\sum_{i=0}^{m} \varphi_p(\gamma_i e)\varphi_p(f^{k_i}) = \varphi_p(\sum_{i=0}^{m} \gamma_i f^{k_i})$.

By assumption, $P(\varphi_p(f^{r_1}), \ldots, \varphi_p(f^{r_n})) = \varphi_p\eta(0)$, the zero of $\varphi_p\eta(C)$; hence, $\varphi_p(\sum_{i=0}^{m} \gamma_i f^{k_i}) = \varphi_p(0)$ and $\sum_{i=0}^{m} \gamma_i f^{k_i} \in M^p$. But $\sum_{i=0}^{m} \gamma_i f^{k_i} = f^{k_0}(\sum_{i=0}^{m} \gamma_i f^{k_i-k_0})$, where $k_i - k_0 > 0$ for each $i \in \{1, \ldots, m\}$. The factor f^{k_0} in the last expression is a unit in R and, therefore, $\sum_{i=0}^{m} \gamma_i f^{k_i-k_0} \in M_p$. We let g_1 denote $\sum_{i=1}^{m} \gamma_i f^{k_i-k_0}$. Then $\gamma_0 e + g_1 \in M^p$. Further, since g_1 is the sum of finitely many elements of $B(X)$ and $\gamma_0 e$ is an element of $B(X)$, we have $\gamma_0 e + g_1 \in M^p \cap B(X) \subset M_1^p$. Hence $(\gamma_0 e + g_1)^\beta(p) = 0$. But $g_1^\beta(p) = 0$; hence, $\gamma_0 = 0$. Suppose we have shown that $\gamma_0 = \ldots = \gamma_{e-1} = 0$. Then $\sum_{i=e}^{m} \gamma_i f^{k_i} = f^{k_e}(\sum_{i=e}^{m} \gamma_i f^{k_i-k_e}) \in M^p$ and $\sum_{i=e}^{m} \gamma_i f^{k_i-k_e} \in M^p$. As above we may conclude that $\gamma_e = 0$. Hence each coefficient of $P(Z_1, \ldots, Z_n)$ is the image under φ_p of $\eta(0)$, and $P(Z_1, \ldots, Z_n)$ is the zero polynomial in $\varphi_p\eta(C)[Z_1, \ldots, Z_n]$. Thus $S(f, T)$ is a transcendence set in R/M^p over $\varphi_p\eta(C)$.

THEOREM 3.3. *If $p \in \beta X - X$, then R/M^p is an extension field of $\varphi_p\eta(C)$ having transcendence degree $\mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum.*

Proof. In Lemma 3.3.1 we demonstrated the existence of a transcendence set having cardinality $\mathfrak{c}$. Therefore the transcendence degree of R/M^p over $\varphi_p\eta(C)$ is at least $\mathfrak{c}$. But the cardinality of R/M^p is itself $\mathfrak{c}$. The theorem follows from these two facts.

THEOREM 3.4. *If $p \in \beta X - X$, then R/M^p is an algebraically closed field.*

Proof. Let $P(Z)$ be a polynomial in $R/M^p[Z]$. We will show that $P(Z)$ has a root in R/M^p. The polynomial $P(Z) = \sum_{i=0}^{n} a_i Z^i$, where $a_i \in R/M^p$.

Since R/M^p is a field we can, without loss of generality, assume that the leading coefficient a_n of $P(Z)$ is the identity element $\varphi_p(e)$ of R/M^p. Now, $a_i \in R/M^p$ implies that there exists $f_i \in R$ such that $\varphi_p(f_i) = a_i$. Thus, it will suffice to show that there exists $g \in R$ such that $\sum_{i=0}^{n} f_i g^i \in M^p$, where $f_n = e$. For then we have $\varphi_p(\sum_{i=0}^{n} f_i g^i) \in \varphi_p(M^p) = \varphi_p\eta(0)$. But $\varphi_p(\sum_{i=0}^{n} f_i g^i) = \sum_{i=0}^{n} \varphi_p(f_i)[\varphi_p(g)]^i = \sum_{i=0}^{n} a_i[\varphi_p(g)]^i$, and $\varphi_p(g)$ is a root of $P(Z)$. We can, therefore, transfer the problem to R by considering the polynomial $\hat{P}(Z) = \sum_{i=0}^{n} f_i Z^i$ in the polynomial ring $R[Z]$. Now, for each $k \in X$, the polynomial $\hat{P}_k(Z) = \sum_{i=0}^{n} f_i(k) Z^i$ is in $C[Z]$, and, therefore, splits into linear factors in C; i. e., there exist complex numbers $\alpha_1, \ldots, \alpha_n$ such that $P_k(Z) = (Z-\alpha_1)\ldots(Z-\alpha_n)$. For each $k \in X$, we let $g_1(k), \ldots, g_n(k)$ denote the n complex roots of $P_k(Z)$ arranged in order of increasing magnitude, where if $|g_{m_1}(k)| = \ldots = |g_{m_q}(k)|$, then these roots are arranged in order of increasing argument. We note that, for each $k \in X$, $\sum_{i=0}^{n} f_i(k)[g_j(k)]^i = 0$ for $j \in \{1, \ldots, n\}$, and $(-1)^n g_1(k)\ldots g_n(k) = f_0(k)$, the constant term of $P_k(Z)$. Thus $(-1)^n g_1\ldots g_n = f_0 \in R$, and $\hat{P}(Z) = (-1)^n g_1\ldots g_n + \sum_{i=1}^{n} f_i Z^i$.

The constant term $(-1)^n g_1\ldots g_1$ is an element of R and, therefore, satisfies $\limsup_k |(-1)^n(g_1\ldots g_n)(k)|^{1/k} \leqslant 1$. Hence, corresponding to each non-negative integer m there is an integer p_m' such that $|(-1)^n(g_1\ldots g_n)(k)|^{1/k} < 1+2^{-m}$ for all $k > p_m'$. Let $p_m = \max(p_m', p_{m-1}+1)$. Then for each $k > p_m$, $|(-1)^n g_1(k)\ldots g_n(k)|^{1/k} < 1+2^{-m}$ or $|g_1(k)|^{1/k}\ldots|g_n(k)|^{1/k} < 1+2^{-m}$. Now, for $k \leqslant p_1$ we define $g(k) = g_1(k)$. For $p_m < k \leqslant p_{m+1}$, $|g_1(k)|^{1/k}\ldots|g_n(k)|^{1/k} < 1+2^{-m}$ and there exists an integer j, $1 \leqslant j \leqslant n$, such that $|g_j(k)|^{1/k} < 1+2^{-m}$. Therefore, there exists a smallest such integer j_k. We define $g(k) = g_{j_k}(k)$. Finally, for $k \in X$ such that $k > p_m$, all m, $|g_1(k)|^{1/k}\ldots|g_n(k)|^{1/k} \leqslant 1$, and there exists a smallest integer j_k, $1 \leqslant j_k \leqslant n$, such that $|g_{j_k}(k)|^{1/k} \leqslant 1$. We define $g(k) = g_{j_k}(k)$. This defines a function g from X to C. Moreover, if $k \in X$, then $\sum_{i=0}^{n} f_i(k) g(k)^i = 0$, and $(\sum_{i=0}^{n} f_i g^i)(k) \equiv 0$ implies that $\sum_{i=0}^{n} f_i g^i$ is the zero of R. Hence, it will suffice to show that g is an element of R. To this end we fix $\varepsilon > 0$. There exists an integer m such that $2^{-m} < \varepsilon$. For $k > p_m$, $|g(k)|^{1/k} < 1+2^{-m} < 1+\varepsilon$; and it follows that $\limsup_k |g(k)|^{1/k} \leqslant 1$ and g is an element of R.

Thus $\sum_{i=0}^{n}\varphi_p(f_i)[\varphi_p(g)]^i = \varphi_p\eta(0)$, and $\varphi_p(g)$ is a root of the polynomial $P(Z)$ in R/M^p.

Theorems 3.3 and 3.4 give us a fairly complete description of R/M^p whenever $p\in\beta X-X$. Moreover, from the next theorem and its corollaries we obtain the somewhat startling result: although R/M^p is an extension field of $\varphi_p\eta(C)$ having an infinite algebraic basis, R/M^p and $\varphi_p\eta(C)$ are still isomorphic fields.

THEOREM 3.5. *If k and k' are isomorphic fields and if K and K' are extension fields of k and k', respectively, satisfying the conditions: 1) K and K' are algebraically closed and 2) the transcendence degrees of K over k and K' over k' are the same, then there exists an isomorphism of K onto K' which extends the isomorphism of k onto k'.*

Proof. Let L be a transcendence basis for K over k, L' a basis for K' over k', and φ an isomorphism of k onto k'. Then there exists an isomorphism $\bar{\varphi}$ from $k(L)$ onto $k'(L')$ which extends φ. Now, K is algebraic over $k(L)$ and K' is an algebraic closure of $k'(L')$. Therefore, by Theorem 3.1, there exists an isomorphism Φ of K into K' which extends $\bar{\varphi}$ and which, therefore, extends φ. Moreover, since K is algebraically closed, $\Phi(K)$ must be K. For if not, then there exists a polynomial $P'(Z)$ in $\Phi(K)[Z]$ which is irreducible and of degree greater than one. There corresponds to $P'(Z)$ under Φ an irreducible polynomial $P(Z)$ of degree greater than one in $K[Z]$. This contradicts the fact that K is algebraically closed. Hence, $\Phi(K) = K'$ and the proof follows.

COROLLARY 3.5.1. *If K is an algebraically closed field of characteristic zero and transcendence degree $\mathfrak{c}$ over its prime field, then K is isomorphic to the complex number field.*

Proof. If K has characteristic zero, then its prime field is isomorphic to the field of rational numbers, the prime field of C. Moreover, C is an algebraically closed extension field of the rational numbers having transcendence degree $\mathfrak{c}$. The corollary now follows from Theorem 3.5.

COROLLARY 3.5.2. *If $p\in\beta X-X$, then R/M^p is isomorphic to the complex number field $\boldsymbol{C}$.*

Proof. We have, from Theorems 2.3 and 2.4, that R/M^p is a transcendental extension of $\varphi_p\eta(C)$ having transcendence degree $\mathfrak{c}$ over $\varphi_p\eta(C)$, and is, moreover, algebraically closed. Hence, R/M^p has transcendence degree $\mathfrak{c}$ over the image of the rational number field under $\varphi_p\eta$. Thus, by Corollary 3.5.1, we have that R/M^p and $\boldsymbol{C}$ are isomorphic.

Although the residue class rings R/M^p are algebraically indistinguishable (i. e., are all isomorphic), the fact that R/M^p and $\varphi_p\eta(\boldsymbol{C})$ are not identical for $p\in\beta X-X$ allows us to distinguish the two classes of residue class rings in terms of the kinds of complex-valued isomorphisms which they admit.

THEOREM 3.6. *If $p\in\beta X$, then there exists an isomorphism σ of R/M^p onto $\boldsymbol{C}$. Moreover, σ can be chosen to be $\boldsymbol{C}$-homogeneous ($\sigma\varphi_p\eta$ = identity) if, and only if, $p\in X$.*

Proof. The existence of the isomorphism σ has been demonstrated above. Moreover, we also showed that if $p\in X$, then the natural homomorphism $\sigma: R/M^p\to \boldsymbol{C}$, defined by $\sigma(\varphi_p(f)) = \bar{\varphi}_p(f)$, is $\boldsymbol{C}$-homogeneous. To prove the converse we assume that there exists $p\in\beta X-X$ and a $\boldsymbol{C}$-homogeneous isomorphism σ of R/M^p onto $\boldsymbol{C}$. By Theorem 3.2 there exists $\xi\in R/M^p-\varphi_p\eta(\boldsymbol{C})$. Consider the element $\varphi_p\eta(\sigma(\xi))\in R/M^p$. By assumption, $\sigma\big(\varphi_p\eta(\sigma(\xi))\big) = \sigma(\xi)$; hence, ξ and $\varphi_p\eta(\sigma(\xi))$ both map into $\sigma(\xi)$ under σ. But $\xi\in R/M^p-\varphi_p\eta(\boldsymbol{C})$, a contradiction to the fact that σ is one-to-one.

COROLLARY 3.6.1. *If φ is a $\boldsymbol{C}$-homogeneous homomorphism ($\varphi\eta$ = identity) of R onto $\boldsymbol{C}$, then φ is continuous and, moreover, $\varphi = \bar{\varphi}_p$ for some $p\in X$.*

Proof. If φ is a $\boldsymbol{C}$-homogeneous homomorphism of R onto $\boldsymbol{C}$, then there exists $p\in\beta X$ such that $\varphi^{-1}(0) = M^p$. If we define $\sigma: R/M^p\to\boldsymbol{C}$ by $\sigma(\varphi_p(f)) = \varphi(f)$, then σ is a $\boldsymbol{C}$-homogeneous isomorphism of R/M^p onto $\boldsymbol{C}$. Therefore, $p\in X$, and $\varphi = \bar{\varphi}_p$. The last statement follows from the fact that if $f\in R$, then $f-\varphi(f)e\in M^p$ and $\bar{\varphi}_p(f-\varphi(f)e) = 0$. But $\bar{\varphi}_p(f-\varphi(f)e) = \bar{\varphi}_p(f)-\varphi(f) = 0$. Hence $\bar{\varphi}_p = \varphi$, and φ is continuous.

4. Closed ideals. In this section we shall consider the closed ideals of R and characterize these ideals in terms of principal ideals. We will consider R to be a ring of functions on X (except in Theorem 4.3), and will use the following topological property: If $f\in R$ and if for each non-negative integer n we define $f_n(p) = f(p)$ for $p\leqslant n$, $f_n(p) = 0$ for $p > n$, then $\{f_n: n = 0, 1, 2, \ldots\}$ converges to f in the topology of R. We shall mean "proper ideal" ($\neq R$) when we say ideal.

THEOREM 4.1. *If I is an ideal in R and if for each $f\in R$ we define f^* by $f^*(p) = \overline{f(p)}$, then $f\in I$ if, and only if, $f^*\in I$.*

Proof. If $f\in R$, then $f(p) = r_p\exp(i\theta_p)$ for each $p\in X$, and the function h_f defined by $h_f(p) = \exp(-2i\theta_p)$ is in R. Then fh_f is in I and $fh_f(p) = r_p\exp(-i\theta_p) = f^*(p)$. Hence, $f\in I$ implies $f^*\in I$. Since $(f^*)^* = f$, the theorem follows.

Theorem 4.1 will be used here as a lemma to the theorems following. However, it is of independent interest. The mapping $f\to f^*$ defined above is an involution in R, and Theorem 4.1 says that every ideal in R is its own image under this mapping (every ideal is symmetric).

THEOREM 4.2. *If I is an ideal in R, then $\bar{I}$ is an ideal in R if, and only if, there exists $n\in X$ such that $I\subset M^n$.*

Proof. The sufficiency is clear, since $I \subset M^n$ implies $\bar{I} \subset \overline{M^n} \neq R$, and the closure of an ideal is an ideal if it is proper. Conversely, suppose I is an ideal in R such that for each $n \in X$ there exists $f'_n \in I - M^n$. Then $f'_n f'^*_n \notin M^n$, since M^n is a prime ideal (for definition, see [3], p. 149). But $f'_n f'^*_n(p) = |f'_n(p)|^2$ for each $p \in X$, and $f'_n \notin M^n$ implies that $|f'_n(n)|^2 > 0$. We now define f_n by $f_n(p) = \sum_{k=0}^{n} |f'_k(p)|^2$. Then $f_n \in I$ and $f_n(p) > 0$ for $p \in \{0, \ldots, n\}$. Finally, we define f''_n by $f''_n(p) = \big(f_n(p)\big)^{-1}$ for $p \in \{0, \ldots, n\}$ and $f''_n(p) = 0$ otherwise. Then $f''_n \in R$ and $f_n f''_n \in I$. But $f_n f''_n = k_{\{0,\ldots,n\}}$, and by subtracting $f_{n-1} f''_{n-1}$ from $f_n f''_n$ we have $k_{\{n\}} \in I$. Hence, I contains the characteristic functions of points. Consequently, $I \supset I_0$ and is dense in R. Moreover, we have that every dense ideal contains the polynomials.

The structure of the closed ideals of R is easily studied by means of the principal ideals (ideals generated by a single element of R) and their closures. Theorem 4.3 was communicated by P. Porcelli. We include a proof here since we have been informed that the result is not specifically stated in the literature. We note that in this theorem R is the ring of analytic functions.

THEOREM 4.3. *If f and g are elements of R such that $f^{(n)}(0) = 0$ implies $g^{(n)}(0) = 0$, then g can be approximated in the compact-open topology by finite linear combinations of the form $\sum_{i=0}^{n} a_i f_{z_i}$, where $a_i \in C$, $|z_i| = 1$, and $f_{z_i}(z) = f(z_i z)$.*

Proof. We shall first consider the case where both f and g have radius of convergence greater than one. We denote the unit circle by C and the restrictions of f and g to C by $\hat{f}$ and $\hat{g}$, respectively. Let $L(\hat{f})$ denote the closure in $L^1(C)$ of the linear manifold generated in $L^1(C)$ by $\hat{f}$ and its translates by members of C. Then $Z(\hat{f})$ is an ideal in $L^1(C)$ (cf. [2], p. 374). Moreover, the space $\mathscr{M}$ of maximal regular ideals of $L^1(C)$ is homeomorphic to the space of integers with the discrete topology. Therefore, if I is a closed ideal in $L^1(C)$, then $h(I) = \{M \in \mathscr{M} : I \subset M\}$ is a discrete set in $\mathscr{M}$. Hence, $k\big(h(I)\big) = I$, where $k(\Delta) = \bigcap \{M \in \Delta\}$ for each subset Δ of $\mathscr{M}$ (cf. [2], p. 221-222, and p. 423). Now, since $f^{(n)}(0) = 0$ implies $g^{(n)}(0) = 0$, we have $\hat{g} \in kh\big(L(\hat{f})\big)$, and $\hat{g} \in L(\hat{f})$. Hence, $\hat{g}$ can be approximated in L^1-norm by sums of the form $\sum_{i=1}^{n} a_i \hat{f}_{z_i}$, where $a_i \in C$ and $z_i \in C$. Now we shall show that f and g satisfy the conclusion of the theorem. Fix $\varepsilon > 0$ and a compact set K in D. Let $\delta = \min\{1 - |x| : x \in K\}$. Then $\varepsilon\delta > 0$ and there exist $\{a_1, \ldots, a_n\} \subset C$ and $\{z_1, \ldots, z_n\} \subset C$ such that $\|\hat{g} - \sum_{i=1}^{n} a_i \hat{f}_{z_i}\|_1 < \varepsilon$. If $x \in K$, then

$$\Big|g(x) - \sum_{i=1}^{n} a_i f_{z_i}(x)\Big| = \Big|\Big(g - \sum_{i=1}^{n} a_i f_{z_i}\Big)(x)\Big| = \Big|(2\pi i)^{-1} \int_C \Big(g - \sum_{i=1}^{n} a_i f_{z_i}\Big)(z) \cdot (z - x)^{-1} dz\Big|$$
$$\leqslant (2\pi)^{-1} \int_C \Big|\Big(g - \sum_{i=1}^{n} a_i f_{z_i}\Big)(z)\Big| \, |z - x|^{-1} d(|z|)$$
$$\leqslant (\delta^{-1}) \Big[(2\pi)^{-1} \int_C \Big|\Big(g - \sum_{i=1}^{n} a_i f_{z_i}\Big)(z)\Big| \, d|z|\Big].$$

But the second factor is $\|g - \sum_{i=1}^{n} a_i f_{z_i}\|_1$. Hence, $x \in K$ implies $|g(x) - \sum_{i=1}^{n} a_i f_{z_i}(x)| < \varepsilon$, and the theorem is proved for f and g having radius of convergence greater than one.

Now we let f and g be arbitrary elements of R satisfying the hypotheses of the theorem. We fix $\varepsilon > 0$ and a compact set K in D. We next choose a sequence $\{r_n : n = 1, 2, \ldots\}$ of elements of $[0, 1)$ converging to one. Then there exists n such that K is contained in the disc about the origin of radius r_n^2. We fix this integer n. Now, f_{r_n} and g_{r_n} satisfy the hypothesis of the first case and there exist $\{a_1, \ldots, a_m\} \subset C$ and $\{z_1, \ldots, z_m\} \subset C$ such that if $|x| < r_n$, then $|g_{r_n}(x) - \sum_{i=1}^{m} a_i f_{r_n z_i}(x)| < \varepsilon$. If $x \in K$, then $|x| < r_n^2$, and there exists z, $|z| < r_n$, such that $x = r_n z$. Then

$$\Big|g(x) - \sum_{i=1}^{m} a_i f_{z_i}(x)\Big| = \Big|g(r_n z) - \sum_{i=1}^{m} a_i f_{z_i}(r_n z)\Big| = \Big|g_{r_n}(z) - \sum_{i=1}^{m} a_i f_{r_n z_i}(z)\Big| < \varepsilon.$$

This theorem makes quite strong statements about approximating analytic functions. For example, any function which is analytic on D can be uniformly approximated by the function $\exp(z)$, since this function has no derivatives which vanish at the origin. Likewise, any other similar function approximates all elements of R.

We now restate Theorem 4.3 in algebraic terms. Here (f) denotes the principal ideal in R generated by f.

COROLLARY 4.3. *If f and g are elements of R such that $n \in X$ and $f \in M^n$ implies $g \in M^n$, then $g \in (\bar{f})$.*

Proof. By Theorem 3.3, g can be approximated by sums of the form $\sum_{i=1}^{n} a_i f_{z_i}$, as an analytic function. In terms of the ring of functions on X, we have $f_{z_i}(p) = z_i^p f(p) = e_{z_i} f(p)$. Thus $\sum_{i=1}^{n} a_i f_{z_i} = \sum_{i=1}^{n} a_i e_{z_i} f = (\sum_{i=1}^{n} a_i e_{z_i}) f \in (f)$. Hence, $g \in (\bar{f})$.

THEOREM 4.4. *If $f \in R$ and $Z(f) = \{p \in X : f(p) = 0\}$, then $(\bar{f}) = R$ if $Z(f) = \emptyset$ and $(\bar{f}) = \bigcap \{M^n : n \in Z(f)\}$ if $Z(f) \neq \emptyset$. Hence, the closure of every principal ideal is principal.*

Proof. If $Z(f) = \emptyset$, then $e \in (\bar{f})$ and $(\bar{f}) = R$. If $Z(f) \neq \emptyset$, then the function $e - k_{Z(f)}$ is zero if, and only if, f is zero; hence, $e - k_{Z(f)} \in (\bar{f})$. But $(e - k_{Z(f)}) = \bigcap \{M^n : n \in Z(f)\}$, and $f \in M^n$ for each $n \in Z(f)$. Therefore, $(\bar{f}) = (e - k_{Z(f)}) = \bigcap \{M^n : n \in Z(f)\}$.

THEOREM 4.5. *If $f \in R$, then (f) is closed if, and only if, $f + k_{Z(f)}$ is a unit in R.*

Proof. We shall first prove the necessity of the condition. If $(f) = (\bar{f})$, then $(f) = (e - k_{Z(f)})$ and there exists $g \in R$ such that $fg = e - k_{Z(f)}$. Then $k_{Z(f)} + fg = e$ and $f(p)g(p) = 1$ if, and only if, $f(p) \neq 0$. Moreover, $(k_{Z(f)} + f)(p)$ is 1 if $p \in Z(f)$ and is $f(p)$ if $p \in X - Z(f)$. We define h by $h(p) = 1$ if $p \in Z(f)$ and $h(p) = g(p)$ if $p \in X - Z(f)$. Then $h \in R$, since $g \in R$; and $h(f + k_{Z(f)}) = e$.

Conversely, if $f + k_{Z(f)}$ is a unit in R, then there exists $g \in R$ such that $g(f + k_{Z(f)}) = e$. Moreover, $g(p) = 1$ if $p \in Z(f)$ and $g(p) = \big(f(p)\big)^{-1}$ if $p \in X - Z(f)$. Also, we have $g(e - k_{Z(f)})(p) = \big(f(-p)\big)^{-1}$ if $p \in X - Z(f)$ and $g(e - k_{Z(f)})(p) = 0$ if $p \in Z(f)$. But then $g(e - k_{Z(f)})f = e - k_{Z(f)}$. Hence, $(e - k_{Z(f)}) \subset (f)$. However, $(\bar{f}) = (e - k_{Z(f)})$.

THEOREM 4.6. *An ideal I in R is closed if, and only if, it is the intersection of closed maximal ideals. In this case it is a principal ideal; hence, closed ideals are principal.*

Proof. The sufficiency is obvious. Conversely, suppose I is a closed ideal in R and let $X_0 = \{p \in X : I \subset M^p\}$. Then $X_0 \neq \emptyset$ and $I \subset \bigcap \{M^n : n \in X_0\}$. Fix $f \in I$. Then $X_0 \subset Z(f)$. Moreover, $|f|^2 \in I$, since $|f|^2 = ff^*$, and $Z(|f|^2) = Z(f) = X_0 \cup \{n_1, n_2, \ldots\}$, where the n_i are arranged in order of increasing magnitude. If the collection $\{n_1, n_2, \ldots\}$ is finite, then the following argument can be terminated in a finite number of steps. We shall now construct an element of I whose zero set is exactly X_0. The closed principal ideal generated by that element will be $\bigcap \{M^n : n \in X_0\}$ and will be contained in I. Now, for each i there exists $g_i \in I$ such that $g_i(n_i) \neq 0$. Let $h_i = g_i g_i^*$. Then $h_i \in I$ and $h_i(n_i) > 0$. The function $h_i k_{\{n_i\}}$ is in I and we define h_i' by $h_i'(p) = \big(h_i(n_i)\big)^{-1}$ if $p = n_i$ and zero otherwise. Then $h_i' \in R$ and $k_{\{n_i\}} h_i h_i' = k_{\{n_i\}} \in I$. Let q denote the characteristic function of $\{n_1, n_2, \ldots\}$, and for each non-negative integer n let $q_n = q k_{\{0,\ldots,n\}}$. Then $\{q_n : n = 0, 1, \ldots\}$ is a sequence of polynomials which converges to q in the topology of R. But, for each n, q_n is the sum of at most a finite number of the functions $k_{\{n_i\}}$. Hence, $q_n \in I$ for each n; and, since I is closed, $q \in I$. Therefore, $|f|^2 + q \in I$. But $Z(|f|^2 + q) = X_0$. Hence $I = \bigcap \{M^n : n \in X_0\}$.

In view of this theorem we can make a connection between the ring being studied and the general theory of topological rings. If A is a topological ring and S is a subset of A, then the sets $\mathscr{L}(S) = \{x \in A : xS = (0)\}$ and $\mathscr{R}(S) = \{x \in A : Sx = (0)\}$ are left and right ideals, respectively. The ring A is called a *dual ring* if 1) $\mathscr{L}(A) = (0) = \mathscr{R}(A)$ and 2) for each closed left ideal I_1 (closed right ideal I_r), $\mathscr{L}\big(\mathscr{R}(I_1)\big) = I_1$ ($\mathscr{R}(\mathscr{L}(I_r)) = I_r$). Dual rings are discussed at length in section 25 of [2]. The ring R under consideration is a dual ring, since each closed ideal is principal, and is, moreover, semi-simple. The results in [2] are obtained under the assumption that the dual rings have either a continuous inverse or a continuous quasi-inverse. Since R has dense ideals it cannot be a ring with continuous inverse. However, the structure of R is very similar to that obtained under these restrictive assumptions: R is a topological direct sum of the annihilators (two-sided minimal ideals) of its closed maximal ideals. Moreover, each minimal ideal is principal; in fact, each is generated by an irreducible idempotent q which has the property that RqR is a field isomorphic to the complex numbers.

In the last section of this chapter we shall concern ourselves with the prime and primary ideals in R (for a discussion of the latter, see [3], p. 152). If I is an ideal in R, then the radical of I, denoted by $\operatorname{rad}(I)$, is the set of all $f \in R$ such that $f^n \in I$ for some non-negative integer n.

THEOREM 4.7. *Every prime ideal (and, hence, every primary ideal) in R is contained in a unique maximal ideal.*

Proof. If P is a prime ideal in R which is contained in distinct maximal ideals M^p and M^q, then $P \cap B(X) \subset [M^p \cap B(X)] \cap [M^q \cap B(X)]$. Hence, $P \cap B(X) \subset M_1^p \cap M_1^q$, a contradiction, since $P \cap B(X)$ is a prime ideal in $B(X)$. For a primary ideal Q, $\operatorname{rad}(Q)$ is prime, and the result is obtained by considering this latter ideal.

THEOREM 4.8. *If P is a closed prime (primary) ideal in R, then P is maximal.*

Proof. If P is prime, then there exists a unique $n \in \beta X$ such that $P \subset M^n$. But, since P is closed, $n \in X$ and $P = M^n$. The theorem follows similarly for closed primary ideals.

An investigation of the dense prime ideals yielded only the result given in the following theorem. The proof depends on the following statement, a proof of which can be found on page 7 of [1]. If I is an ideal in a commutative ring with identity, then $\operatorname{rad}(I) = \bigcap \{P \subset R : P \text{ is a prime ideal}, P \supset I\}$.

THEOREM 4.9. *There exist non-maximal prime ideals in R.*

Proof. From Corollary 2.3.1, we have that the intersection of the dense maximal ideals of R is exactly the set of elements having radius of convergence greater than one, a set which property contains I_0. However, $I_0 = \operatorname{rad}(I_0) = \bigcap \{P \subset R : P \text{ is a prime ideal}, P \supset I_0\}$. Therefore, the set of dense prime ideals properly contains the set of dense maximal ideals.

Bibliography

[1] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton 1960.

[2] M. A. Naimark, *Normed rings*, Groningen 1960.

[3] O. Zariski and P. Samuel, *Commutative algebra*, Princeton 1958.

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On higher gradients of harmonic functions

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Chapter I

1. Let $U(x) = U(x_1, x_2, \ldots, x_n)$ be a real-valued harmonic function defined in a domain D of the n-dimensional Euclidean space ($n \geqslant 2$). Consider the norm $W(x)$ of the gradient of $U(x)$,

$$W(x) = |\mathrm{grad}\, U| = \left\{\sum_{j=1}^{n}\left(\frac{\partial U}{\partial x_j}\right)^2\right\}^{1/2}.$$

It is a classical fact that $W(x)$ is subharmonic in D, and therefore $\{W(x)\}^p$ is also subharmonic for any $p \geqslant 1$. E. M. Stein and G. Weiss [3] established a remarkable fact that $\{W(x)\}^p$ is subharmonic in D for some values of p less than 1, more precisely, subharmonic for any

$$p \geqslant \frac{n-2}{n-1}. \tag{1.1.1}$$

The example $U(x) = \left(\sum x_j^2\right)^{-(n-2)/2}$ shows that the result is false for p less than $(n-2)/(n-1)$. The case $n = 2$ is, of course, classical if we interpret the result as the subharmonicity of $\log W$.

In this chapter we extend the Stein-Weiss result to higher gradients.

2. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be any multi-index of weight m, that is, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers and $m = |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. We write $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!$ and

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\ldots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Given any harmonic function $U(x)$ we consider its gradient of order m, that is, the set of all distinct derivatives of order m (arranged in any fixed way)

$$\mathrm{grad}_m U(x) = \{D^\alpha U\}_{|\alpha| = m}$$

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