STUDIA MATHEMATICA, T. XXIV. (1964)

On a theorem of K. Maurin

by

W. F. DONOGHUE, Jr. (Lawrence)

In a recent paper [6] K. Maurin has shown that the embedding of certain functional Hilbert spaces in one another under certain circumstances is a Hilbert-Schmidt mapping. This result he applies to obtain significant results in the theory of generalized eigenfunction expansions. Our interest is in the Hilbert-Schmidt character of the embedding, and we consider Maurin's theorem in the context of the theory of Bessel potentials, where it reads as follows:

THEOREM. *Let D be a bounded open set in R^n and $P_0^a(D)$ the space of all Bessel potentials of order a vanishing outside D. The natural embedding of $P_0^a(D)$ in $P_0^\beta(D)$ is a Hilbert-Schmidt mapping if $a-\beta > n/2$.*

The theorem has been established by Maurin in the special case when a and β are integers; we extend the result to all values. Our method of proof is quite different from that of [6] which uses the integral character of a and β in an essential way, and which moreover depends on a difficult estimate of Sobolev's of a certain reproducing kernel.

1. Bessel potentials. The positive function $G_a(x)$ is defined on the space R^n by the formula

$$G_a(x) = [2^{(n+a-2)/2}\pi^{n/2}\Gamma(a/2)]^{-1}|x|^{(a-n)/2}K_{(n-a)/2}(|x|),$$

the function $K_\nu(z)$ being the modified Bessel function of the third kind.

For positive a the function $G_a(x)$ is integrable over R^n, it is integrable square if $a > n/2$ and it is continuous if $a > n$. Moreover, for all positive a and β the convolution equation $G_a * G_\beta = G_{a+\beta}$ holds, as well as the differential equation $(1-\Delta)G_a = G_{a-2}$.

The Bessel potential of order $a > 0$ form the space $P^a = P^a(R^n)$ of all functions which coincide except for a set of $2a$-capacity zero with convolutions of the form $u = G_a * f$ where f is in L^2; the integral exists, except, perhaps, for a set of the corresponding capacity zero, and we write $u = G_a f$. The norm of u in P^a equals the L^2 norm of the corresponding f and P^a is a Hilbert space which also appears as the perfect functional completion of the space of all (Bessel) potentials of order 2α of measures of finite 2α-energy. In contradistinction to Riesz potentials the Bessel potentials are always L^2 functions, and we have the following convenient formula for the norm in terms of the Fourier transform:

$$\|u\|_\alpha^2 = \int (1+|\xi|^2)^\alpha |\hat{u}(\xi)|^2 d\xi .$$

For $0 < 2\alpha < n$ the potentials coincide locally with the Riesz potentials of the same order and have exactly the same exceptional sets, similarly for $2\alpha = n$ the potentials are locally logarithmic potentials, and the sets of n-capacity zero are precisely those of the usual logarithmic capacity zero. For $2\alpha > n$ the potentials are continuous functions and only the empty set has capacity zero. For $\alpha = 0$ we have $P^0 = L^2$ and the capacity is the usual Lebesgue measure.

If D is an open set in R^n we define $P_0^\alpha(D)$ as the space of all potentials in P^α which vanish outside D except, perhaps, for an exceptional set. This is a closed subspace of P^α. In the special case that α is an even integer, $\alpha = 2k$, and $u = G_{2k}$ the function f belongs to $P_0^{2k}(D)$, then f may be obtained from u by the equation $f = (1-\Delta)^k u$. In this case, then, the L^2 function f vanishes outside D.

We also define the space $P^\alpha(D)$ as the space of all restrictions to D of potentials in P^α; this space appears in a natural way as a quotient space of P^α and we take on it the quotient norm. We will be concerned with $P^\alpha(D)$ mostly in the very special case when D is a hypercube W in R^n. In this case, as has been shown in [3] in a very general context, there exists a Lichtenstein extension, that is to say, a linear mapping $u \to \tilde{u}$ of $P^\alpha(W)$ into P^α such that $\tilde{u}$ coincides with u on W and for which there exists a constant M so that $\|\tilde{u}\|_\beta \leqslant M\|u\|_\beta$ for all u in $P^\alpha(W)$ and all β in the interval $0 \leqslant \beta \leqslant \alpha$. For any such β then, the Lichtenstein extension occurs as a continuous linear mapping of $P^\beta(W)$ into P^β (at first defined only on the dense subspace $P^\alpha(W)$). We may also require that the extension take its values in $P_0^\alpha(W')$ where W' is some larger hypercube containing the closure of W in its interior, and if D is an open set, the closure of which is contained in W, we may suppose the Lichtenstein extension so determined that potentials in $P_0^\alpha(D)$ have extensions which vanish outside W.

2. Hilbert-Schmidt mappings. A continuous linear transformation T which carries the Hilbert space H_1 into the Hilbert space H_0 is called *Hilbert-Schmidt* (abbreviated H-S) if and only if $\sum_{ij} |(Tu_i, v_j)_0|^2$ is finite where $\{u_i\}$ is some complete orthonormal set in H_1 and $\{v_j\}$ is some complete orthonormal set in H_0. The definition is independent of the choice of the orthonormal systems, and we prefer to write the series above in

the form $\sum_i \|Tu_i\|_0^2$. Thus the embedding of $P_0^\alpha(D)$ into $P_0^\beta(D)$ will be H-S if and only if $\alpha > \beta$ and

$$\sum_i \|u_i\|_\beta^2 < \infty$$

where the $\{u_i\}$ form a complete orthonormal system in $P_0^\alpha(D)$.

It is evident that the restriction of a H-S mapping to a subspace is again an H-S mapping. We emphasize the fact that the passage in the spaces H_1 and H_0 to equivalent quadratic norms does not change the class of H-S mappings.

3. Proof of Maurin's theorem. We remark first that if W is a hypercube containing the closure of D in its interior, the space $P_0^\alpha(D)$ appears as a closed subspace of $P^\alpha(W)$: a sequence of potentials u_n in $P_0^\alpha(D)$ converging in the norm of $P^\alpha(W)$ corresponds to a convergent sequence of extensions $\tilde{u}_n$ all of which vanish outside D and which converge to a potential in $P_0^\alpha(D)$. Thus, if the embedding of $P^\alpha(W)$ into $P^\beta(W)$ is H-S, so also is its restriction to $P_0^\alpha(D)$.

On the other hand, if W' is a small hypercube with closure contained in D, the extension mapping of $P^\alpha(W')$ into P^α may be supposed to take its values in $P_0^\alpha(D)$. In this way, $P^\alpha(W')$ may be identified with a closed subspace of $P_0^\alpha(D)$, and if the mapping of the latter space into $P_0^\beta(D)$ is H-S, so is the embedding of $P^\alpha(W')$ in $P^\beta(W')$.

Finally it is clear that if the embedding of $P^\alpha(W)$ in $P^\beta(W)$ is H-S for some hypercube W, it must be so for all hypercubes. Thus it is sufficient to prove the theorem for the spaces $P^\alpha(W)$.

In the special case when α is an even integer, $\alpha = 2k$, it is quite easy to give a direct proof of the theorem for the spaces $P_0^{2k}(D)$. The potentials u in this space are of the form $u = G_{2k}f$ where f belongs to a certain closed subspace $\mathscr{M}$ of $L^2(D)$. This may be written $u = G_\beta G_{2k-\beta} f$, whence $\|u\|_\beta = \|G_{2k-\beta}f\|_0$. A complete orthonormal set $\{u_i\}$ in $P_0^{2k}(D)$ can be obtained from any complete orthonormal set $\{f_i\}$ in $\mathscr{M}$, whence

$$\sum \|u_i\|_\beta^2 = \sum \|G_{2k-\beta}f_i\|_0^2 = \sum_{i,j} \Big| \int_D\int_D G_{2k-\beta}(x-y) f_i(x)\overline{f_j(y)}\,dx\,dy \Big|^2 .$$

The sum on the right is finite, since the function $G_{2k-\beta}(x-y)$ is square integrable over the product space $D\times D$ because D has finite Lebesgue measure and $G_{2k-\beta}(x)$ is L^2 when $2k-\beta > n/2$.

Let k be a large integer and consider the Hilbert space $P^{2k}(W)$. The square of the L^2 norm appears as a positive definite continuous quadratic form on this space, and is therefore represented by a positive bounded operator H, i. e. $\|u\|_0^2 = (Hu, u)_{2k}$. We introduce a family of quadratic norms by the equation

$$|u|_\alpha^2 = (H^{1-\alpha/2k}u, u)_{2k} = \int \lambda^{1-\alpha/2k} d(E_\lambda u, u)_{2k},$$

where E_λ is the resolution of the identity associated with H. The norms $|u|_\alpha$, $0 \leqslant \alpha \leqslant 2k$, form an interpolatory sequence on the space $P^{2k}(W)$, as has been shown by Aronszajn [1], Lions [5], and Stein and Weiss [7].

The spaces P^α themselves also form an interpolation sequence in the sense of those authors.

The restriction mapping, which carries a potential in P^α into one in $P^\alpha(W)$ is clearly a continuous transformation of bound 1 for any α and in particular for the values $\alpha = 0$ and $\alpha = 2k$. By the interpolation theorem then, $|u|_\alpha \leqslant \|\tilde{u}\|_\alpha$ for all α in the interval $0 \leqslant \alpha \leqslant 2k$. Similarly, the extension mapping from $P^\alpha(W)$ into P^α is a continuous linear transformation for all such α with a common bound M, and in particular for the values $\alpha = 0$ and $\alpha = 2k$. Again invoking the interpolation theorem we conclude that $\|\tilde{u}\|_\alpha \leqslant M|u|_\alpha$ for all α in the interval. Thus we have

$$|u|_\alpha \leqslant \|\tilde{u}\|_\alpha \leqslant M|u|_\alpha$$

while we already knew $\|u\|_\alpha \leqslant \|\tilde{u}\|_\alpha \leqslant M\|u\|_\alpha$.

It evidently follows that the norms $|u|_\alpha$ and $\|u\|_\alpha$ are equivalent on $P^{2k}(W)$ for such α. We can therefore prove our theorem making use of the interpolation norms, since the passage to equivalent norms does not affect the H-S character of the embedding.

Since k is large, the mapping of $P^{2k}(W)$ into $P^0(W) = L^2(W)$ is H-S by what has already been shown, in particular, the operator H introduced above is completely continuous, and its eigenfunctions form a complete orthonormal set in $P^{2k}(W)$. This system of eigenfunctions is particularly convenient for our purposes, since it is a complete orthogonal system for all of the norms $|u|_\alpha$, $0 \leqslant \alpha \leqslant 2k$.

Let $\{u_i\}$ be the set of those eigenfunctions and λ_i the corresponding eigenvalues; we have $(Hu_i, u_i)_{2k} = \|u_i\|_0^2 = \lambda_i$. Thus we obtain a complete orthonormal system for the norm $|u|_\alpha$ by taking the sequence of functions $v_i = \lambda_i^{(\alpha-2k)/4k} u_i$. Accordingly

$$|v_i|_\beta^2 = (H^{1-\beta/2k} v_i, v_i)_{2k} = \lambda_i^{(\alpha-\beta)/2k}$$

and the series $\sum |v_i|_\beta^2$ converges if and only if the series $\sum \lambda_i^{(\alpha-\beta)/2k}$ does. We already know that the last series converges if $\alpha - \beta > n/2$, since we know it in the special case $\alpha = 2k$. This completes the proof of the theorem.

4. Remarks. Our argument shows that the condition $\alpha - \beta > n/2$ is a necessary one since the asymptotic distribution of the eigenvalues λ_i has been given by Gårding [4]. If $\mu_i = 1/\lambda_i$ then $N(\mu)$, the number of μ_i which are $\leqslant \mu$, is equivalent to $\mu^{n/4k}$, and the sum $\sum \lambda_i^\xi$ is finite if and only if the Stieltjes integral

$$\int_0^\infty (\mu - \xi)\, dN(\beta)$$

converges. The integral evidently diverges if $\xi \leqslant n/4k$ and hence the embedding is not H-S if $\alpha - \beta < n/2$.

A review of our argument shows that the proof that the Maurin theorem holds for the spaces $P^\alpha(W)$ depends only on the fact that those spaces have a Lichtenstein extension, or more precisely, that there exists a linear mapping $u \to \tilde{u}$ of $P^\alpha(W)$ into P^α for which $\tilde{u} = u$ on W and which is continuous simultaneously for all α in the interval. Subsets of R^n with this property are said to be of class $\mathscr{E}[0, 2k]$ in [3] where they are investigated at length. We may therefore state the following theorem, a slightly weaker form of which is stated in [6] for integer values of the parameters:

THEOREM. *If D is a bounded open subset of R^n belonging to the class $\mathscr{E}[0, p]$, then the natural embedding of $P^\alpha(D)$ into $P^\beta(D)$ is H-S for $\alpha \leqslant p$ if and only if $\alpha - \beta > n/2$.*

Even in an indirect way our argument has not made use of the theorem of Rellich, given in an extended form in [2] which guarantees that the passage from $P_0^\alpha(D)$ to $P_0^\beta(D)$ is completely continuous if $\alpha > \beta$. Using the machinery which is here set up it is easy to obtain a somewhat unnatural proof of that theorem as follows.

On the space $P^{2k}(W)$ the passage from the quadratic norm $\|u\|_\alpha$ to $\|u\|_\beta$ is completely continuous if and only if the passage from $|u|_\alpha$ to $|u|_\beta$ is. For the second pair of norms the functions $\{u_i\}$ form a complete set of functions, and the form $|u|_\beta^2$ is represented on the completion of $P^{2k}(W)$ in the norm $|u|_\alpha$ by an operator which has the $\{u_i\}$ as eigenfunctions. The eigenvalues are then easily seen to converge to 0, and the embedding of $P^\alpha(W)$ in $P^\beta(W)$ is completely continuous. Restricting this embedding to the subspace $P_0^\alpha(D)$, where the closure of D is inside W, we obtain the Rellich theorem.

References

[1] N. Aronszajn, *The Berkeley Conference 1960.*

[2] N. Aronszajn and K. T. Smith, *Theory of Bessel potentials*, Part L, Ann. Inst. Fourier 11 (1961), p. 385-475.

[3] — *Theory of Bessel potentials*, Part II, to appear in Annales de l'Institut Fourier.

[4] L. Gårding, *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand. 1 (1953), p. 55-72.

[5] J. L. Lions, *Espace intermédiares entre espaces Hilbertiens*, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine 50 (1958), p. 419-432.

[6] K. Maurin, *Abbildungen vom Hilbert-Schmidtschen Typus und ihre Anwendungen*, Math. Scand. 9 (1961), p. 359-371.

[7] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), p. 159-172.

Reçu par la Rédaction le 28. 10. 1962