Punkte x_0 ist der Existenz in x_0 der endlichen extremen n-ten Differential-koeffizienten (2) von der rechten bzw. linken Seite gleichwertig.

Es gilt

SATZ 1. *Sei E die Menge der Punkte, in denen die Distribution T mindestens einseitig beschränkt ist. Dann hat T einen Wert fast überall in E.*

Hieraus folgt

SATZ 2. *Die Menge S aller singulären Punkte einer Distribution ist eine Vereinigung*

$$S = S_\infty \cup S_b$$

worin S_∞ ein G_δ und S_b ein $G_{\delta\sigma}$ vom Maße Null ist.

Mit S_∞ wurde die Menge der Unbeschränktheit der Distribution bezeichnet, während S_b aus allen singulären Punkten besteht, in denen die Distribution beschränkt ist.

Umgekehrt gilt

SATZ 3. *Zu jeder Vereinigung S einer G_δ-Menge S_1 und einer $G_{\delta\sigma}$-Menge S_2 vom Maße Null existiert eine Distribution, für welche S die Menge singulärer und CS die Menge regulärer Punkte ist.*

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Some remarks on topological algebras

by

W. ŻELAZKO (Warszawa)

1. Introduction

The development of the theory of any class of topological linear spaces yields the developement of the theory of analogical class of topological algebras. The eminent Gelfand's theory of Banach algebras was followed by the theory of Banach spaces; the same is the case with locally convex and other classes of topological algebras. In this paper a short account of the basic properties of normed algebras and of some of their generalizations is given. The following well-known theorem of Kolmogoroff is a starting point of our classification: a topological linear space is a normed space if and only if it is a locally bounded and at the same time locally convex space. There follow two generalizations of normed algebras: locally bounded algebras and locally convex algebras. We give a comparison of some basic properties of these algebras. The following aspects will be considered: continuity of multiplication, structure of division algebras, some problems connected with representations, involutions. Some unsolved problems will be formulated or recalled.

2. Definitions and notations

A *topological algebra* is an algebra over the real or complex scalars which is a topological linear space with an associative and separately continuous multiplication (i. e. xy is continuous in one variable with fixed another). Every topological algebra may be topologically imbedded in an algebra with unit. In the sequel we shall assume that every considered algebra possesses the unit e.

A *radical* of an algebra A is the intersection of all its maximal left ideals.

A *topological division algebra* is an algebra in which for every $x \neq 0$ there exists an inverse x^{-1}. The continuity of inversion is not assumed. A class of topological algebras is said to possess the property M [30], if every division algebra belonging to this class is isomorphic and hom-

comorphic either with the field of real numbers, or with the complex field, or with the division algebra of real quaternions.

The term *normed algebra* (locally convex algebra, locally bounded algebra, or generally *q*-algebra, where *q* is any class of linear topological spaces) means: topological algebra which is a normed space (locally convex space, etc.), as a linear topological space.

A topological algebra is called *m-convex*, or *idempotent* if it possesses a basis $\{U\}$ of neighbourhoods of zero which consists of idempotent sets (i. e. $UU \subset U$).

3. Continuity of multiplication

3.1. Normed algebras. In the case when a normed algebra is incomplete the multiplication needs not to be jointly continuous (continuous in the two variables simultaneously). An example of such an algebra is the algebra of all complex sequences $x = (x_n)$ such that $\sum |x_n| < \infty$ with multiplication defined as convolution and with the norm defined as $\|x\| = (\sum |x_n|^2)^{1/2}$. The continuity of multiplication in this algebra follows from the inequality

$$\|xy\| \leqslant \left(\sum |x_n| \right) \|y\|.$$

Such a class of algebras was considered by Rohlin [21], who calls them *unitary rings* (assuming that they are unitary spaces; more strictly unitary rings are completions of such algebras). In a Banach algebra (complete normed algebra) the multiplication is jointly continuous. This theorem, due to Gelfand [9, Satz 1], was later generalized by Arens [3, theorem 5], who stated that in every completely metrisable topological algebra the multiplication is jointly continuous. If A is a normed algebra, then the following statements are equivalent:

1° The multiplication in A is jointly continuous.

2° Algebra A is m-convex.

3° In A there exists an equivalent norm satisfying $\|xy\| \leqslant \|x\| \|y\|$.

4° The completion of A is a topological algebra.

3.2. Locally bounded algebras. A linear topological space is called *locally bounded* if it possesses a basis of neighbourhoods which are bounded sets. A set E of topological linear spaces is bounded if for every neighbourhood U of zero there exists a scalar t_U, such that $t_U E \subset U$. It is proved by Rolewicz [22], that the topology in a locally bounded topological linear space may be introduced by the means of a norm $\|x\|$ satisfying $\|ax\| = |a|^p \|x\|$ for every scalar a, where p is a fixed number satisfying $0 < p \leqslant 1$. For the locally bounded algebras there exists the following theorem, due to Pełczyński [30]: Let A be a complete metric algebra. Then the following statements are equivalent:

1° The topology in A may be introduced by mean of the submultiplicative metric $\varrho(xy, 0) \leqslant \varrho(x, 0) \varrho(y, 0)$ (this condition was also considered by Kaplansky [12]).

2° A is a locally bounded algebra.

3° The topology in A may be introduced by means of the submultiplicative *p*-homogeneous norm:

$$\|xy\| \leqslant \|x\| \|y\|, \qquad \|ax\| = |a|^p \|x\|, \qquad 0 < p \leqslant 1.$$

(We recall that A is assumed to possess the unit element.)

A metric algebra complete in the *p*-homogeneous submultiplicative norm is called *p-normed algebra*. Hence every locally bounded complete metric algebra is a *p*-normed algebra.

3.3. Locally convex algebras. A locally convex completely metrisable topological linear space is called B_0-space. The topology in a B_0-space may be introduced by the means of a denumerable family of pseudonorms:

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \ldots \leqslant \|x\|_n \leqslant \ldots$$

A sequence x_n tends to x_0 if and only if $\|x_n - x_0\|_k$ tends to zero as $n \to \infty$, $k = 1, 2, 3, \ldots$

A B_0-algebra is a topological algebra which as a topological linear space is B_0-space. For any B_0-algebra there exists an equivalent system of pseudonorms such that

$$(3.31) \qquad \|xy\|_i \leqslant \|x\|_{i+1} \|y\|_{i+1} \qquad (i = 1, 2, \ldots).$$

The m-convexity of a locally bounded algebra is a natural consequence of its completeness. For the B_0-algebras it need not to be so. The first example of a non m-convex B_0-algebra was constructed by Arens [1], see also [30]. A B_0-algebra is m-convex if and only if there exists an equivalent system of pseudonorms such that

$$(3.32) \qquad \|xy\|_i \leqslant \|x\|_i \|y\|_i \quad (i = 1, 2, \ldots).$$

Very little is known about B_0-algebras which are non m-convex. It would be interesting to discover for B_0-algebras the necessary and sufficient conditions for being m-convex. If A is an m-convex B_0-algebra, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is entire function of complex variable z, then for every element $x \in A$ the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent. On the other hand, in the known examples of non-m-convex B_0-algebras there exist elements for which the series of e^x is divergent. It may be supposed that it is generally so; hence arises the question:

Is a B_0-algebra A m-convex if and only if for every entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and for every $x \in A$, the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent? [1]

In a similar way the following question may be brought forward:

Let A be a commutative complex B_0-algebra. Is it m-convex if and only if for every non inversible element $x \in A$ there exists a multiplicative linear functional f such that $f(x) = 0$? (We recall that A is assumed to possess the unit element [2].)

It may be noticed that a B_0-algebra is m-convex if and only if there exists an equivalent system of pseudonorms such that multiplication is continuous in respect to each of them [18].

4. Structure of division algebras

4.1. Normed algebras. The fundamental theorem of Gelfand's theory of Banach algebras is the Gelfand-Mazur theorem stating that the class of normed algebras possesses the property M. This theorem was first announced by Mazur [17] in 1938, and then it was proved by Gelfand [9] in 1941 (for complete complex division algebras). The Mazur's proof (unpublished till to-day) is based upon the Liouville theorem for harmonic functions and the Frobenius theorem. Some other proofs appeared [11], [20], [26], [27]. In Mazur's paper [17] it was also announced that if A is a normed algebra whose norm satisfies $\|xy\| = \|x\|\|y\|$, for every $x, y \in A$, then A is either a real field, or a complex field, or a division algebra of real quaternions. The same conclusion was then obtained by Edwards [7] under assumption that A is a Banach algebra and $\|x^{-1}\| = \|x\|^{-1}$ for every x inversible in A.

4.2. Locally bounded algebras. The class of locally bounded algebras possesses the property M [30]. If A is a topological algebra in which topology is introduced by means of the norm $\|x\|$ satisfying the following conditions:

1^0 $\|x\| = 0$ if and only if $x = 0$,

2^0 $\|x\| = \|-x\|$,

3^0 $\|x+y\| \leqslant \|x\| + \|y\|$,

4^0 if (a_n) is a sequence of scalars tending to 0, then for every $x \in A$ $\lim \|a_n x\| = 0$,

5^0 if $\lim \|x_n\| = 0$, then $\lim \|ax_n\| = 0$ for every scalar a (see [5], p. 35-37), and additionally

6^0 $\|xy\| = \|x\|\|y\|$,

then A is either a field, or real numbers, or a complex field, or a division

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(1) The answer is positive. For the proof and detailed discussions, see [25].
(2) The answer is negative, see [25], p. 300.

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algebra of real quaternions [30]. If A is a topological algebra complete in the norm satisfying 1^0-5^0, and additionally

$6'$ $\|xy\| \leqslant \|x\|\|y\|$,

7^0 $\|x^{-1}\| = \|x\|^{-1}$ for every x inversible in A, then A satisfies the conclusion of the previous theorem [30]. We notice that by 6^0, or $6'$ the considered algebras are locally bounded.

4.3. Locally convex algebras. The class of B_0-algebras possesses the property M. This was proved in [3] for separable division algebras, and generally in [31]. It is interesting that for non complete locally convex metrisable topological algebras this theorem is false. Williamson [28] constructed an example of non-trivial topological field in which topology is introduced by a denumerable family of pseudonorms satisfying (3.31). Consequently its completion is a B_0-algebra containing a non-trivial field. Such a situation is impossible for normed or locally bounded algebras.

4.4. We pose here a general question concerning complete metric algebras:

Does the class of completely metrisable topological algebras possess the property M?

It is sufficient to prove this fact in the commutative case; in this case it may be assumed that the inversion is continuous in the considered algebra [31]. It may be also assumed that the topology in the considered algebra is given by means of the norm $\|x\|$ satisfying 1^0-5^0 of 4.2. In connection with this problem we recall the following theorem due to Shafarevitsh [25]: a topological field K may be normed if and only if the set $R = \{x \in K : x^n \to 0\}$ is an open set, and the set $\bar{R} = \{x \in K : x^{-1} \notin R\}$ is a bounded set.

5. Representation

5.1. Banach algebras. The fundamental facts of the now classical theory of commutative complex Banach algebras are as follows:

Every maximal ideal in a Banach algebra A is closed, and there is a 1-1 correspondence $M \leftrightarrow f_M$ between maximal ideals and multiplicative linear functionals of A, given by the relation $M = \{x \in A : f_M(x) = 0\}$. The set $\mathfrak{M}$ of all maximal ideals of A is compact in a weak topology and there exists a homomorphism of A onto $\hat{A} \subset C(\mathfrak{M})$, given by the formula $x \to \hat{x} = x(M) = f_M(x)$. It is $\|\hat{x}\| = \max_{M \in \mathfrak{M}} |x(M)| = \lim \sqrt[n]{\|x^n\|}$. If A is semi-simple (the radical of A is the zero ideal), then the homomorphism $x \to \hat{x}$ is an algebraic isomorphism. It follows that the radical of A is characterized by the relation $\mathrm{rad}\, A = \{x \in A : \lim \sqrt[n]{\|x^n\|} = 0\}$. Very little is known about the structure of $\hat{A}$, and it would be interesting to charac-

terize those subalgebras of $C(X)$ which are algebraically isomorphic with the commutative Banach algebras having X as maximal ideals space. The representation theorem for the commutative algebras was extended to some classes of non-commutative Banach algebras (Luchins [16]): a Banach algebra A is called *strictly semi-simple*, or *sss-algebra* if its strict radical defined as intersection of those of its two-sided ideals which are regular maximal right ideals is only the zero ideal. Every sss-real Banach algebra is algebraically isomorphic with a subalgebra of $C(X, Q)$ of all continuous quaternion-valued functions defined on some compact X.

5.2. Locally bounded algebras. The classical Gelfand's theory is also true for commutative complete p-normed algebras. In fact Gelfand's theory is based upon submultiplicativity of the norm $\|xy\| \leqslant \|x\|\|y\|$, Gelfand-Mazur theorem, and characterization of radicals $\mathrm{rad}\, A = \{x \in A: \lim \sqrt[n]{\|x^n\|} = 0\}$. All these facts are true also for p-normed algebras. Consequently a semi-simple commutative p-normed algebra A is algebraically isomorphic with a subalgebra of $C(X)$ for some compact X [30], [32].

5.3. Locally convex algebras. Every m-convex B_0-algebra is a projective limit of Banach algebras. The problem of representation of non m-convex B_0-algebras is open and seems to be of great interest.

6. Involutions

Let A be a complex topological algebra. An involution $x \to x^*$ is defined as operation $A \to A$ satisfying the following conditions:

i. $x^{**} = x$,

ii. $(xy)^* = y^*x^*$,

iii. $(ax + by)^* = \bar{a}x^* + \bar{b}y^*$,

where $x, y \in A$, a, b are complex scalars (bar means complex conjugate).

6.1. Banach algebras. The paper [6] is devoted to general discussion of involutions in Banach algebras. It establishes the existence of commutative and non-commutative Banach algebras possessing no involution. But if in a non-commutative Banach algebra there exists a continuous involution then there exists an uncountable set of involutions. The same is false for commutative algebras. There exist also Banach algebras possessing involutions with arbitrarily large norms.

The classical theory of representation of Banach algebras with involutions is due to Gelfand and Neumark [10]. They considered the Banach algebras with involution satisfying the following conditions:

a. $\|x\| = \|x^*\|$,

b. $\|x^*x\| = \|x^*\|\|x\|$,

c. $(e + x^*x)^{-1}$ exists for every element x.

Algebras satisfying these conditions are called C^*-*algebras*.

Every C^*-*algebra* is isomorphic and isometric with a closed subalgebra of the algebra of all bounded operators of Hilbert space [10]. This isomorphism $x \leftrightarrow A_x$ sends x^* onto A_x^*. Moreover, Gelfand and Neumark proved that in the commutative case every C^*-algebra is isomorphic and isometric which $C(\mathfrak{M})$, where $\mathfrak{M}$ is its maximal ideal space. This isomorphism is given by the formula $x \leftrightarrow \hat{x}$. It is also $x^*(M) = \overline{x(M)}$ [10]. They proved this theorem assuming only b. Hence in the commutative case b implies a and c. The conjecture that it is so in the general case was stated by Kaplansky [13]. And in fact, Fukamiya [8], and Kelley and Vaught [15] proved that in non commutative case a and b implies c. Recently Yood [29] proved that b implies a. He considered some more general condition introduced by Arens [2], namely

b'. $\|x^*x\| \geqslant k\|x^*\|\|x\|$, $\quad k > 0$,

calling a Banach algebra with involution satisfying b' an *Arens algebra. Yood proved that if in an *Arens algebra $k > t_0$, where $t_0 = 0,676\ldots$ is the real root of the equation $4t^3 + 2t^2 - t - 1 = 0$, then in this algebra there exists an equivalent norm satisfying a and b. Hence every *Arens algebra with $k > t_0$ is a C^*-algebra. A question may brought forward:

Is every *Arens algebra a C^*-algebra?

In the commutative case the answer is positive (Arens [2]).

6.2. Locally bounded algebras. It seems that the great part of results concerning Banach algebras is true also for p-normed complete algebras, but no interesting examples have been discovered.

6.3. Locally convex algebras. In this section we give a report of the results obtained by Sia Do-shing [23], [24]. We assume here A is a complete m-convex locally convex algebra with topology introduced by means of the family $\|x\|_a$, $a \in \mathfrak{A}$, of submultiplicative (i. e. satisfying (3.32)) pseudonorms. It is assumed that on A an involution $x \to x^*$ is defined. We recall that linear functional f is called *positive* if $f(x^*x) \geqslant 0$ for every $x \in A$. A multiplicative linear functional f is called a *real functional* if $f(x^*) = \overline{f(x)}$ for every $x \in A$. The following theorems are proved in [24]:

6.31. If A is a B_0 algebra with a continuous inversion, then every positive functional defined on A is continuous.

6.32. Let $\mathfrak{M}$ be the set of all continuous multiplicative linear functionals defined on A and let $\mathfrak{M}$ be the set of all real functionals on A. Then if the involution on A is continuous and if f is a positive functional such that $f(e) = 1$, then there exists a Borel measure μ defined on a com-

pact in the weak topology subset $\hat{\mathfrak{M}}_f \subset \hat{\mathfrak{M}}$, $\mu(\hat{\mathfrak{M}}_f) = 1$, such that

$$f(x) = \int_{\mathfrak{M}_f} x(u)\, d\mu(u).$$

As a corollary it is obtained that if the real functionals are continuous, then so are the positive functionals.

6.33. If the involution in A is continuous, then A satisfies condition c of 6.1 if and only if

$$\sup_{f(e)=1} f(x^*x) = \sup_a \lim_n \sqrt[n]{\|(x^*x)^n\|_a}.$$

6.34. If A is a commutative algebra and

$$\sup_a \|x^*x\|_a = \sup_a \|x^*\|_a \cdot \sup_a \|x\|_a \quad \text{for every } x \epsilon A,$$

then

i. $\|x^*x\|_a = \|x^*\|_a \|x\|_a$ for every $x \epsilon A$ and every $a \epsilon \mathfrak{A}$,

ii. A is complete with respect to $\|x\|_a$ for every $a \epsilon \mathfrak{A}$,

iii. $A = C(\mathfrak{M})$ ($\mathfrak{M}$ is considered as a topological space with the weak topology) and there is a decomposition $\mathfrak{M} = \bigcup_a \mathfrak{M}_a$ such that the norm $\|x\|_a$ is equivalent to the norm $\sup_{M \epsilon \mathfrak{M}_a} |x(M)|$ for every $a \epsilon \mathfrak{A}$.

6.35. If $\sup_a \|x^*x\|_a = \sup_a \|x\|_a^2$ for every $x \epsilon A$, then $\|x^*x\|_a = \|x\|_a^2$ for every $a \epsilon \mathfrak{A}$, and there exists a Hilbert space H and a family of projective operators P_a, $a \epsilon \mathfrak{A}$, of H into H, such that:

i. There exists a symmetric isomorphism $x \leftrightarrow A_x$ of A into the algebra of all operators $H \to H$.

ii. $A_x P_a \subset P_a A_x$, and there is a bounded operator for every $x \epsilon A$, and every $a \epsilon \mathfrak{A}$.

iii. $\|x\|_a = \|A_x P_a\|$.

It would be interesting to have 6.31 and 6.32 also for some non-m-convex B_0-algebras. By a slight modification of the proof of Siu Doshing we get theorem 6.31 for an arbitrary B_0-algebra, or even for an arbitrary completely metrizable topological algebra with a continuous involution. Theorem 6.32 cannot be true in this form, because there are non m-convex B-algebras with positive functionals and possessing no multiplicative linear functionals.

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