conditions on the z_n in order that x_n shall have properties (i) and (ii) above. In other words, we want: $\sum |f(z_n)|^2/K_n(z_n) < M\|f\|$ for all f in H, and, given (c_n) in l_2 there is f in H such that $f(z_n) = c_n K_n(z_n)$ and $\|f\| \leqslant m\sum |c_n|^2$.

For example, in H_2 we want the necessary and sufficient condition on z_n in order that the mapping T from H_2 to the space of all sequences of complex numbers given by

$$Tf = \{f(z_n)(1-|z_n|^2)^{1/2}\}$$

shall actually be a mapping of H_2 onto l_2.

THEOREM. *T maps H_2 onto l_2 if and only if there exist functions f_k in H_2 $(k = 1, 2, \ldots)$ of uniformly bounded norm such that*

$$Tf_k = \{\delta_{nk}\}_n,$$

where $\{\delta_{nk}\}_n$ is the sequence that is zero everywhere except at the k^{th} position where it is 1.

The condition may be restated as follows:

$$\prod_{n\neq k}\left|\frac{z_k-z_n}{1-\bar{z}_n z_k}\right| \geqslant \delta > 0 \qquad (k = 1, 2, \ldots).$$

This is also the necessary and sufficient condition that T_p maps H_p onto p $(1 \leqslant p \leqslant \infty)$ where T_p is defined by

$$T_p f = \{f(z_n)(1-|z_n|^2)^{1/p}\}.$$

In the case $p = \infty$ (interpolation of arbitrary bounded values by bounded analytic functions) the result was proved by L. Carleson, and his result could be used to prove the general case. Our method is different and yields a new proof of his results.

In examples II i III it is easy to give sufficient conditions for property (i), but so far we have not obtained necessary and sufficient conditions for both (i) and (ii).

Added in proof. The lemma is due to N. Bari (Uč. Zap. Mos. Gos. Univ. 148, Matem. 4 (1951), p. 69-107).

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Determinants in Banach spaces

by

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§ 1. Terminology and notation. X is a Banach space, Ξ is the dual of X. x, y, z denote elements of X, and ξ, η, ζ — elements of Ξ. ξx is the value of ξ at x. E is the Banach algebra of all bounded endomorphisms in X (called also *operators*), with the unit I. If $y = Ax$ is a bounded endomorphism in X, then $\eta = \xi A$ denotes the adjoint endomorphism in Ξ. Endomorphisms (operators) will be often interpreted as bilinear functionals $\xi Ax = \xi(Ax) = (\xi A)x$. For fixed x_0, ξ_0 the symbol $x_0\cdot\xi_0$ denotes the one-dimensional operator $Ax = x_0\cdot\xi_0 x$.

$T\in E$ is said to be *quasinuclear* if there exists a bounded linear functional F on E such that $\xi Tx = F(x\cdot\xi)$. Then T is denoted by T_F, and F is called *quasinucleus* of T. E. g. if x_0, ξ_0 are fixed, then $F(A) = \xi_0 A x_0$ $(A\in E)$ is a functional on E, denoted by $\xi_0\otimes x_0$, which is a quasinucleus of $x_i\cdot\xi_i$. All functionals in the closure of the set of all finite sums $\sum_{i=1}^{m}\xi_i\otimes x_i$ are called *nucleus*. If F is nucleus, then $T = T_F$ is called *nuclear* and F is the nucleus of T. For any quasinucleus F, $F(I)$ is called *trace* of F and denoted by $\mathrm{Tr}F$. The space Q of all quasinucleus is a Banach algebra with multiplication: $F_1\circ F_2(A) = \frac{1}{2}\big(F_1(T_{F_2}A)+F_2(AT_{F_1})\big)$. The *canonical mapping* $F\to T_F$ is a ring homomorphism. We write sometimes $F_{\xi x}(\xi Tx)$ instead of $F(T)$.

§ 2. The determinant system for an $A\in E$ is an infinite sequence

$$D_0,\ D_1\binom{\xi_1}{x_1},\ D_2\binom{\xi_1,\ \xi_2}{x_1,\ x_2},\ \ldots,\ D_n\binom{\xi_1,\ \ldots,\ \xi_n}{x_1,\ \ldots,\ x_n},\ \ldots$$

such that: (1) D_0 is a scalar; (2) for $n>0$, $D_n\binom{\xi_1,\ \ldots,\ \xi_n}{x_1,\ \ldots,\ x_n}$ is a $2n$-linear functional on $\Xi^n\times X^n$, skew symmetric in variables $\xi_1, \ldots, \xi_n$ and skew symmetric in $x_1, \ldots, x_n$; (3) $D_n\binom{\xi_1,\ \ldots,\ \xi_n}{x_1,\ \ldots,\ x_n}$, interpreted as a function of ξ_1 and x_1 only is a bilinear functional on $\Xi\times X$ of the form $\xi_1 C x_1$ where $C\in E$; (4) for an integer r, D_r does not vanish identically (the smallest r

is called the *order* of the determinant system); (5) the following identities hold:

$$(\mathrm{D}_n)\quad D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_n\\ x_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=0}^{n}(-1)^i\,\xi_0 x_i\cdot D_n\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots\ldots, \xi_n\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix},$$

$$(\mathrm{D}_n')\quad D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_n\\ Ax_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=0}^{n}(-1)^i\,\xi_i x_0\cdot D_n\begin{pmatrix}\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n\\ x_1, \ldots\ldots\ldots\ldots\ldots\ldots, x_n\end{pmatrix}.$$

§ 3. First fundamental theorem. If $A \in E$ has a determinant system, then A is Fredholm. Viz. if the order r is 0, then $A^{-1} = D_1/D_0$. If $r > 0$ and $\eta_1, \ldots, \eta_r, y_1, \ldots, y_r$ are fixed elements such that $\delta = D_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix} \neq 0$, let $\zeta_1, \ldots, \zeta_r, z_1, \ldots, z_r$ and $B \in E$ be such that, for all ξ, x:

$$\xi Bx = \delta^{-1}D_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r\\ x, y_1, \ldots, y_r\end{pmatrix},$$

$$\zeta_i x = D_r\begin{pmatrix}\eta_1, \ldots\ldots\ldots\ldots\ldots\ldots, \eta_r\\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_r\end{pmatrix},$$

$$\xi z_i = D_r\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_r\\ y_1, \ldots\ldots\ldots\ldots\ldots\ldots, y_r\end{pmatrix}.$$

Then $\zeta_1, \ldots, \zeta_r$ are linearly independent in Ξ, and so are $z_1, \ldots, z_r$ in X. The linear equation $Ax = x_0$ has a solution x if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$; then $x = Bx_0 + c_1z_1 + \ldots + c_rz_r$ is the general form of the solution. The adjoint equation $\xi A = \xi_0$ has a solution ξ if $\xi_0 z_i = 0$ for $i = 1, \ldots, r$; then $\xi = \xi_0 B + c_1\zeta_1 + \ldots + c_r\zeta_r$ is the general form of the solution.

§ 4. The second fundamental theorem. If A is Fredholm, then A has a determinant system. The determinant system of A is determined by A uniquely up to a scalar factor $\neq 0$.

§ 5. Examples of determinant systems. (a) Let X be the m-dimensional space, and A an endomorphism in X, determined by a square matrix $(a_{i,j})$. Let $D_0 = \det(a_{i,j})$. For $0 < n < m$, the set of all algebraic minors obtained from $(a_{i,j})$ by ommiting n rows and n columns is an n-covariant and n-contravariant tensor, i. e. a $2n$-linear functional $D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$ on $\Xi^n \times X^n$. Let $D_m\begin{pmatrix}\xi_1, \ldots, \xi_m\\ x_1, \ldots, x_m\end{pmatrix} = \det(\xi_i x_j)$, and $D_n = 0$ for $n > m$. Then $D_0, D_1, D_2, \ldots$ is a determinant system for A.

(b) If X is any Banach space, and $A \in E$ has the inverse A^{-1}, then

$$(1)\qquad D_0^* = 1 \quad\text{and}\quad D_n^*\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = \det(\xi_i A^{-1}x_j)$$

is a determinant system for A.

§ 6. Effective analytic formulae for a determinant system. For any quasinucleus F, let

$$(2)\qquad D_n(F) = \sum_{m=0}^{\infty}\frac{1}{m!}D_{n,m}(F)\qquad (n = 0, 1, 2, \ldots),$$

where

$$(3)\qquad D_{n,m}(F) = D_{n,m}(F)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$$

$$= F_{\xi_{n+1}x_{n+1}}\ldots F_{\xi_{n+m}x_{n+m}}\begin{vmatrix}\xi_1x_1, \ldots, \xi_1x_{n+m}\\ \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\\ \xi_{n+m}x_1, \ldots, \xi_{n+m}x_{n+m}\end{vmatrix}.$$

The sequence (called the *determinant system of* F)

$$(4)\qquad D_0(F), D_1(F), D_2(F), \ldots$$

is a determinant system *for* $A = I + T_F$. Moreover,

$$(5)\qquad D_{0,m}(F) = \begin{vmatrix}\sigma_1 & m-1 & 0 & 0 & 0\\ \sigma_2 & \sigma_1 & m-2 & 0 & 0\\ \sigma_3 & \sigma_2 & \sigma_1 & m-3\ \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \sigma_{m-1} & \sigma_{m-2} & \ldots\ldots & \sigma_1 & 1\\ \sigma_m & \sigma_{m-1} & \ldots\ldots & \sigma_2 & \sigma_1\end{vmatrix},$$

$$D_{n,m}(F) = \left|\begin{array}{c|c} T_n^0 & n\quad 0\ \ldots\ldots\ 0\\ \hline T_n^1 & \\ T_n^2 & D_{0,m}(F)\\ \vdots & \\ T_n^m & \end{array}\right|\qquad (n = 1, 2, \ldots),$$

where $\sigma_n = \mathrm{Tr}(F^n) = F(T_F^{n-1})$, and T_n^m is the $2n$-linear functional

$$(6)\qquad T_n^m\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = \sum_{\substack{i_1+\ldots+i_n=m\\ i_1,\ldots,i_n\geq 0}}\det(\xi_j T_F^{i_j} x_k) = \sum_{\substack{i_1+\ldots+i_n=m\\ i_1,\ldots,i_n\geq 0}}\det(\xi_j T_F^{i_k} x_k).$$

§ 7. Some identities for the determinant system (4). Let $D_0(F; F_1) = \lim_{\varepsilon\to 0}\big(D_0(F+\varepsilon F_1)-D_0(F)\big)/\varepsilon$, and, by induction

$$(7)\qquad D_0^{(n)}(F; F_1, \ldots, F_n) = \lim_{\varepsilon\to 0}\big(D_0^{(n-1)}(F+\varepsilon F_n; F_1, \ldots, F_{n-1})-D_0(F; F_1, \ldots, F_{n-1})\big)/\varepsilon .$$

We have

$$(8)\qquad D_n(F)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = D_0^{(n)}(F; \xi_1\otimes x_1, \ldots, \xi_n\otimes x_n).$$

$D_0(F)$ is the only analytic solution (in Q) of the differential equation

$$(8')\qquad D_0'(F; F_1+F\cdot F_1) = D_0(F)\cdot \operatorname{Tr}(F_1)$$

with the initial condition $D_0(0) = 1$.

For $|F|<1$,

$$(9)\qquad D_0(F) = \exp \operatorname{Tr}\log(J+F),$$

where J is the abstract unit added to the algebra Q. For all $F, F_1, F_2 \in Q$,

$$(10)\qquad D_0(F_1+F_2+F_1\cdot F_2) = D_0(F_1)D_0(F_2)$$

(theorem on multiplication of determinants), and

$$(11)\qquad D_n(F) = D_0(F)\cdot D_n^*,$$

where D_n^* is defined by (1) with $A = I+T_F$.

§ 8. The case where X is the m-dimensional space. Then formulas (2), (3) yield the algebraic determinant system described in § 5 (a).

§ 9. The case where X, Ξ, are spaces of measurable fnnctions defined on a set Γ with a measure μ, $\xi x = \int_\Gamma \xi(t)x(t)d\mu(t)$. Let $\tau(s, t)$ be a function such that the functional $F_0(K) = \iint_{\Gamma\Gamma}\tau(t, s)\varkappa(s, t)d\mu(t)d\mu(s)$ is continuous on the class of integral operators K: $Kx(s) = \int_\Gamma \varkappa(s, t)d\mu(t)$. Let F be any extension of F_0 over the whole E. Then (4) is the determinant system for the integral equation

$$(12)\qquad x(s)+\int_\Gamma \tau(s, t)x(t)d\mu(t) = x_0(s),$$

and T_F is the integral endomorphism with the kernel $\tau(s, t)$. The determinant system (4) does not coincide, in the case $X = C$, with the original Fredholm determinant and subdeterminants. The Fredholm determinant system coincides with the sequence $\{\overline{D}_n(F)\}$, where

$$(13)\qquad \overline{D}_n(F)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = D_n(F)\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = D_n(F)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix}$$

and $T = T_F$. The same integral formulas for (13) can be written as in the case investigated by Fredholm.

§ 10. The case of infinite square matrices. Substituting in § 9: $\Gamma =$ the set of positive integers with a trivial measure, we get a generalization Koch's theory of determinants and subdeterminants of infinite square matrices.

§ 11. The non-uniqueness effect. Observe that the canonical mapping $F\to T_F$ is not one-to-one; consequently the determinant system (4) for A is uniquely determined by F, but it is not uniquely determined by $A = I+T_F$. In many concrete cases we know that the canonical mapping is one-to-one on the class of all nuclei. Then, if we restrict ourselves to examine only the operators $A = I+T$, where T is nuclear, we can uniquely assign, to every A of this form, a determinant system, viz. the system (4) where F is the only nucleus of T. In the general case we cannot prove that the canonical mapping is one-to-one on the set of all nuclei. The problem whether the canonical mapping is one-to-one on the set of all nuclei is equivalent to the problem whether every compact endomorphism is a uniform limit of a sequence of finitely dimensional operators.

Observe that, for every quasinucleus F, the sequence

$$(14)\qquad \{D_n(F)\exp(-\operatorname{Tr}F+\tfrac{1}{2}\operatorname{Tr}F^2)\}$$

is also a determinant system for $A = I+T_F$ and is uniquely determined by A only! The sequence

$$(15)\qquad \{D_n(F)\cdot\exp(-\operatorname{Tr}F)\}$$

has the same property, provided T_F is a uniform limit of finitely dimensional operators. However, in the case where X is a finitely dimensional space, neither (15) nor (14) coincides with the algebraic determinant system § 5 (a).

§ 12. The Carleman determinant system in $L^2(\Gamma, \mu)$. If the kernel $\tau(s, t)$ of the integral equation (12) is such that $\iint_{\Gamma\Gamma}|\tau(s, t)|^2 d\mu(t)d\mu(s) < \infty$, the determinant system (4) does not exist, in general. However, the expressions (15) remain sensible and give a whole determinant system for (12).

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О кубатурных формулах

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Формулой механических кубатур называют обычно приближенную формулу

(1) $$\int_\Omega \varphi\,dx \cong \sum_{k=1}^{N} C_k \varphi(x^{(k)}),$$

где Ω — некоторая область n-мерного пространства, точки $x^{(k)}$ суть какие то точки внутри этой области, а коэффициенты C_k — заданная система чисел. Ошибка формулы зависит от функции φ. Для различных классов функций эту ошибку можно оценивать по разному. В пространствах $C^{(m)}$ $(m \geqslant 1)$ и $W_p^{(m)}$ $(m > n/p)$, как это следует из теорем вложения, функционал

(2) $$(l, \varphi) = \int_\Omega \varphi\,dx - \sum_{k=1}^{N} C_k \varphi(x^{(k)})$$

является линейным. Максимум такого функционала на единимной сфере в $W_2^{(m)}$ может быть найден эффективно. Ниже будет показано, как это произвести. Имея выражение для максимума можно поставить задачу о нахождении

(3) $$\min_{x^{(k)}, C_k} [\max_{\|\varphi\|=1} (l, \varphi)],$$

т. е. о построении оптимальной формулы механических кубатур с заданным числом точек, что представляет собою задачу о нахождении экстремума функции конечного числа переменных.

Взяв за норму в $W_2^{(m)}$ величину

(4) $$\|\varphi\|^2_{W_2^{(m)}} = \|\Pi\varphi\|^2_{S^{(m-1)}} + \|\varphi\|^2_{L_2^{(m)}},$$

где $S^{(m-1)}$ пространство многочленов степени $m-1$, $L_2^{(m-1)}$ фактор-пространство

(5) $$L_2^{(m-1)} = W_2^{(m-1)}/S^{(m-1)},$$