STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I (1963)

Isomorphic properties of Banach spaces of continuous functions

by

Z. SEMADENI (Poznań)

One of the most important classes of Banach spaces is that of the spaces of continuous functions. The purpose of this paper is to give a review of new results concerning isomorphic properties of these spaces (i. e. properties invariant with respect to linear bicontinuous operations). Of course, only some questions are chosen from several interesting ones investigated in last years.

Kakutani (1939), M. Krein and S. Krein (1940) and others have established various characterizations of the spaces of continuous functions in terms of a lattice, ring or in purely linear metric terms. Assuming these results as well known, we shall treat the space m of bounded sequences, its subspace c of convergent sequences and the space L_∞ of essentially bounded measurable functions as spaces of continuous functions. E. g. m is equivalent to $C[\beta(N)]$, where $\beta(N)$ is the Stone-Čech compactification of the set N of integers. In the sequel $S, S_0, S_1, \ldots$ will always denote compact Hausdorff spaces and $C(S)$ will denote the space of all real-valued continuous functions defined on S. However, the term "a space $C(S)$" will stand also for spaces isometric to spaces $C(S)$.

Among Banach spaces, spaces $C(S)$ have especially interesting isomorphic properties, and the results are not complete. Many general problems concerning Banach spaces, including some problems raised by Banach in his book, have been solved negatively by suitable counter-examples of spaces $C(S)$.

On the other hand, some of Banach's problems concerning spaces $C(S)$ are yet unsolved. E. g. it is not known whether the spaces $C(J)$ and $C(J^2)$ are isomorphic, J being a closed interval. We do not know whether $C(J) \sim C(2^{\aleph_0})$ either. The well-known Banach-Stone theorem establishes that S is topologically determined by isometrical properties of the space $C(S)$, but two spaces $C(S_1)$ and $C(S_2)$ may be isomorphic although S_1 and S_2 are not homeomorphic. We have no general criterion for isomorphism of such spaces, and proofs of isomorphism or non-isomorphism of two concrete spaces of continuous functions are often difficult. It has been proved, however, that all separable and infinite dimensional spaces are homeomorphic (Bessaga and Pełczyński, 1960).

The structure of conjugate spaces is simpler. The space conjugate to a space $C(S)$ can be represented as a space $L(\mu)$ of all functions integrable with respect to a measure (Kakutani, 1941). However, this measure is finite or σ-finite if and only if S is countable (in this case the space conjugate to $C(S)$ is equivalent to the space l). The space $L(J)$ of functions integrable with respect to the ordinary Lebesgue measure is not isomorphic to the conjugate space of any Banach space (Gelfand, 1938). If S is not countable, then (by Kakutani's theorem) this measure μ may be considered as a Radon measure defined on the discrete union of an uncountable family of compact spaces and, by a theorem of Maharam (1942), any finite measure on such a compact space may be represented uniquely by product measures on a countable union of Tychonov cubes. This representation does make it easier to prove some properties of L-spaces; however, it has been used in few papers only and it does not seem to be exploited enough; e. g. it could be used to establish a classification of the conjugate spaces of spaces $C(S)$.

The content of the paper is divided into 5 parts. The first concerns simultaneous extensions of continuous functions i. e. extensions which are linear and positive operations from the space of continuous functions on a closed subset to that on the whole topological space; we discusse generalizations of Borsuk's extension theorem and typical applications.

Section 2 deals with the spaces $C(S)$ in the case where S is dispersed i. e. if no non-empty subset of S is dense-in-itself. These spaces of continuous functions have special properties and are considerably different from the other spaces $C(S)$. Their isomorphical properties are rather singular and some general problems have been solved negatively by suitable examples of spaces $C(S)$ with dispersed S.

Section 3 is devoted to Schauder bases. There are considered only separable spaces $C(S)$ (separability of $C(S)$ is equivalent to metrisability of S), although some slight results concerning uncountable bases have been published. The original Schauder basis $\{\varphi_n(t)\}$ of polygonal functions defined in $[0,1]$ is there discussed especially, as constructions for other spaces have a similar idea. After Schauder had proved that this system is a basis for $C[0,1]$, few mathematicians investigated its properties, and only some first functions are usually written in books on the subject. However, in recent years some interesting papers were published and Schauder functions were applied e. g. in the theory of Brownian motion. There are also discussed: Waher's proof of existence of a basis in any separable space $C(S)$ and unconditional bases.

Section 4 concerns Grothendieck's and Bartle-Dunford-Schwartz's theorems on weakly compact operators in the spaces $C(S)$, and Section 5 is devoted to projections from spaces $C(S)$ onto their subspaces and projections onto spaces $C(S)$. Several interesting questions in this subject are unsolved and seem to be rather difficult.

References: Banach [1]; Bessaga and Pełczyński [4]; Day [2]; Dieudonné [1]; Dunford and Schwartz [1]; Eilenberg [1]; Gel'fand [1]; Kakutani [1], [3], [4]; M. Krein and S. Krein [1], [2]; Maharam [1]; Pełczyński [4]; Stone [1].

1. Simultaneous extension of continuous functions

There is a natural question whether a smaller space S corresponds to a smaller space $C(S)$, both notions "smaller" being suitable defined. The term "a smaller Banach space" will mean a space which is isomorphically contained in the comparized one. From the topological point of view, "smaller" may mean e. g. topological embedding or existence of a continuous map. The second case is clear by the following theorem of M. H. Stone (1937): in order that S_1 be a continuous image of S_2 it is necessary and sufficient that $C(S_1)$ be isometric and ring-isomorphic to a subalgebra of $C(S_2)$. However, the case of topological embedding is not so simple, and neither implication between both embeddings is true in general.

However, one of the implications holds for metric spaces. Namely, Borsuk (1933) has proved the following deep theorem, being a modification of the classical Tietze extension theorem with the requirement that the operation of extension be linear.

Let E_0 be a separable closed subset of a metric space E. To every bounded real-valued continuous function $x(t)$ defined on E there corresponds a bounded continuous function $x^{\uparrow}(t)$ defined on E and satisfying the following conditions:

(1) $x^{\uparrow}(t) = x(t)$ *for* $t \in E_0$,

(2) $(x+y)^{\uparrow} = x^{\uparrow} + y^{\uparrow}$,

(3) *if* $x \geqslant 0$, *then* $x^{\uparrow} \geqslant 0$ (*which means that if* $x(t) \geqslant 0$ *for* $t \in E_0$, *then* $x^{\uparrow}(t) \geqslant 0$ *for* $t \in E$),

(4) $\|x^{\uparrow}\| = \|x\|$,

(5) *if* $x(t) = 1$ *for* $t \in E_0$, *then* $x^{\uparrow}(t) = 1$ *for* $t \in E$.

Any extension satisfying conditions (1)-(3) is called a *simultaneous extension*.

If such an extension exists, then the space $C(\beta(E_0))$ is isomorphic to a subspace of $C(\beta(E))$. In particular, if $S_0 \subset S$ are metrisable and compact, then $C(S_0)$ is isometric to the subspace of $C(S)$ consisting of the extended functions:

$$X_0 = \{x \in C(S): x = z^{\uparrow},\ z \in C(S_0)\}.$$

Moreover, the transformation $y = Px$, where $y(\cdot)$ is the extension of the function $x(\cdot)$ restricted to E_0, is a linear lattice-homomorphism projection of $X = C(S)$ onto X_0, i.e.

$$P(X) = X_0, \quad P^2 = P, \quad \|P\| = 1, \quad P(x \vee y) = (Px) \vee_0 (Py)$$

where $\vee_0$ means the relative supremum in X_0.

The complementary subspace consists of all functions such that $P(x) = 0$, i. e. such that $x(t) = 0$ on S_0. So we may write (if $S_0 \subset S$)

$$C(S) \sim C(S_0) \times C(S \parallel S_0)$$

where the sign $\sim$ denotes an isomorphism and $C(S_1 \parallel S_2)$ denotes the set of all functions of $C(S_1)$ vanishing on S_2, i. e.

$$C(S_1 \parallel S_2) = \bigcap_{u \in S_2} \{x \in C(S_1) : x(u) = 0\}.$$

Similarly, if $S_1 \subset S_2 \subset S_3$, then $C(S_1 \parallel S_3) = C(S_1 \parallel S_2) \times C(S_2 \parallel S_3)$.

Such formulas are the starting point for the method of algebraic calculations in proofs of isomorphisms of certain spaces. This method was used first by Borsuk and developed by Pełczyński. As an example of this method let us show that if S_1 and S_2 are metrisable and uncountable and if either space differs by a countable set from the other, then $C(S_1) \sim C(S_2)$. It is enough to prove this when $S_1 \subset S_2$ and $S_2 \setminus S_1$ is non-compact. Let $S_3 = (S_2 \setminus S_1) \cup t_\infty$ be the one-point compactification of the difference $S_2 \setminus S_1$ and let S_4 be a subset of S_1 homeomorphic to S_3 (it exists because S_1 contains a subset homeomorphic to the Cantor set). Then

$$\begin{aligned} C(S_1) &\sim C(S_4) \times C(S_1 \parallel S_4) \sim [C(S_4) \times C(S_4)] \times C(S_1 \parallel S_4) \\ &\sim C(S_3) \times [C(S_4) \times C(S_1 \parallel S_4)] \sim C(S_3 \parallel t_\infty) \times C(S_1) \\ &\sim C(S_2 \parallel S_1) \times C(S_1) \sim C(S_2). \end{aligned}$$

(We have applied two known theorems: If S_0 is countable and infinite, then $C(S_0)$ is isomorphic to its Cartesian square and $C(S_0) \sim C(S_0 \parallel t_0)$ for each $t_0 \in S_0$.)

This theorem is not true for arbitrary compact S_1 and S_2; e. g. if $S_1 = \beta N$ and $S_2 = \beta N \cup S_0$, where S_0 is the one-point compactification of integers and βN and S_0 are considered as disjoint, then $C(S_1)$ is not isomorphic to $C(S_2) = C(\beta N) \times C(S_0)$ because there is a projection of $C(S_2)$ onto $C(S_0) = c$ and a projection from $C(\beta N) = m$ onto c is impossible (Phillips, 1940; Sobczyk, 1941).

The original proof of Borsuk gives an integral formula for the extended function $x^\dagger(t)$ and is founded on the existence of a continuous map of a set of positive linear Lebesgue measure onto E_0. Kakutani (1940), Dugundji (1951), Arens (1952) and Michael (1953) discussed and generalized the theorem of Borsuk. Dugundji has established that if E is metrisable, then the assumption of separability of E_0 is superflous; Arens has proved the theorem assuming that E_0 is metrisable and compact and E is paracompact. On the other hand, Arens has established that in the general case the theorem is not true even if both sets E and E_0 are compact. Other examples were found in following years.

The simplest one is the following: the space $\beta(N) \setminus N$ does not admit a simultaneous extension on $\beta(N)$. Indeed, if it were so, there would exist a decomposition of the space $m = C(\beta(N))$ into the direct product

$$C(\beta N) \sim C[\beta(N) \setminus N] \times C[\beta(N) \parallel \beta(N) \setminus N] = C[\beta(N) \setminus N] \times c_0,$$

contradicting the theorem of Phillips quoted above.

An interesting example, due to Pełczyński, is a consequence of Day's results on strictly convex spaces. The space $\beta(N_1)$, where N_1 is an isolated set of power $\aleph_1$, is topologically contained in a Tychonoff cube $J^{\aleph_c}$ although $C[\beta(N_1)]$ is not isomorphic to any subspace of $C(J^{\aleph_c})$, for $C(J^{\aleph_c})$ is isomorphic to a strictly convex space and $C[\beta(N_1)]$ is not.

One can prove that if $C(S)$ contains isometrically the space m, then there exists a closed subset S_0 of S which does not admit simultaneous extensions on S. The negative solution of the general problem of simultaneous extensions yields several particular questions concerning existence and behavior of possible extensions.

The conversions of the simultaneous extension theorem concern investigations of embeddings of $C(S_0)$ into $C(S)$. If $C(S_0)$ is isometrically and isotonically isomorphic to a subspace X_0 of $C(S)$ (which need not be a sublattice of $C(S)$), then a notion of *lattice boundary* of S with respect to X_0 may be introduced; it is a closed subset S_1 of S with the following properties: (i) $(x \vee_0 y)(t) = \max[x(t), y(t)]$ for each $t \in S_1$, $x \in X_0$ and $y \in X_0$ ($\vee_0$ denotes the relative supremum i. e. $x \vee_0 y$ is the smallest element of X_0 which majorize both x and y), (ii) $e_0(t) = 1$ for each $t \in S_1$, (where e_0 denotes the relative unit of X_0 i. e. the smallest element of X_0 which majorizes all elements of the unit ball of X_0), (iii) $\sup\{|x(t)| : t \in S_1\} = \|x\|$ for each $x \in X_0$.

Such an S_1 always exists but need not be unique (Gęba and Semadeni, 1959). If X_0 separates S_1, then S_0 and S_1 are homeomorphic and the given isomorphism $C(S_0) \to X_0$ is a simultaneous extension from S_1 to S.

References: Arens [1]; Bauer [1], p. 108-114; Bessaga and Pełczyński [2]; Borsuk [1]; Day [1]; Dugundji [1]; Gęba and Semadeni [1], [2]; Kakutani [2]; Michael [1]; Pełczyński [3]; Phillips [1],; Šilov [1]; Sobczyk [1]; Stone [1], p. 475; Yoshizawa [1].

2. The spaces of continuous functions on dispersed compact sets

A topological space is said to be *dispersed* (= *clairsemé*) if it contains no perfect non-void subset. Typical examples of dispersed compact spaces are closed intervals of ordinal numbers

$$\Gamma_a = \{\xi\colon \xi \leqslant a\}$$

provided with the order topology (with open intervals as neighborhoods). Γ_a is metrisable if and only if it is countable, i. e. if a is smaller than the first uncountable ordinal ω_1. Mazurkiewicz and Sierpiński (1920) have proved the following important theorem: *any dispersed compact metric space is homeomorphic to a certain space Γ_a with $a < \omega_1$.* For non-metrisable spaces this theorem is not valid as the example $\Gamma_\omega \times \Gamma_{\omega_1}$ shows, and the structure of non-metrisable compact dispersed spaces is more complicated.

Rudin (1957) has proved that *every regular finite Borel measure on a dispersed compact space S is purely atomic;* the converse statement is also true and this establishes a useful characterization of dispersedness of a compact space.

Dispersedness of S may be also characterized by isomorphic properties of $C(S)$. Each of the following conditions is necessary and sufficient in order that a compact space S be dispersed (Pełczyński and Semadeni, 1959):

(1) $C(S)$ does not contain isomorphically the space l,

(2) Any infinite dimensional subspace of $C(S)$ contains a subspace isomorphic to c,

(3) $C(S)$ is conditionally weakly compact.

Banach has raised the following problem: Is every infinite dimensional Banach space isomorphic to its Cartesian square? Bessaga and Pełczyński have established that a well-known non-reflexive space constructed by James is a counter-example and the question arose whether the relation $X \sim X^2$ was true for more special classes of Banach spaces, e. g. for the spaces $C(S)$. In this case the answer is also negative, namely the space $C(\Gamma_{\omega_1})$ is not isomorphic to its Cartesian square $C(\Gamma_{\omega_1}) \times C(\Gamma_{\omega_1}) = C(\Gamma_{\omega_1 \cdot 2})$ (Semadeni, 1960).

The proof is founded on the following notion: X being a Banach space, X_s will denote the set of all linear functionals x^{**} in the second conjugate space X^{**} which are sequentially continuous with respect to the weak topology $\sigma(X^*, X)$, i. e.

$$X_s = \{x^{**} \in X^{**}\colon \text{ if } x_n^*(x) \to 0 \text{ for all } x \in X, \text{ then } x^{**}(x_n^*) \to 0\}.$$

Of course, $\varkappa X \subset X_s \subset X^{**}$ where $\varkappa X$ denotes the canonical image of X in X^{**}. Now, if $X = C(\Gamma_{\omega_1})$, then the codimension of $\varkappa X$ in X_s turns out to be equal exactly to 1; of course, for the Cartesian square the corresponding codimension is equal to 2, whence these spaces cannot be isomorphic.

Another of Banach's questions was: are the spaces $C(\Gamma_a)$ isomorphic mutually for $\omega \leqslant a < \omega_1$? Pełczyński has answered it in the negative; he proved that the space $C(\Gamma_\omega)$, which is obviously equivalent to the space c of convergent sequences, is not isomorphic to the space $C(\Gamma_{\omega^\omega})$. Some other proofs of this theorem are also known now; one of them is founded on the following theorem due essentially to Schreier (1933): there exists a weakly convergent sequence x_n in $C(\Gamma_{\omega^\omega})$ such that the sequence

$$y_k = \frac{x_{n_1} + \ldots + x_{n_k}}{k}$$

is not strongly convergent for any sequence $n_1 < n_2 < \ldots$ of indices; such a sequence cannot exist in $C(\Gamma_\omega)$. Bessaga and Pełczyński (1960) established the following classification theorem: in order that $C(\Gamma_{a_1})$ be isomorphic to $C(\Gamma_{a_2})$, where $\omega \leqslant a_1 \leqslant a_2 < \omega_1$, it is necessary and sufficient $a_2 < a_1^\omega$. Thus, the numbers ω, ω^ω, $(\omega^\omega)^\omega$, ... determine the successive isomorphic types.

Banach also raised the question whether the linear isometry of conjugate spaces X_1 and X_2 implies isomorphism of X_1 and X_2. The quoted theorem of Pełczyński gives the negative answer. Indeed, the spaces $C(\Gamma_\omega)$ and $C(\Gamma_{\omega^\omega})$ are not isomorphic although their conjugate spaces are equivalent to the space l. Moreover, the spaces $C(\Gamma_a)$ with $\omega \leqslant a \leqslant \omega_1$ form $\aleph_1$ different isomorphic types with the same conjugate space.

Two questions are still open: 1° Is any separable infinite dimensional space $C(S)$ isomorphic to its square? 2° What are necessary and sufficient conditions for $C(\Gamma_a)$ and $C(\Gamma_\beta)$ to be isomorphic?

References: Banach [1], p. 194; Bessaga and Pełczyński [1], [2], [3]; James [1]; Lindenstrauss [1]; Mazurkiewicz and Sierpiński [1]; Pełczyński [3]; Pełczyński and Semadeni [1]; Phelps [1]; Rudin [1]; Schreier [1]; Semadeni [1].

3. Bases

Any separable space $C(S)$ possesses a basis (Waher, 1955). The Schauder polygonal basis for $C(J)$ consists of the infinite integrals of the Haar orthogonal functions

$$\varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \varphi_{2^n+k}(t) = \int_0^t \chi_{2^n+k}(u)\,du,$$

for $0 \leqslant t \leqslant 1$, $n = 0, 1, \ldots$, $k = 1, \ldots, 2^n$. Assuming standard normalization of Haar functions we get $\|\varphi_{2^n+k}\| = 2^{-\frac{1}{2}n-1}$.

Every continuous function $x(t)$ on $[0, 1]$ can be uniquely expanded into a series

$$x(t) = \sum_{m=1}^{\infty} a_m \varphi_m(t)$$

and the partial sums of this series are polygonal functions interpolating the values of $x(t)$ at a suitable division of the interval. By the addition of a new function $a_{m+1}\varphi_{m+1}$ we obtain one new angle point, and the coefficient a_{2^n+k} depends only on values of the function $x(t)$ at the three corresponding angle-points of φ_{2^n+k}, namely

$$a_{2^n+k} = -\int_0^1 x(u)\,d\chi_{2^n+k} = 2\sqrt{2^n}\left\{x\left(\frac{2k+1}{2^{n+1}}\right) - \frac{1}{2}\left[x\left(\frac{k}{2^n}\right) + x\left(\frac{k+1}{2^n}\right)\right]\right\}$$

for all coefficients with the exception of the first two. These formulas, written explicitly by Ciesielski (1959), show that the Haar functions χ_m form a system biorthogonal to the Schauder system. The coefficents give a measure of the deviation from linearity, e. g. if a function x is concave, then $a_n \geqslant 0$ for $n \geqslant 2$.

Schauder's construction can be generalized by a triangulation method to n-dimensional cubes ($n = 2, 3, \ldots$) and to the Hilbert cube. In the general case, Waher's proof is founded on Urysohn's embedding of the considered metric compact space S into the Hilbert cube and uses finite-dimensional triangulations of the cube. However, as it was remarked by C. Bessaga, her proof is valid only for uncountable spaces S. The case of countable S must be considered separately; in this case S is homeomorphic to a compact set Q of dyadic numbers of the unit interval such that if a number $t \in Q$ satisfies $k2^{-n} < t < (k+l)2^{-n}$, then $k2^{-n} \in Q$ and $(k+1)2^{-n} \in Q$, and $C(S)$ is equivalent to the space Y of all functions continuous on Q and linearly extended on the interval. The set of Schauder functions with the angle points in Q is a basis for Y.

A basis is called *unconditional* if the expansion of any element of the space is unconditionally convergent. The unconditional (strong) convergence in $C(S)$ is simply characterized by the following theorem of Sierpiński (1910): a series $\sum x_n$ of elements of $C(S)$ is convergent for any arrangement of its elements if and only if the series $\sum |x_n|$ is strongly convergent. Here $|x_n|$ denotes the function $|x_n(t)|$ belonging to $C(S)$ as well.

Karlin (1948) has proved that if S is uncountable, then $C(S)$ cannot possess an unconditional basis. Bessaga and Pełczyński (1960) have proved that the space $C(\Gamma_{\omega^\omega})$ cannot be isomorphic to a subspace of a space with an unconditional basis. Consequently, any separable space $C(S)$ with an unconditional basis is isomorphic to the space c (or is finite dimensional).

References: Bessaga [1]; Bessaga and Pełczyński [2]; Ciesielski [1], [2]; Day [3], p. 58-76; Ellis and Kuehner [1]; James [1]; Karlin [1]; Kaczmarz and Steinhaus [1]; Orlicz [2]; Pełczyński and Semadeni [1]; Schauder [1]; Sierpiński [1]; Waher [1].

4. Weakly compact linear operators

A linear operator T from a Banach space X to Y is said to be *weakly compact* if it maps bounded sets into weakly compact sets, i. e. if the weak closure of the image of the unit sphere is weakly compact in Y. An equivalent formulation says that T is weakly compact if and only if its second adjoint T^{**} maps X^{**} into the canonical image of Y in Y^{**} i. e. $T\colon X^{**} \to \varkappa Y$.

One may assume other definitions of weak compactness of a linear operator, but the above one is the most appropriate. Investigations of such operators

$$T\colon C(S) \to Y$$

have led to some excellent results.

Some representation theorems for weakly compact operators in $C(S)$ are due to Gelfand (1938) and Sirvint. Dunford and Pettis (1940) have established the main properties of weakly compact operators defined in an L-space, and Grothendieck (1953) and Bartle, Dunford and Schwartz (1955) have proved analogous theorems for operators in the spaces $C(S)$. By a theorem of Gantmacher, an operator is weakly compact if and only if its adjoint is weakly compact, so the Dunford-Pettis theorem and those concerning the spaces $C(S)$ are closely related.

The proofs of two basic theorems concerning weakly compact operators in $C(S)$ are founded on the following representation theorem:

The general form of a weakly compact linear operator from a space $C(S)$ to a Banach space Y is

$$Tx = \int_S x(s)\,d\mu(s), \qquad x \in C(S),$$

where μ is a vector measure defined on the Borel subsets of S with values in the space Y, and the norm of T is equal to the semivariation of μ, i. e.

$$\|T\| = \sup\{\|\sum_{i=1}^{n} a_i \mu(E_i)\|\colon E_j \cap E_k = \emptyset \text{ for } j \neq k,\ |a_i| \leqslant 1\}.$$

First basic theorem may be formulated as follows:

A linear operator $T\colon C(S) \to Y$ is weakly compact if and only if it maps weakly convergent sequences onto strongly convergent ones.

Sufficiency follows from the Lebesgue theorem on bounded convergence which is also valid for vector valued measures. Indeed, weak convergence in $C(S)$ means uniform boundedness together with pointwise convergence which imply convergence in mean, i. e. strong convergence in Y.

Consequently, if this is the case, then T maps conditionally weakly compact sets into strongly compact sets and if the operators

$$T_1: C(S_1) \to C(S_2) \quad \text{and} \quad T_2: C(S_2) \to Y$$

are weakly compact, then their product T_2T_1 is compact (i. e. completely continuous). Indeed, in this case T_1 maps bounded sets into weakly compact sets which are transformed, in turn, into strongly compact sets. In particular, if $T: C(S) \to C(S)$ is weakly compact, T^2 is compact.

The second theorem completes the first:

If $T: C(S) \to Y$ is an arbitrary linear operator and if Y is weakly complete, then T is weakly compact.

Pełczyński (1960) has generalized the latter theorem assuming only that the weak and strong unconditional convergence of a series are equivalent in Y. This assumption is essentially weaker than that of weak completeness, by a theorem of Orlicz (1929), but it is equivalent to the assumption that Y does not contain isomorphically the space c. A partially converse result is also true: If Y is separable and contains isomorphically the space c, then there exists a linear operator T from a space $C(S)$ to Y which is not weakly compact.

References: Bartle, Dunford and Schwartz [1]; Bessaga and Pełczyński [1]; Day [3], p. 108; Dunford and Pettis [1]; Dunford and Schwartz [1]; Gelfand [1]; Grothendieck [1], [2]; Lindenstrauss [2]; Orlicz [1]; Pełczyński [1], [2]; Pettis [1]; Sirvint [1].

5. Projections and injective Banach spaces

A subspace X_0 of a Banach space X is said to be *complemented* in X if there exists a projection of X onto X_0; if this is the case, then

$$X \sim X_0 \times X_1 = \{x: Px = x\} \times \{x: Px = 0\}.$$

Banach and Mazur (1933) established the first example of a non-complemented subspace, namely they proved that L is not complemented in C.

A Banach space is said to be *injective* if it is complemented in any Banach space containing it. Phillips (1940) has proved that this property is equivalent to the following one: For every linear operator $T: Z_0 \to X$ from a subspace Z_0 of a Banach space Z, there exists a linear extension $T^\dagger$ of T to the whole space Z. If $\|T^\dagger\| = \|T\|$ is required additionally, then X is said to have the property $\mathfrak{P}_1$; this property, called also the *property of Nachbin*, is equivalent to existence of a projection of norm 1 from every Banach space containing X.

Nachbin (1950), Goodner (1950) and Kelley (1952) have established the following fundamental theorem:

A Banach space has the property $\mathfrak{P}_1$ if and only if it is equivalent to a space $C(S)$ with S extremaly disconnected.

In particular, the spaces m and L_∞ have this property.

Every Banach space can be embedded into a space with property $\mathfrak{P}_1$, namely into the space of all bounded functions on the unit cell of the conjugate space. Further, any complemented subspace of an injective space is injective. Consequently, a space has this property if and only if it complemented in the space of bounded functions on the unit cell in X^*.

Unfortunately, no characterization of the injective spaces is known and the problem of existence of an injective space non-isomorphic to a space $C(S)$ is still open.

Grothendieck (1953) has proved that *if X is injective, then both weak topologies in the conjugate space are sequentially equivalent*, i. e. if $x_n^*(x) \to 0$ for all $x \in X$, then $x^{**}(x_n^*) \to 0$ for all $x^{**} \in X^{**}$. This is a very interesting consequence of a lemma of Phillips. By Banach theorems on regularly closed linear sets it follows that a separable Banach space has Grothendieck's property if and only if it is reflexive. Thus, though spaces $C(S)$ are quite different from the reflexive ones, some properties of injective spaces do resemble reflexivity. On the other hand, if a reflexive space is injective, it is finite dimensional. Indeed, suppose that X is a reflexive subspace of an injective $C(S)$ (e. g. of a suitable space of all bounded functions) and $P: C(S) \to X$ is a projection onto X. By the latter of two theorems quoted in Section 4, P is weakly compact and, by the former one, P^2 is compact. But $P^2 = P$ characterizes projections and P maps the unit cell of X onto itself. Hence this cell must be strongly compact and X must be finite-dimensional. In such a way Grothendieck has proved that neither a reflexive nor a separable space can be injective unless it is finite-dimensional.

In particular, the space c_0 is not injective; nevertheless it has a restricted projection property, namely for each separable Banach space X containing c_0 there exists a projection $P: X \to c_0$ with the norm $\|P\| \leqslant 2$ (Sobczyk, 1941).

Pełczyński (1959) has proved that the spaces m and L_∞ are isomorphic. His proof is very simple and is another good example the calculation method mentioned in Section 1. Namely we apply well-known facts:

$$m \sim m \times m, \quad L_\infty \sim L_\infty \times L_\infty, \quad m \subset L_\infty, \quad L_\infty \subset m$$

(inclusions mean isometric embeddings) and either space is complemented in the other. So we can write $m \sim X \times L_\infty$, and $L_\infty \sim m \times Y$, whence $L_\infty \sim m \times Y \sim (m \times m) \times Y \sim m \times (m \times Y) \sim m \times L_\infty \sim (X \times L_\infty) \times L_\infty \sim$ $\sim X \times (L_\infty \times L_\infty) \sim X \times L_\infty \sim m$.

A quite analogous method can be used to show that the space of all Baire functions on an interval considered up to sets of the first category is also isomorphic to m. However, both proofs apply the axiom of choice (using the Hahn-Banach theorem) and the problem of effectivity of isomorphism between L_∞ and m is still open (it might be true that if an isomorphism $T\colon L_\infty \to m$ is given, we can construct effectively a non-measurable subset of the interval). The calculation method can be used, after some modification, to several proofs of isomorphisms (or homeomorphisms even) of Banach spaces whenever we can find suitable isomorphisms onto complemented subspaces.

The theorems quoted above enable us to conjecture that every space with property $\mathfrak{P}_1$ is isomorphic to the space $m(A)$ of all bounded functions on a set A, but this is not true. E. g. the space $L_\infty(J^{\aleph_\tau})$ of essentially bounded functions on a Tychonov cube, measurable with respect to the product measure, has the property $\mathfrak{P}_1$ and is isomorphic to a strictly convex space whence it cannot be isomorphic to $m(A)$ if A is uncountable; on the other hand, if τ is large enough, $L_\infty(J^{\aleph_\tau})$ cannot be isomorphic to $m = C(\beta N)$ either.

Let us notice that Ciesielski (1960) has proved that a continuous function x on $[0,1]$ satisfies the Hölder condition with an exponent α $(0<\alpha<1)$ if and only if its Schauder polygonal expansion (cf. Section 3) is of the form

$$x(t) \to \sum a_n n^{-\alpha+1/2}\varphi_n(t)$$

where (a_n) is a bounded sequence. Hence, the map $x \to (a_n)$ is an isomorphism from H_α onto m which shows that H_α is injective for each $0<\alpha<1$. Another example of a space with property $\mathfrak{P}_1$ is the space of bounded harmonic functions on an open relatively compact subset of a Green space. (The method of superharmonic majorants gives us the least harmonic function majorizing the given family of uniformly bounded harmonic functions.) The same is true for solutions of the heat equation.

Finally, let us recall two problems.

1° Let X be an injective Banach space. Must X be isomorphic to a space with property $\mathfrak{P}_1$?

2° Let X be complemented in a space $C(S)$. Must X be isomorphic to a space $C(S_0)$?

References: Akilov [1]; Amir [1]; Aronszajn and Panitchpakdi [1]; Banach [1]; Banach and Mazur [1]; Bauer [2]; Ciesielski [2]; Day [1], [3]; Dilworth [1]; Dixmier [1], [2]; Gęba and Semadeni [1]; Goodner [1]; Grothendieck [1], [2]; Grünbaum [1], [2]; Isbell and Semadeni [1]; Kantorowitch, Vulih and Pinsker [1]; Kakutani [4]; Kelley [1]; Lindenstrauss [2]; Nachbin [1], [2]; Nakano [1]; Pełczyński [2], [3]; Pełczyński and Sudakov [1]; Semadeni [2], [3]; Semadeni and Zbijewski [1]; Sierpiński [2]; Sobczyk [1]; Stone [1], [2]; Wada [1].

Bibliography

D. Amir

[1] *Continuous functions spaces with the bounded extension property*, Bull. Res. Council Israel, to appear.

C. P. Akilov

[1] *On extension of linear operators* (in Russian), Leningrad. Gos. Univ. Utch. Zapiski 144, Ser. Mat. 23 (1952), p. 47-84.

R. Arens

[1] *Extensions of function on fully normal spaces*, Pacif. J. Math. 2 (1952), p. 11-22.

N. Aronszajn and P. Panitchpakdi

[1] *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6 (1956), p. 405-441.

S. Banach

[1] *Théorie des opérations linéaries*, Warszawa 1932.

S. Banach and S. Mazur

[1] *Zur Theorie der Linearen Dimension*, Studia Math. 4 (1933), p. 100-112.

R. G. Bartle, N. Dunford and J. T. Schwartz

[1] *Weak compactness and vector measures*, Canadian J. Math. 7 (1955), p. 289-305.

H. Bauer

[1] *Silovscher Rand und Dirichletsches Problem*, Ann. Inst. Fourier 11 (1961), p. 89-136.

[2] *Axiomatische Behandlung des Dirichletschen Problems für elliptische und parabolische Differentialgleichungen*, Math. Annalen 146 (1962), p. 1-59.

C. Bessaga

[1] *Basis in certain spaces of continuous functions*, Bull. Acad. Pol. Sci. 5 (1957), p. 11-14.

C. Bessaga and A. Pełczyński

[1] *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), p. 151-169.

[2] *Spaces of continuous functions (IV) (On isomorphical classification of spaces $C(S)$)*, Studia Math. 19 (1960), p. 53-62.

[3] *Banach spaces non-isomorphic to their Cartesian squares I*, Bull. Acad. Pol. Sci. 8 (1960), p. 77-80.

[4] *Some remarks on homeomorphisms of Banach spaces*, Bull. Acad. Pol. Sci. 8 (1960), p. 757-761.

K. Borsuk

[1] *Über Isomorphie der Funktionalräume*, Bull. Int. Acad. Pol. Sci. 1933, p. 1-10.

Z. Ciesielski

[1] *On Haar functions and on the Schauder basis of the space $C_{\langle 0,1\rangle}$*, Bull. Acad. Pol. Sci. 7 (1959), p. 227-232.

[2] *On the isomorphism of the spaces H_α and m*, Bull. Acad. Pol. Sci. 8 (1960), p. 217-222.

M. M. Day

[1] *Strict convexity and smoothness of normed spaces*, Trans. Amer. Math. Soc. 78 (1955), p. 516-528.

[2] *Every L-space is isomorphic to a strictly convex space*, Proc. Amer. Math. Soc. 8 (1957), p. 415-417.

[3] *Normed Linear Spaces*, Ergebnisse der Math., 1958.

R. P. Dilworth

[1] *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc. 68 (1950), p. 427-438.

J. Dixmier

[1] *Sur un théorème de Banach*, Duke Math. J. 15 (1948), p. 1057-1071.

[2] *Sur certains espaces considérés par M. H. Stone*, Summa Brasil. Math. 2 (1951), p. 151-182.

J. Dieudonné

[1] *Sur les espaces L^1*, Arch. der Math. 10 (1959), p. 151-152.

J. Dugundji

[1] *An extension of Tietze's theorem*, Pacific Math. J. 1 (1951), p. 353-367.

N. Dunford and B. J. Pettis

[1] *Linear operators on summable functions*, Trans. Amer. Math. Soc., 47 (1940), p. 323-392.

N. Dunford and J. T. Schwartz

[1] *Linear Operators.*

S. Eilenberg

[1] *Banach space methods in topology*, Ann. of Math. 43 (1942), p. 568-579.

H. W. Ellis and D. G. Kuehner

[1] *On Schauder bases for spaces of continuous functions*, Canad. Math. Bull. 3 (1960), p. 173-184.

I. M. Gelfand

[1] *Abstrakte Funktionen und lineare Operatoren*, Mat. Sbornik 4 (1938), p. 235-286.

K. Gęba and Z. Semadeni

[1] *On the M-subspaces of the Banach spaces of continuous functions*, Zeszyty Nauk. Uniw. Pozn. 2 (1960), p. 53-68.

[2] *Spaces of continuous functions (V), (On linear isotonical embedding of $C(\Omega_1)$ into $C(\Omega_2)$*, Studia Math. 19 (1960), p. 303-320.

D. A. Goodner

[1] *Projections in normed linear spaces*, Trans. Amer. Math. Soc. 69 (1950), p. 89-108.

A. Grothendieck

[1] *Une caractérisation vectorielle métrique des espaces L^1*, Canadian J. Math. 7 (1955), p. 552-561.

[2] *Sur les applications linéaires faiblement compactes d'espace du type $C(K)$*, Canadian J. Math. 5 (1953), p. 129-173.

B. Grünbaum

[1] *Some application of expansion constants*, Pacific J. Math. 10 (1960), p. 193-201.

[2] *Projections onto some function spaces*, Proc. Amer. Math. Soc. 13 (1962), in print.

J. R. Isbell and Z. Semadeni

[1] *Projection constants and spaces of continuous functions*, Trans. Amer. Math. Soc., to appear.

R. C. James

[1] *Bases and reflexivity of Banach spaces*, Ann. of Math. 52 (1951), p. 518-527.

S. Kakutani

[1] *Weak topology, bicompact sets and the principle of duality*, Proc. Imp. Acad. Tokyo 16 (1940), p. 63-67.

[2] *Simultaneous extension of continuous functions considered as a positive linear operation*, Japan. J. Math. 17 (1940), p. 1-4.

[3] *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Annals of Math. 42 (1941), p. 523-537.

[4] *Concrete representation of abstract (M)-spaces*, ibidem, p. 994-1024.

L. V. Kantorovitch, B. E. Vulih, A. G. Pinsker

[1] *Functional analysis in semiordered spaces* (in Russian), Moscow 1950.

S. Kaczmarz and H. Steinhaus

[1] *Theorie des Orthogonalreihen*, Warszawa 1935.

S. Karlin

[1] *Bases in Banach spaces*, Duke Math. J. 15 (1948), p. 971-985.

J. L. Kelley

[1] *Banach spaces with the extension property*, Trans. Amer. Math. Soc. 72 (1952), p. 323-326.

M. Krein and S. Krein

[1] *On an inner characteristic of the set of all continuous functions defined on a bicompact Hausdorff space*, Doklady 27 (1940), p. 427-430.

[2] *Sur l'espace des fonctions continues définies sur un bicompact de Hausdorff et ses sousespaces semiordonnés*, Mat. Sbornik 13 (1943), p. 1-38.

J. Lindenstrauss

[1] *On some subspaces of c_0 and l_1*, Bull. Res. Council Israel 10 (1951), p. 74-80.

[2] *Extensions of compact operators*, Thesis, Hebrew University, 1962.

D. Maharam

[1] *On homogeneous measure algebra*, Proc. Nat. Acad. Sci. USA 28 (1942), p. 108-111.

E. Michael

[1] *Some extension theorem for continuous functions*, Pacific Math. J. 3 (1953), p. 789-806.

L. Nachbin

[1] *A theorem of Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc. 68 (1950), p. 28-46.

[2] *Some problems in extending and lifting continuous linear transformations*, Proc. Internat. Symp. on Linear Spaces, Jerusalem 1960, p. 340-351.

W. Orlicz

[1] *Beiträge zur Theorie der Orthogonalentwicklungen (II)*, Studia Math. 1 (1929), p. 241-255.

[2] *O szeregach doskonale zbieżnych w pewnych przestrzeniach funkcyjnych*, Prace Matem. 1 (1955), p. 393-414.

A. Pełczyński

[1] *On B-spaces containing subspaces isomorphic to the space c_0*, Bull. Acad. Pol. Sci. 5 (1957), p. 797-798.

[2] *On the isomorphism of the spaces M and m*, ibidem, 66 (1958), p. 695-696.

[3] *Projections in certain Banach spaces*, Studia Math. 19 (1960), p. 209-228.

[4] *On the impossibility of embedding of the space L in certain Banach spaces*, Coll. Math. 8 (1961), p. 199-203.

A. Pełczyński and Z. Semadeni

[1] *Spaces of continuous functions (III)*, Studia Math. 18 (1959), p. 211-222.

A. Pełczyński and V. N. Sudakov

[1] *Remark on non-complemented subspaces of the space $m(S)$*, Coll. Math. 19 (1961), p. 85-88.

B. J. Pettis

[1] *On integration in vector spaces*, Trans. Amer. Math. Soc. 44 (1958), p. 277-304.

R. R. Phelps

[1] *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. 95 (1960), p. 238-255.

R. S. Phillips

[1] *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), p. 516-541.

W. Rudin

[1] *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. 8 (1957), p. 39-42.

J. Schauder

[1] *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Zeits. 26 (1927), p. 47-65.

J. Schreier

[1] *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. 2 (1930), p. 58-62.

Z. Semadeni

[1] *Banach spaces non-isomorphic to their Cartesian squares*, Bull. Acad. Pol. Sci. 8 (1960), p. 81-84.

[2] *Functions with sets of points of discontinuity belonging to a fixed ideal*, Fund. Math., to appear.

[3] *Spaces of continuous functions (VI) (Localization of multiplicative linear functionals)*, Studia Math., to appear.

Z. Semadeni and P. Zbijewski

[1] *Przestrzenie funkcji ciągłych na kompaktach*, Prace Matem. 3 (1958), p. 17-47.

W. Sierpiński

[1] *L'influence de l'ordre des termes sur la convergence uniforme d'une série*, Sprawozdania Tow. Nauk. Warsz. 3 (1910), p. 353-357.

[2] *Fonctions additives non complètement additives et fonctions non mesurables*, Fund. Math. 30 (1938), p. 96-99.

R. Sikorski

[1] *Boolean Algebras*, Ergebn. Math. 25, Berlin 1960.

G. Silov

[1] *Ideals and subrings of the ring of continuous functions*, Doklady 22 (1939), p. 7-10.

A. Sobczyk

[1] *Projections of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. 47 (1941), p. 938-947.

M. H. Stone

[1] *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), p. 375-481.

[2] *Boundedness properties in function-lattices*, Canadian Math. J. 1 (1949), p. 176-186.

J. Wada

[1] *Stonian spaces and the second conjugate spaces of AM-spaces*, Osaka Math. J. 9 (1957), p. 195-200.

F. S. Waher

[1] *On basis in a space of continuous functions on a compact space* (in Russian), Doklady 101 (1955), p. 589-582.

H. Yoshizawa

[1] *On simultaneous extension of continuous functions*, Proc. Imp. Acad. Tokyo 20 (1944), p. 653-654.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I (1963)

Interpolation in Hilbert spaces of analytic functions

by

H. S. SHAPIRO and A. L. SHIELDS (Ann Arbor, Mich.)

Let H be a separable Hilbert space and let x_n be a sequence of unit vectors that span H. We wish to know when the sequence x_n will have the following two properties.

(i) $\sum |(x, x_n)|^2 \leqslant M\|x\|^2$ for some constant M and all x in H.

(ii) For each sequence $\{c_n\}$ in l_2 there is an x in H with $(x, x_n) = c_n$ and $\|x\|^2 \leqslant m\sum|c_n|^2$, where m is a constant.

Let $a_{ij} = (x_i, x_j)$ and let A be the infinite matrix (a_{ij}). Then it is not difficult to prove the following lemma:

LEMMA. *Property* (i) *is equivalent to each of the following two statements.*

1. *Given any orthonomal basis e_n, there is a bounded operator $T\colon H\to H$ with $T(e_n) = x_n$ (all n).*

2. *The matrix A is a bounded transformation from l_2 to itself.*

Property (ii) *is equivalent to each of the following two statements.*

1. *Given any orthonormal basis e_n, there is a bounded operator $T\colon H\to H$ with $T(x_n) = e_n$ (all n).*

2. *The matrix A is bounded below on l_2.*

Suppose now that H is a Hilbert space of analytic functions in some domain D of the complex plane. For example:

I. H is the set of $f = \Sigma a_n z^n$ with $\Sigma|a_n|^2 < \infty$ (this is the Hardy space H_2).

II. H is the set of f with $\sum|a_n|^2/(n+1) < \infty$ (this is the Bergman space of f analytic in the unit circle for which $\iint|f|^2dx\,dy < \infty$).

III. H is the set of f with $\sum|a_n|^2 n! < \infty$.

We assume that H has a *reproducing kernel*, that is, a function $K_\zeta(z)$ (z, ζ in D) such that $(f, K) = f(\zeta)$ for all f in H and ζ in D. In the three examples we have:

I. $K = 1/(1-\bar{\zeta}z)$.

II. $K = 1/(1-\bar{\zeta}z)^2$.

III. $K = \exp(\bar{\zeta}z)$.

Let z_n be a sequence of points in D, and form the unit vectors $x_n = K_n/\|K_n\|$ where $K_n = K_{z_n}(z)$. We now seek necessary and sufficient