is induced by commuting the set $(A_\beta)_{\beta\in B}$. *Then*

$$\hat{\varphi}_k(\lambda) = \langle \varphi, e_k(\lambda)\rangle, \quad k = 1, 2, \ldots, \dim \hat{H}(\lambda),$$

where $e_k(\lambda)\in\Phi'$ *(dual of* Φ*). The* $e_k(\lambda)$ *are generalized simultaneous eigenfunctions of* (A_β).

COROLLARY 3 (Berchanskiĭ). *If the operator* B *with a dense domain* $D(B)$ *has an inverse* B^{-1} *of H.-S.-type, then putting* $H_1 = D(B)$ *with* $(u, v)_B = (Bu, Bv)+(u, v)$ *we get the thesis of theorem* 4.

COROLLARY 4. *Other spectral theorems given by Berchanskiĭ.*

COROLLARY 5. *Put* $H_p = H^m(\Omega^p)$, $H = H^0 = L^2(\Omega_N)$, *then the eigenelements of partial differential operators* (A_β) *are distributions* $e_k(\lambda)\in H^{-m}(\Omega) = H^m$ *of an order* $\leqslant N/2$.

Let $B(\varphi, \psi)$ be scalar product of an order $\leqslant r$ $(=1, 2, \ldots)$, $H(B)$ completion of $C_0^\infty(\Omega)$ in $B(\cdot, \cdot)$.

Put in theorem 4: $H = H(B)$, $\Phi = H^{m+r}(\Omega)$; then we get the following

COROLLARY 6 (a sharper form of a theorem of Gårding). *Let* (A_β) *be a commuting set of observables in* $H(B)$; *then the Fourier transform* $\Phi\ni\varphi\to\hat{\varphi}_k(\lambda) = \langle\varphi, e_k(\lambda)\rangle$, *where the simultaneous eigenfunctions of* (A_β) *are elements of* $H^{-(m+r)}(\Omega_N)$, *i. e. distributions of an order not exceeding* $N/2+r$.

Concluding remark. All proofs are exceedingly simple, which shows that the instrument of H.-S.-mappings is suitable for mastering the problem considered above.

STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I. (1963)

Continuous selections in Banach spaces

by

E. MICHAEL (Seattle, Wash.)

Dedicated to the Memory of Stefan Banach

One of Stefan Banach's many interests was the interrelationship of topological and linear phenomena. This paper is a summary of some recent work in that direction.

Let X and Y be topological spaces, and Φ a function from X to the collection of non-empty subsets of Y. Then a *selection* for Φ is a continuous $f\colon X\to Y$ such that $f(x)\in\Phi(x)$ for every $x\in X$. Our problem is to find conditions which insure the existence of a selection for Φ.

For continuity, it suffices to assume that Φ is *lower semi-continuous*, that is, for every open $V\subset Y$ the set $\{x\in X \mid \Phi(x)\cap V\neq 0\}$ is open in X. As for the sets $\Phi(x)$, they will usually be closed, and either *convex* subsets of a *Banach space* or something similar; this will assure not only that these sets are individually well behaved, but that they are properly interrelated. Finally, we shall usually assume that X is paracompact.

We begin with our simplest and most basic result, and will then consider various refinements.

THEOREM 1 [1]. *If* X *is paracompact,* Y *a Banach space, and* $\mathcal{C}(Y)$ *the family of non-empty, closed, convex subsets of* Y, *then any lower semi-continuous* $\Phi\colon X\to\mathcal{C}(Y)$ *admits a selection.*

It should be remarked that Theorem 1 actually characterizes paracompact spaces.

COROLLARY 1 [1]. *If* E *is a Banach space,* F *a closed subspace, and* $u\colon E\to E/F$ *the natural projection, then there exists a continuous* $f\colon E/F\to E$ *such that* $f(x)\in u^{-1}(x)$ *for every* $x\in E/F$.

There are various ways of strengthening Theorem 1. The simplest is to replace the Banach space Y by a locally convex F-space. Let us outline three other possible improvements.

First, the requirement that the sets $\Phi(x)$ be closed can, under suitable circumstances, be somewhat relaxed. For instance, if X is *perfectly normal* (not necessarily paracompact) and Y *separable*, then $\mathcal{C}(Y)$ can be replaced in Theorem 1 by a somewhat larger class of convex sets which includes, in particular, all those which are either *closed*, or *finite dimensional*, or possess an *interior point* [1].

Our second method of strengthening Theorem 1 is to weaken the requirement of convexity. Let us call a subset A of a Banach space *a-convex* ($0 \leqslant a < \frac{1}{2}$) if, whenever $S \subset A$ and x is in the convex hull of S, then $\varrho(x, S) \leqslant a$ (diam S). (Note that if A is closed, 0-convex = convex.) The sets which are a-convex for some $a < \frac{1}{2}$ are a strange lot; they include, for instance, all finite unions of intervals radiating from a point, but do not include a semi-circle in the plane (where the subset S consisting of the two end poir ts causes trouble). Anyhow, Theorem 1 remains true if $\mathscr{C}(Y)$ is replaced by the family of all closed, a-convex subsets of Y, for some *fixed* $a < \frac{1}{2}$. (See [3].)

Finally, Theorem 1 may be strengthened by no longer requiring Y to be a Banach space, but only a complete metric space with an axiomatically defined *convex structure*, which permits one to take "convex combinations" of some (but not necessarily all) n-tuples in a suitably continuous fashion. (This includes the locally convex F-spaces mentioned earlier.)

With convex sets then defined in the obvious way, Theorem 1 remains true in this considerably more general context. Among the consequences of this new result, let us mention that Corollary 1 remains true if E is merely assumed to be a metrizable group, and F a closed subgroup which is isomorphic to a Banach space. (See [2]).

In conclusion, let us mention that, for *finite dimensional* X, it is possible to place purely topological conditions on the sets $\Phi(x)$ which are not only sufficient but, in a sense, also necessary. Without dimensional restrictions on X, however, the search for such a set of conditions has so far remained unsuccessful.

References

[1] E. Michael, *Continuous selections I*, Annals of Math. 63 (1956), p. 361-382.

[2] — *Convex structures and continuous selections*, Canadian J. Math. 11 (1959), p. 556-575.

[3] — *Paraconvex sets and continuous selections*, Math. Scand. 7 (1959), p. 372-376.

STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I. (1963)

Operators and distributions

by

J. MIKUSIŃSKI (Warszawa)

I recall first the definition of operator which I have introduced in my book *Operational Calculus*.

We start from the ring of continuous functions of a real variable t ($0 \leqslant t < \infty$), the product being defined by the convolution

$$fg = \int_0^t f(t-\tau)g(\tau)d\tau.$$

By a Titchmarsh theorem, that ring has no divisors of zero. The elements of the quotient field, obtained from that ring are by definition operators.

This algebraic method seems to be the most adequate for the notion of operators. However an equivalent definition can be given by the sequential method, as for distributions. Then instead of distributional convergence we should introduce the following one:

We say that a sequence of continuous functions $\varphi_n(x)$ is *convergent* or *fundamental* if, given any non-vanishing identically function ω, the sequence $\omega\varphi_n$ (convolution) converges almost uniformly.

Since this convergence is more general than the distributional one, the notion of operator is more general than that of distribution. But this is only so when the interval where the functions are defined is bounded from below. It if is not, distributions can be obtained from operators by an additional limiting process.

It is important to answer the question whether the algebraic (or sequential) method of introducing operators is essentially more general than that of the Laplace transform. In order to give a positive answer we should show that there exist operators or functions which can not be represented in the form

$$x = \frac{f}{g}$$

(the division being meant here as the inversion of the convolution), where f and g are Laplace transformable. In other words, we should