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STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I. (1963)

Mappings of Hilbert-Schmidt type; their applications to eigenfunction expansions and elliptic boundary problems

by

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Definition. The linear mapping $A: E \to F$ of the form

$$E \ni u \to Au = \sum_i (u, e_i)_E f_i \in F$$

where $(e_i)^\infty$ is orthonormal basis of (pre) Hilbert space E, $f_i \in F$, $\sum \|f_i\|_F^2 < \infty$ is called *H.-S.-mapping*.

THEOREM 1. *Let G, H be Hilbert spaces. $B: G \to E$; $C: F \to H$ continuous, $A: E \to F$ of H.-S.-type. Then $B \circ A \circ C$ and the adjoint $A^*: F \to E$ are of H.-S.-type.*

THEOREM 2. *If A, B are H.-S., then $A \circ B$ is nuclear.*

Definition. $(u, v)_m = \sum_{|\alpha| \leqslant m} (D^\alpha u, D^\alpha v)_0$, where $(u, v)_0 = \int_{\Omega_N} u\bar{v}\,dx$, $D^\alpha = D_{\alpha_1} \dots D_{\alpha_N}$, $D_{\alpha_k} = (\partial/\partial x_k)^{\alpha_k}$, $|\alpha| = \alpha_1 + \dots + \alpha_N$.

H^m (resp. H_0^m) completion of $C^\infty(\bar{\Omega}_N)$ (resp. $C_0^\infty(\Omega_N)$ in $\| \|_m$-norm, Ω_N bounded, $H^m(\Omega) = 1.\operatorname{ind}_p H_0^m(\Omega^p)$, where $\Omega^p \nearrow \Omega_N$.

THEOREM 3. *The embeddings $H^{m+k} \to H^k$ (resp. $H_0^{m+k} \to H_0^k$) for Ω_N with strong cone property (respectively every bounded Ω_N) are H. S. for $m > N/2$, $k \geqslant 0$.*

COROLLARY. *Embeddings $H^{2m+k} \to H^k$, $m > N/2$, $k \geqslant 0$, are nuclear.*

COROLLARY 2. *If*

$$Au = f, \quad Bju/\partial\Omega_N = 0 \quad (j = 1, \dots, p) \tag{1}$$

is correctly posed elliptic boundary problem, i. e. there is such a constant c_k that $\|u\|_{r+k} \leqslant c_k \|Au\|_k$, r — order of A, R — resolvent of (1), $R: H^0 \to H^0$; then R^j is of H.-S. type for $jr > N/2$.

THEOREM 4. *If $\Phi = 1.\operatorname{ind}_p H_p$, where canonic embeddings $i_p: H_p \to H$ are H.-S.; then the Fourier transform*

$$\Phi \ni \varphi \to F\varphi = \hat{\varphi} \in \hat{H} = \int_\Lambda \hat{H}(\lambda)\,d\mu(\lambda)$$

is induced by commuting the set $(A_\beta)_{\beta\in B}$. Then

$$\hat{\varphi}_k(\lambda) = \langle\varphi, e_k(\lambda)\rangle, \qquad k = 1, 2, \ldots, \dim \hat{H}(\lambda),$$

where $e_k(\lambda)\in\Phi'$ (dual of Φ). The $e_k(\lambda)$ are generalized simultaneous eigenfunctions of (A_β).

COROLLARY 3 (Berchanskiĭ). *If the operator B with a dense domain $D(B)$ has an inverse B^{-1} of H.-S.-type, then putting $H_1 = D(B)$ with $(u, v)_B = (Bu, Bv)+(u, v)$ we get the thesis of theorem 4.*

COROLLARY 4. *Other spectral theorems given by Berchanskiĭ.*

COROLLARY 5. *Put $H_p = H^m(\Omega^p)$, $H = H^0 = L^2(\Omega_N)$, then the eigenelements of partial differential operators (A_β) are distributions $e_k(\lambda)\in$ $\in H^{-m}(\Omega) = H^m$ of an order $\leqslant N/2$.*

Let $B(\varphi, \psi)$ be scalar product of an order $\leqslant r$ $(=1, 2, \ldots)$, $H(B)$ completion of $C_0^\infty(\Omega)$ in $B(\cdot, \cdot)$.

Put in theorem 4: $H = H(B)$, $\Phi = H^{m+r}(\Omega)$; then we get the following

COROLLARY 6 (a sharper form of a theorem of Gårding). *Let (A_β) be a commuting set of observables in $H(B)$; then the Fourier transform $\Phi\ni\varphi\to\hat{\varphi}_k(\lambda) = \langle\varphi, e_k(\lambda)\rangle$, where the simultaneous eigenfunctions of (A_β) are elements of $H^{-(m+r)}(\Omega_N)$, i. e. distributions of an order not exceeding $N/2+r$.*

Concluding remark. All proofs are exceedingly simple, which shows that the instrument of H.-S.-mappings is suitable for mastering the problem considered above.

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Continuous selections in Banach spaces

by

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Dedicated to the Memory of Stefan Banach

One of Stefan Banach's many interests was the interrelationship of topological and linear phenomena. This paper is a summary of some recent work in that direction.

Let X and Y be topological spaces, and Φ a function from X to the collection of non-empty subsets of Y. Then a *selection* for Φ is a continuous $f\colon X\to Y$ such that $f(x)\in\Phi(x)$ for every $x\in X$. Our problem is to find conditions which insure the existence of a selection for Φ.

For continuity, it suffices to assume that Φ is *lower semi-continuous*, that is, for every open $V\subset Y$ the set $\{x\in X \mid \Phi(x)\subset V\neq 0\}$ is open in X. As for the sets $\Phi(x)$, they will usually be closed, and either *convex* subsets of a *Banach space* or something similar; this will assure not only that these sets are individually well behaved, but that they are properly interrelated. Finally, we shall usually assume that X is paracompact.

We begin with our simplest and most basic result, and will then consider various refinements.

THEOREM 1 [1]. *If X is paracompact, Y a Banach space, and $\mathscr{C}(Y)$ the family of non-empty, closed, convex subsets of Y, then any lower semi-continuous $\Phi\colon X\to C(Y)$ admits a selection.*

It should be remarked that Theorem 1 actually characterizes paracompact spaces.

COROLLARY 1 [1]. *If E is a Banach space, F a closed subspace, and $u\colon E\to E/F$ the natural projection, then there exists a continuous $f\colon E/F\to E$ such that $f(x)\in u^{-1}(x)$ for every $x\in E/F$.*

There are various ways of strengthening Theorem 1. The simplest is to replace the Banach space Y by a locally convex F-space. Let us outline three other possible improvements.

First, the requirement that the sets $\Phi(x)$ be closed can, under suitable circumstances, be somewhat relaxed. For instance, if X is *perfectly normal* (not necessarily paracompact) and Y *separable*, then $\mathscr{C}(Y)$ can