[16] P. Malliavin, *Impossibilité de la synthèse spectrale sur les groupes abéliens non compacts*, Publ. Math. de l'Institut des Hautes Études Scientifiques, No. 2 (1959), p. 61-68.

[17] S. Mazur, *Sur les anneaux linéaires*, C. R. Acad. Sci. Paris 207 (1938), p. 1025-1027.

[18] Mitio Nagumo, *Einige analytische Untersuchungen in linearen metrischen Ringen*, Jap. J. Math. 13 (1936), p. 61-80.

[19] J. v. Neumann, *Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren*, Math. Ann. 102 (1929), p. 370-427.

[20] W. Rudin, *Idempotent measures on abelian groups*, Pacific J. Math. 9 (1959), p. 195-209.

[21] — *Measure algebras on abelian groups*, Bull. Amer. Math. Soc. 65 (1959), p. 227-247.

[22] I. E. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc. 61 (1947), p. 69-105.

[23] Yu. A. Šreĭder, *On an example of a generalized character*, Mat. Sbornik (N. S.) 29 (71) (1951), p. 419-426.

[24] — *The structure of maximal ideals in rings of measures with convolution*, Mat. Sbornik (N. S.) 27 (69) (1950), p. 297-318.

[25] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), p. 375-481.

[26] K. Stromberg, *A note on the convolution of regular measures*, Math. Scand. 7 (1959), p. 347-352.

[27] N. Wiener, *Tauberian theorems*, Ann. of Math. (2) 33 (1932), p. 1-100.

[28] — *The Fourier integral and certain of its applications*, Cambridge 1933.

[29] N. Wiener and H. R. Pitt, *On absolutely convergent Fourier-Stieltjes transforms*, Duke Math. J. 4 (1938), p. 420-436.

[30] J. H. Williamson, *A theorem on algebras of measures on topological groups*, Proc. Edinburgh Math. Soc. 11 (1959), p. 195-206.

[31] K. Yosida, *On the group embedded in the metrical complete ring*, Jap. J. Math. 13 (1936), p. 7-26.

STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I. (1963)

Linear differential equations in Banach algebras

by

E. HILLE (New Haven, Conn.)

1. We denote by B a complex B-algebra with unit element e. Let ζ be a complex variable, and $f(\zeta)$ a function whose values are in B. Then

$$w'(\zeta) = f(\zeta)w(\zeta) \tag{1}$$

may be regarded as a generalization of the classical system of first order linear differential equations. It reduces to such a system if B is the algebra of n by n matrices.

If $f(\zeta)$ is holomorphic in a simply-connected domain Δ, and if w_0 is a given element of B, then there exists a unique solution $w(\zeta; \zeta_0, w_0)$ of (1) which is holomorphic in Δ and which reduces to w_0 when $\zeta \to \zeta_0$. This solution is a regular element of algebra if and only if w_0 has this property. We have

$$w(\zeta; \zeta_0, w_0) = w(\zeta; \zeta_0, e)w_0. \tag{2}$$

2. Suppose next that Δ is a sector with $\zeta = 0$ as vertex and that $f(\zeta)$ is not holomorphic at $\zeta = 0$. The behavior of the solution as $\zeta \to 0$ depends upon the integrability properties of $\|f(\zeta)\|$. If this function is integrable down to 0, the solutions have finite limits as $\zeta \to 0$ and the initial value problem may be set also at the singular point $\zeta = 0$.

If the integral diverges, several cases arise. Thus, if $\zeta = 0$ is a simple pole of $f(\zeta)$, then a priori estimates of the form

$$a^{-1}|\zeta|^{a} \leqslant \|w(\zeta)\| \leqslant a|\zeta|^{-a} \tag{3}$$

hold and we may speak of a regular singular point in analogy with the classical case. If the pole is of order k, $k > 1$, we have the analogue of an irregular singular point of rank $k-1$ and $\|w(\zeta)\|$ now lies between two exponential functions of $|\zeta|^{-1}$.

3. In the regular singular case, suppose that

$$f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^{n-1}, \quad c_n \in B,\ |\zeta| < \varrho. \tag{4}$$

As in the classical case, we try to find a solution of the form

$$w(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{c_0+nc}, \tag{5}$$

where the powers are defined by the operational calculus. This substitution leads to a system of equations

$$\begin{aligned} c_0 a_0 - a_0 c_0 &= 0, \\ n a_n - (c_0 a_n - a_n c_0) &= \sum_{k=1}^{n} c_k a_{n-k}, \quad n = 1, 2, \ldots \end{aligned} \tag{6}$$

The properties of this system depend upon the spectral properties of c_0 as an element of B and these in their turn determine the spectral properties of the commutator

$$T(x) = c_0 x - x c_0. \tag{7}$$

If the spectrum of $T(x)$ does not contain any integer $\neq 0$, then we can take $a_0 = e$ and determine the a_n's successively from (6). The resulting series converges for $0 < |\zeta| < \varrho$ and satisfies the differential equation. This case arises if the spectrum of c_0 does not contain any two numbers which differ by an integer.

If the spectrum of $T(x)$ does contain integers, but these are poles of the resolvent of $T(x)$, then a modification of the classical method of Frobenius leads to the desired result. If N is the sum of the orders of the poles of the resolvent at the positive integers, then the solution is a polynomial in $\log\zeta$ of degree N where each power of $\log\zeta$ is multiplied by a power series of type (5). The analysis breaks down if singularities of more general nature are allowed.

For the case of an irregular singular point, the reduction of G. D. Birkhoff using Laplace transforms offers a possible mode of approach.

STUDIA MATHEMATICA, SERIA SPECJALNA, Z. I. (1963)

Characterizations of reflexive Banach spaces

by

R. C. JAMES (Claremont, Calif.)

A Banach space B is isometric with a subspace of its second conjugate space under the "natural mapping" for which the element of B^{**} which corresponds to the element x_0 of B is the linear functional F_{x_0} defined by $F_{x_0}(f) = f(x_0)$ for each f of B^*. If every F of B^{**} is of this form, then B is said to be *reflexive* and B is isomorphic with B^{**} under this natural mapping. It is known that for a Banach space B each of the following conditions is equivalent to reflexivity:

(i) *the unit sphere of B is weakly compact* (Eberlein and Šmulian);

(ii) *each decreasing sequence of bounded closed convex sets has a non-empty intersection* (Šmulian).

If the unit sphere is weakly compact, then each continuous linear functional attains its sup on the unit sphere. Klee used (ii) to show that if each continuous linear functional attains its sup on the unit sphere of any isomorph of the space, then the unit sphere is weakly compact. It was shown by the author that if each continuous linear functional attains its sup on the unit sphere of a separable Banach space, then the space is reflexive. The proof used a characterization of reflexivity which states essentially that a Banach space is reflexive if and only if its unit sphere does not contain a "large flat region", that is,

(iii) *there does not exist a sequence of elements $\{z_i\}$ and positive numbers, Δ, δ, and σ, for which the following are true:*

$$\Delta(\Sigma a_i) > \|\Sigma a_i z_i\| > \delta(\Sigma a_i) \text{ if each } a_i \text{ is positive};$$

$$\|\Sigma_1^n a_i z_i - \Sigma_{n+1}^p \beta_i z_i\| > \sigma \|\Sigma_1^n a_i z_i\| \text{ if each } \beta_i \text{ is positive.}$$

This characterization of reflexivity and an extension of the author's original proof of the theorem about sups of continuous linear functionals for separable Banach spaces has been used to prove that a Banach space is reflexive if and only if:

(iv) *each continuous linear functional attains its sup on the unit sphere.*