$$\varphi_1(t) \leqslant f(t) \leqslant \varphi_2(t) \quad \text{überall auf } D,$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\varphi_2(t)-\varphi_1(t)]\,dt < \varepsilon.$$

Vermöge der Korrespondenz zwischen den stetigen Funktionen auf K_Λ und den Bohrschen fp. Funktionen mit Exponenten aus Λ beweist man nun mühelos den

SATZ 1. *Damit eine B-Klasse eine R-fp. Funktion enthält, ist notwendig und hinreichend, daß die entsprechende μ-Klasse eine nach Riemann integrierbare Funktion enthalte.*

Um sich klarzumachen, welchen Platz die R-fp. Funktionen unter den bekannten Typen von verallgemeinerten fp. Funktionen einnehmen, bemerke man zuerst, daß jede Bohrsche fp. Funktion trivialerweise auch R-fastperiodisch ist und daß eine fp. Funktion von Stepanoff nicht R-fastperiodisch zu sein braucht. Wohl aber gilt der leicht beweisbare

SATZ 2. *Eine R-fp. Funktion ist für jedes $p \geqslant 1$ W^p-fastperiodisch, d. h. der Limes einer Folge trigonometrischer Polynome mit reellen Exponenten nach der Norm*

$$\left\{\lim_{l\to\infty} \sup_{x} \frac{1}{l} \int_{x}^{x+l} |f(t)|^p\,dt\right\}^{1/p}$$

Die R-fp. Funktionen erscheinen bei der Untersuchung gewöhnlicher Bohrscher Funktionen, es gilt nämlich der

SATZ 3. *Wird $c_a(y) = 1$ oder 0 gesetzt, je nachdem man $y < a$ oder $y \geqslant a$ hat und ist f eine fp. Funktion von Bohr mit Exponenten aus Λ, so ist die Funktion $c_a(f(t))$ für jedes a bis auf eine höchstens abzählbare Menge R-fastperiodisch mit Exponenten aus Λ.*

Satz 3 erhält man am leichtesten durch einen Umweg über das Kompaktum K_Λ.

Es ist beinahe evident, daß die Grenzfunktion f einer gleichmäßig auf D konvergenten Folge $\{f_n\}$ von R-fp. Funktionen selbst R-fp. ist. Darüber hinaus gilt aber

SATZ 4. *Konvergiert die Folge $\{f_n\}$ auf D gleichmäßig und sind die Funktionen f_n mit R-fp. Funktionen B-äquivalent, so ist es die Limesfunktion auch.*

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Measure algebras on locally compact groups: a case history in functional analysis

by

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The study of function algebras as such and of abstractions arising therefrom appeared only at a comparatively late period in the development of functional analysis. The fundamental work of Hilbert, F. Riesz, Hahn, and others had already produced the famous examples of normed linear spaces that every student is familiar with today; Banach had axiomatized the theory of these spaces into the very important class of spaces that bear his name; Banach and other workers had established the fundamental facts of the theory of Banach spaces and applied them to many important problems in analysis: all of this before anyone apparently thought of studying *algebras* that are also topological structures.

Algebras that are Banach spaces and in which $\|xy\| \leqslant \|x\|\,\|y\|$ are called Banach algebras [*нормированные кольца* in Soviet terminology]. The general theory of these algebras is the creation of I. M. Gel′fand, who in his famous 1941 memoir [3] defined Banach algebras, established their basic properties, and pointed the way for the future development of the subject. His axiomatic development was anticipated by several writers. In 1936, Nagumo [18] and Yosida [31] gave independently the definition of a Banach algebra. Yosida used it to prove that a locally compact multiplicative subgroup of a Banach algebra is a Lie group. In 1938, Mazur [17] defined real normed algebras and proved that a real normed division algebra is the real number field, the complex number field, or the quaternions.

As is so frequently the case, the general theory of Banach algebras was preceded by the analysis of a number of special examples of Banach algebras. The first study of an infinite-dimensional algebra with an accompanying topological structure seems to have been v. Neumann's work [19] on rings of bounded linear operators in Hilbert spaces. In the years since v. Neumann's work, this theory has grown into a vast and complicated field. It is not subsumed under the ordinary theory of Banach algebras [desirable though this would be], however, and we shall not consider it here.

Chronologically, the next Banach algebra to have been closely studied is the algebra $\mathfrak{L}_1(R)$, the class of all absolutely integrable complex functions on the real line R, with pointwise linear operations, the product of two functions f and g defined by convolution: $f*g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$, and $\|f\| = \int_{-\infty}^{\infty} |f(t)|\,dt$. Wiener's work on $\mathfrak{L}_1(R)$ [27, 28], culminating in his famous Tauberian theorem, is universally interpreted today as algebraic in nature, although Wiener himself did not use the jargon or the techniques of algebra. [It is worth while to note that none of the proofs of Wiener's Tauberian theorem is algebraic: all of them require analytic methods.] For example, Wiener's Tauberian theorem asserts that every proper closed ideal in the algebra $\mathfrak{L}_1(R)$ is contained in some regular maximal ideal. Also, the Lévy-Wiener theorem on analytic functions of functions with absolutely convergent Fourier series is a special case of a theorem true for all commutative Banach algebras. The algebraic structure of $\mathfrak{L}_1(R)$ is not yet completely known; P. Malliavin has recently shown that it has great complexity [16].

Considerably more success has attended the study of all bounded continuous functions on a topological space. Let X be a compact space [it may as well be taken to satisfy Hausdorff's separation axiom]. Let $\mathfrak{c}(X)$ denote the set of all complex-valued continuous functions on X. Under pointwise operations and the uniform norm, $\mathfrak{C}(X)$ is plainly an algebra which is also a Banach space and in which the fundamental inequality $\|fg\| \leqslant \|f\|\cdot\|g\|$ obtains. In a 1937 paper [25], M. H. Stone, among other things, completely analyzed the structure of this algebra. [He dealt with $\mathfrak{C}_r(X)$, the real-valued functions in $\mathfrak{C}(X)$, but all of his results and some of his arguments go over, *mutatis mutandis*, to the complex case.] The maximal ideals, closed ideals, and uniformly closed subalgebras of $\mathfrak{C}_r(X)$ were completely described, and the fact that the algebraic structure of $\mathfrak{C}_r(X)$ identifies X among all compact Hausdorff spaces was proved. In the complex case, one apparent technicality has led to much research. For a uniformly closed subalgebra $\mathfrak{A}$ of $\mathfrak{C}(X)$ to be the algebra $\mathfrak{c}(Y)$ for a continuous image Y of X, it is necessary and sufficient that $\mathfrak{A}$ should be closed under the formation of complex conjugates. The study of uniformly closed subalgebras of $\mathfrak{c}(X)$ *not* closed under the formation of complex conjugates involves some very delicate questions, and the theory is far from complete at the present day. A useful survey, containing also new results, has recently been published by K. Hoffman and I. M. Singer [15].

The next specific Banach algebra to be studied, so far as the writer is aware, was the algebra of complex functions of finite total variation on the real line R, which constitutes a superalgebra of $\mathfrak{L}_1(R)$. Let $\mathcal{M}(R)$ denote the set of all complex-valued functions on R having finite total variation, that are continuous on one side [let us say on the left], and have limit 0 at $-\infty$. Addition and scalar multiplication of such functions are defined pointwise. The product $f*g$ [convolution] of two functions f, g in $\mathcal{M}(R)$ is defined by the formula $f*g(x) = \int_{-\infty}^{\infty} f(x-t)\,dg(t)$. The norm $\|f\|$ of $f \in \mathcal{M}(R)$ is the total variation of f over R. [The proviso that functions in $\mathcal{M}(R)$ should be left continuous is a harmless normalization. As the integral defining $f*g$ is necessarily a Lebesgue-Stieltjes integral, if two functions f and f' of finite variation on R differ only at points of discontinuity, we will have $g*f = g*f'$ for all bounded Borel measurable functions g. The proviso $f(-\infty) = 0$ is less trivial. If it were omitted, our algebra would contain the function $\boldsymbol{l}$ which is identically equal to 1. We would have $f*\boldsymbol{l} = \boldsymbol{O}$ and $\boldsymbol{l}*f = \big(f(\infty)-f(-\infty)\big)\boldsymbol{l}$ for all functions f of finite variation. Thus our algebra would be noncommutative and would have a one-dimensional radical, consisting of all constant functions.]

The algebra $\mathcal{M}(R)$ was studied by A. Beurling in a paper published in 1938 [1]. He used the Fourier-Stieltjes transform $\hat{f}(t) = \int_{-\infty}^{\infty} \exp(itx)\,df(x)$ pointing out that the mapping $f \to \hat{f}$ is an isomorphism of $\mathcal{M}(R)$. He decomposed $\mathcal{M}(R)$ into the direct sum of the classes of absolutely continuous functions, singular continuous functions, and saltus functions, pointing out that the difficulties in studying $\mathcal{M}(R)$ arise because of the existence of singular continuous functions. He introduced the famous limit $\lim_{n\to\infty} \|f^n\|^{1/n}$, which later became a vitally important instrument in Gel'fand's theory of commutative Banach algebras. He also proved that if $f \in \mathcal{M}(R)$ and f has no singular part, then $\sup\{|\hat{f}(t)|: t \in R\} = \lim_{n\to\infty} \|f^n\|^{1/n}$. In terms of Gel'fand's theory, this equality states that a function in $\mathcal{M}(R)$ with no singular part cannot be mapped by a multiplicative linear functional of $\mathcal{M}(R)$ into a number of absolute value greater than $\sup\{|\hat{f}(t)|: t \in R\}$. A simple proof of Beurling's equality can be obtained from this observation.

An important contribution to the theory of $\mathcal{M}(R)$ by N. Wiener and H. R. Pitt [29] also appeared in 1938. Their main result is as follows. Let f be in $\mathcal{M}(R)$ and have singular part f_s and saltus part f_d. If $\|f_s\| < \inf\{|\hat{f}_d(t)|: t \in R\}$ and $\hat{f}(t)$ vanishes for no t [this relaxes their hypothesis somewhat], then f has an inverse in $\mathcal{M}(R)$. This result is also easy to obtain by the general theory of Banach algebras. Wiener and Pitt also constructed a function $f \in \mathcal{M}(R)$ such that $\inf\{|\hat{f}(t)|: t \in R\} > 0$ but f^{-1} does not exist. This construction seems to be obscure, however, and the situation was completely cleared up only by a construction of Yu. A. Šreĭder [24] published in 1950.

Gel'fand [4] has also commented on $\mathscr{M}(R)$, and further contributions to its theory have been made by D. A. Raĭkov [5]. I. E. Segal in [22] proved Gel'fand's theorem on analytic functions of elements of Banach algebras, for functions in $\mathscr{M}(R)$ without singular part.

The real Banach algebra $\mathscr{M}_r(R)$, consisting of all real-valued functions in $\mathscr{M}(R)$, admits a lattice ordering: nonnegative functions are the nondecreasing functions. Then sums and products of positive functions are positive, and one might expect that this ordering would play a rôle in the structure of $\mathscr{M}_r(R)$ comparable to the vital rôle played by positivity in the theory of $\mathfrak{C}_r(X)$. This is not the case. The mappings $f \to \int_{-\infty}^{\infty} \exp(itx)\,df(x)$ are homomorphisms of $\mathscr{M}_r(R)$ onto the complex number field for all real $t \neq 0$. This phenomenon does not occur in the case of $\mathfrak{C}_r(X)$, where nonnegative functions go into non-negative *real* numbers under every homomorphism of $\mathfrak{C}_r(X)$ onto a field. There are $2^{\mathfrak{c}}$ homomorphisms of $\mathscr{M}_r(R)$ onto the real number field, but they do not determine $\mathscr{M}_r(R)$.

Unlike algebras $\mathfrak{c}(X)$, the algebra $\mathscr{M}(R)$ is very far from being completely analyzed. The remainder of this essay is a summary of the present knowledge about this algebra and about some of its generalizations.

It was recognized long ago that the algebra $\mathscr{M}(R)$ is a concrete example of a large class of Banach algebras. With a function $f \in \mathscr{M}(R)$ one can associate a unique complex-valued Borel measure λ on R: $\lambda(]-\infty, t[) = f(t)$ for all $t \in R$. As is well known, λ is completely determined for all Borel sets by the values $\lambda(]-\infty, t[)$ and the requirement of countable additivity. If $g \in \mathscr{M}(R)$ and $\mu(]-\infty, t[) = g(t)$, then the measure $\lambda * \mu$ corresponding to the convolution $f*g$ is given by $\lambda*\mu(E) = \int_{-\infty}^{\infty} \lambda(E-x)\,d\mu(x) = \int_{-\infty}^{\infty} \mu(E-x)\,d\lambda(x) = \mu*\lambda(E)$, for all Borel sets $E \subset R$.

This leads to the following general definition. Let G be a locally compact group, written multiplicatively, and with identity element e. Let B denote the Borel sets of G, that is, the smallest σ-algebra of subsets of G containing all compact sets. Let $\mathscr{M}(G)$ denote the set of all complex-valued, bounded, regular, countably additive measures defined on B. [This definition is slightly redundant.] For $\lambda \in \mathscr{M}(G)$, let $|\lambda|(E) = \sup\{\sum_{j=1}^{m} |\lambda(E_j)| : E_1, E_2, \ldots, E_m$ is a partition of E into Borel sets$\}$. Then $|\lambda|$ is the smallest nonnegative real measure in $\mathscr{M}(G)$ that majorizes $|\lambda(E)|$ for all Borel sets E. [The properties of $|\lambda|$ are described in detail in [8].] Let $\|\lambda\|$ be defined as $|\lambda|(G)$. For $\lambda, \mu \in \mathscr{M}(G)$, let $(\lambda+\mu)(E) = \lambda(E) + \mu(E)$ and for a complex number α, $(\alpha\lambda)(E) = \alpha(\lambda(E))$ for all $E \in \mathsf{B}$. Let $\lambda*\mu(E) = \int_G \lambda(Ex^{-1})\,d\mu(x) = \int_G \mu(x^{-1}E)\,d\lambda(x)$, for all $E \in \mathsf{B}$. The set-function $\lambda*\mu$ is again regular, as K. Stromberg has shown [26], and thus is an element of $\mathscr{M}(G)$. With the algebraic operations and norm just described, $\mathscr{M}(G)$ is a Banach algebra, commutative if and only if G is Abelian. For $a \in G$, let ε_a be the measure such that $\varepsilon_a(E) = 0$ or 1 as $a \notin E$ or $a \in E$. Then ε_e is the identity of $\mathscr{M}(G)$, and the measures ε_a form a group under convolution isomorphic with G.

Actually, $\mathscr{M}(G)$ itself is a particular example of a large class of algebras, called *convolution algebras*, introduced and studied by E. Hewitt and H. S. Zuckerman in a series of papers [12], [11], [14], [13]. These algebras appear in the most diverse branches of mathematics, all the way from number theory to mathematical statistics. Their theory has as yet been only slightly developed.

We return to the algebra $\mathscr{M}(G)$. A survey of the theory of $\mathscr{M}(G)$ for locally compact Abelian G as of 1958 has recently been published by W. Rudin [21], and a survey of the theory of $\mathscr{M}(G)$ for arbitrary locally compact G as of 1956 by E. Hewitt [7]. For the state of the theory at the times of writing, the reader is referred to these memoirs. We shall now describe some of the developments in the theory of $\mathscr{M}(G)$ that have been made in the past two years.

Paul J. Cohen [2] has settled a long outstanding problem by identifying all of the idempotent measures $[\mu*\mu = \mu]$ in $\mathscr{M}(G)$ for locally compact Abelian G. Cohen's result, which corroborates a conjecture of W. Rudin [18], can be stated simply enough. Let H be a compact subgroup of G, and let λ_H denote normalized Haar measure on H. Then, defining $\lambda_H(E)$ as $\lambda_H(E \cap H)$ for all $E \in \mathsf{B}$, we can regard λ_H as an element of $\mathscr{M}(G)$. It is obvious that $\lambda_H * \lambda_H = \lambda_H$, since $\lambda_H(Ex^{-1}) = \lambda_H(E)$ for Borel sets $E \subset H$ and $x \in H$. Furthermore, if $\chi_1, \chi_2, \ldots, \chi_m$ are [continuous] characters of G, no two of which are equal on H, a simple computation shows that the measure $(\chi_1 + \chi_2 + \ldots + \chi_m)\lambda_H$ is idempotent. [For a measure φ and a measurable function g, the measure $g\varphi$ is defined by $g\varphi(E) = \int_E g(x)\,d\varphi(x)$.] It is also obvious that if μ_1 and μ_2 are idempotent measures, then $\mu_1 * \mu_2$, $\varepsilon_e - \mu_1$, and $\mu_1 + \mu_2 - \mu_1 * \mu_2$ are also idempotent measures. Cohen's theorem asserts that *every* idempotent measure in $\mathscr{M}(G)$ can be obtained from idempotents of the form $(\chi_1 + \chi_2 + \ldots + \chi_m)\lambda_H$ by iterating these three operations. For the proof, which is long, see Cohen, *loc. cit.*

The functions that "operate" on $\mathscr{M}(G)$ have also recently been identified, by H. Helson, J.-P. Kahane, Y. Katznelson, and W. Rudin [6]. Again, let G be Abelian and this time infinite. [The theorem to be stated is meaningless for non-Abelian G and false for finite Abelian G.] Consider all of the measures μ in $\mathscr{M}(G)$ such that the Fourier-Stieltjes transform

$\hat{\mu}(\chi) = \int_G \chi(x)\,d\mu(x)$ is real and of absolute value $\leqslant 1$ for all characters χ of G. Suppose that F is a complex-valued function defined on the closed interval $[1, 1]$ and that $F(0) = 0$. Suppose that the composite function $F\circ\hat{\mu}$ has the form $\hat{\lambda}$ for some $\lambda \in \mathscr{M}(G)$ for all μ of the sort just described. Then F is said to *operate on* $\mathscr{M}(G)$. The theorem of Helson, Kahane, Katznelson, and Rudin asserts that every F operating on $\mathscr{M}(G)$ can be extended throughout the complex plane so as to be an entire function. Thus $\mathscr{M}(G)$ is grossly different from an algebra $\mathfrak{C}(X)$, in which F can be *any* continuous function. For the proof, see Helson, Kahane, Katznelson, and Rudin, *loc. cit.*

It has been known since 1950, and was shown by Yu. A. Šreĭder, that $\mathscr{M}(R)$ is asymmetric in the following sense. Given any locally compact group G, $\mathscr{M}(G)$ admits an involution $\mu \to \tilde{\mu}$, where $\tilde{\mu}(E) = \overline{\mu(E^{-1})}$ for all $E \in \mathsf{B}$. This involution satisfies the usual axioms: $(\lambda+\mu)^{\sim} = \tilde{\lambda}+\tilde{\mu}$; $(a\mu)^{\sim} = \bar{a}\tilde{\mu}$ for complex numbers a; $(\lambda*\mu)^{\sim} = \tilde{\mu}*\tilde{\lambda}$; $(\mu)^{\sim\sim} = \mu$. We also have $\int_G \chi(x)\,d\tilde{\mu}(x) = \overline{\int_G \chi(x)\,d\mu(x)}$ for all characters of G. Thus, if G is Abelian and $\mathscr{M}(G)$ admits any involution $^+$ such that $\tau(\mu^+) = \overline{\tau(\mu)}$ for all multiplicative linear functionals τ on $\mathscr{M}(G)$, the involution $^+$ must be $^{\sim}$. This is apparent from the fact that if $\int_G \chi(x)\,d\lambda(x) = \int_G \chi(x)\,d\mu(x)$ for all characters χ of G, then $\lambda = \mu$. Šreĭder [24] constructed a multiplicative linear functional τ on $\mathscr{M}(R)$ and a measure σ in $\mathscr{M}(R)$ such that $\tau(\sigma) = 1$ and $\tau(\tilde{\sigma}) = 0$. This proved of course that $\mathscr{M}(R)$ is asymmetric, and also established the validity of the assertion of Wiener and Pitt referred to above: the measure $\gamma = \sigma - \tilde{\sigma} - \varepsilon_0$ has the property that $|\int_{-\infty}^{\infty} \exp(itx)\,d\gamma(x)| \geqslant 1$ for all real t, but γ^{-1} does not exist. Šreĭder's construction was extended by Hewitt [9] to all locally compact Abelian groups in which every neighborhood of e contains elements of infinite order. J. H. Williamson [28] has given a quite different construction, based according to Williamson [oral communication] on the original construction of Wiener and Pitt, which proves the asymmetry of $\mathscr{M}(G)$ for all nondiscrete locally compact Abelian groups G. A particularly simple proof is found in Rudin [21].

Constructions of the sort described in the preceding paragraph have been carried considerably further in a paper of E. Hewitt and S. Kakutani [10]. This paper shows that "most" linear functionals on certain infinite-dimensional linear subspaces of $\mathscr{M}(G)$ [if G is nondiscrete] are in fact multiplicative: they can be extended over all of $\mathscr{M}(G)$ so as to be multiplicative linear functionals on $\mathscr{M}(G)$. These results are thus consonant with the oldest tradition of functional analysis, namely, the computation and detailed study of the linear functionals on specific normed linear spaces. One may with propriety therefore present these results in a conference dedicated to the memory of Stefan Banach, and express a hope that Banach himself would have found them of interest.

We proceed to a precise statement of the results of [10]. Let G be a nondiscrete locally compact Abelian group. A subset A of G is called *independent* if, whenever $x_1, x_2, \ldots, x_n$ are distinct elements of A and $q_1, q_2, \ldots, q_n$ are integers, the equality $x_1^{q_1}x_2^{q_2}\ldots x_n^{q_n} = e$ implies that $q_1 = q_2 = \ldots = q_n = 0$. Let a be an integer >1. A subset A of G is said to be *a-independent* if all elements of A have order a and if, whenever $x_1, x_2, \ldots, x_n$ are distinct elements of A and $q_1, q_2, \ldots, q_n$ are integers, the equality $x_1^{q_1}x_2^{q_2}\ldots x_n^{q_n} = e$ implies that $q_1 \equiv q_2 \equiv \ldots \equiv q_n \equiv 0 \pmod{a}$.

For an arbitrary closed subset F of G, let $\mathscr{M}(F)$ denote the set of all $\lambda \in \mathscr{M}(G)$ such that $|\lambda|(F') = 0$; these are the measures all of whose mass is confined to the set F. Let $\mathscr{M}_c(F)$ denote the continuous measures in $\mathscr{M}(F)$ and $\mathscr{M}_d(F)$ the purely discontinuous measures in $\mathscr{M}(F)$. It is easy to see that $\mathscr{M}(F)$ is the direct sum of $\mathscr{M}_c(F)$ and $\mathscr{M}_d(F)$, and that all three of these sets are closed linear subspaces of $\mathscr{M}(G)$.

Given a closed subset F of G, and a linear functional L not identically 0 defined on $\mathscr{M}(F)$, one may ask for conditions under which there is a *multiplicative* linear functional M defined on all of $\mathscr{M}(G)$ that is an extension of L. Plainly L must be bounded and of norm $\leqslant 1$. Also L must behave properly on the measures ε_x for $x \in F$. Specifically, if $x_1, x_2, \ldots, x_n$ are elements of F, not necessarily distinct, and $q_1, q_2, \ldots, q_n$ are integers such that $x_1^{q_1}x_2^{q_2}\ldots x_n^{q_n} = e$, then $L(\varepsilon_{x_1}), L(\varepsilon_{x_2}), \ldots, L(\varepsilon_{x_n})$ are defined because $\varepsilon_{x_1}, \varepsilon_{x_2}, \ldots, \varepsilon_{x_n}$ are measures in $\mathscr{M}(F)$. If L is to have a multiplicative linear extension M defined on the entire algebra $\mathscr{M}(G)$, we must have $1 = M(\varepsilon_e) = M(\varepsilon_{x_1}^{q_1}\varepsilon_{x_2}^{q_2}\ldots\varepsilon_{x_n}^{q_n}) = M(\varepsilon_{x_1})^{q_1}M(\varepsilon_{x_2})^{q_2}\ldots M(\varepsilon_{x_n})^{q_n} = L(\varepsilon_{x_1})^{q_1}L(\varepsilon_{x_2})^{q_2}\ldots L(\varepsilon_{x_n})^{q_n}$. Now let P be a closed subset of G that is independent or a-independent for some integer $a>1$, and let $F = P \cup P^{-1}$. For this choice of F, we find that the simple necessary conditions for multiplicative extensibility are also sufficient. The exact statement follows:

THEOREM A. *Let G be a nondiscrete, locally compact, Abelian group and let P be any closed subset of G that is either independent or a-ind pendent for some integer $a>1$. Let L be any linear functional of norm $\leqslant 1$ on the linear space $\mathscr{M}(P \cup P^{-1})$ such that if $x_1, x_2, \ldots, x_n$ are elements of P [not necessarily distinct], $q_1, q_2, \ldots, q_n$ are integers, and $x_1^{q_1}x_2^{q_2}\ldots x_n^{q_n} = e$, then*

$$L(\varepsilon_{x_1})^{q_1}L(\varepsilon_{x_2})^{q_2}\ldots L(\varepsilon_{x_n})^{q_n} = 1.$$

Then there is a multiplicative linear functional M on $\mathscr{M}(G)$ such that $\mathscr{M}(\lambda) = L(\lambda)$ for all $\lambda \in \mathscr{M}(P \cup P^{-1})$.

Theorem A would of course be vacuous if there were no closed independent or a-independent sets, and trivial if there were no independent or a-independent sets with plenty of measures on them. It turns out that both of these requirements can be met abundantly. If every neighborhood of e in G contains an element of infinite order, then every nonvoid open subset of G contains an independent set P that is homeomorphic with Cantor's ternary set. Thus $\mathscr{M}(P \cup P^{-1})$ is an infinite-dimensional space, as is $\mathscr{M}_c(P \cup P^{-1})$. If some neighborhood of e in G contains only elements of finite order, then a slight technicality must be surmounted. In this case, there is at least one integer $a > 1$ such that every neighborhood of e contains an a-independent set P homeomorphic with Cantor's ternary set. Other open sets may not contain such sets P. However, every nonvoid open subset of G contains a translate of such a set P, for which the conclusion of Theorem A holds.

The proof of Theorem A is simple in principle. It consists in showing that the set $\{\mu - L(\mu)\varepsilon_e : \mu \in \mathscr{M}(P \cup P^{-1})\}$ is contained in some proper ideal $\mathscr{I}$ of $\mathscr{M}(G)$. Then $\mathscr{I}$ can be extended to a maximal ideal $\mathscr{J}$ of $\mathscr{M}(G)$, which by one of Gel'fand's fundamental theorems is the kernel of a multiplicative linear functional M. Thus $\mu - L(\mu)\varepsilon_e \in \mathscr{J}$ for all $\mu \in \mathscr{M}(P \cup P^{-1})$, and hence $M(\mu) = L(\mu)M(\varepsilon_e) = L(\mu)$ for all $\mu \in \mathscr{M}(P \cup P^{-1})$. Thus we must show that the equality $\sum_{l=1}^{m}(\mu_l - L(\mu_l)\varepsilon_e) * \lambda_l = \varepsilon_e$ can hold for no $\mu_1, \mu_2, \ldots, \mu_m \in \mathscr{M}(P \cup P^{-1})$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathscr{M}(G)$. The details of the proof are somewhat complicated, and for them we refer the reader to [10].

Theorem A has a number of consequences. For example, let L be any linear functional on $\mathscr{M}_c(P \cup P^{-1})$ of norm not exceeding 1. Then there is a multiplicative linear functional M on $\mathscr{M}(G)$ that is an extension of L. To see this, take any character ψ of G, continuous or discontinuous. For $\lambda = \lambda_c + \sum_{l=1}^{\infty} \alpha_l \varepsilon_{x_l} \in \mathscr{M}(P \cup P^{-1})$, let $L_1(\lambda) = L(\lambda_c) + \sum_{l=1}^{\infty} \alpha_l \psi(x_l)$.

Then apply Theorem A to the functional L_1. If P is homeomorphic with Cantor's ternary set, the conjugate space of $\mathscr{M}_c(P \cup P^{-1})$ is a very complicated Banach space [it is the direct sum of $2^{\aleph_0}$ L_1 spaces], so that the linear functionals L on $\mathscr{M}_c(P \cup P^{-1})$ are very numerous and complicated. As a simple case, however, consider any complex-valued Borel measurable function f defined on $P \cup P^{-1}$, of absolute value $\leqslant 1$ everywhere. Then the mapping $\lambda \to \int_{P \cup P^{-1}} f(x)\,d\lambda(x)$ $[\lambda \in \mathscr{M}_c(P \cup P^{-1})]$ is a suitable L. Taking $f(x) = i$ for all $x \in P \cup P^{-1}$, we construct a multiplicative linear functional M on $\mathscr{M}(G)$ such that $M(\lambda) = \overline{M(\lambda)}$ for $\lambda \in \mathscr{M}_c(P \cup P^{-1})$ if and only if $\lambda(G) = 0$. This is asymmetry of $\mathscr{M}(G)$ in an extreme form!

By taking even more special sets than perfect independent or a-independent sets, we can obtain a result stronger than Theorem A. Let S denote the compact Hausdorff space consisting of all multiplicative linear [nonzero] functionals on $\mathscr{M}(G)$, topologized with Gel'fand's topology. The dual group X of G forms an open subspace of S, if we associate with each [continuous] character χ of G the multiplicative linear functional $\lambda \to \int_G \chi(x)\,d\lambda(x)$. The closure X^- of X in S is a proper subset of S, in view of the asymmetry of $\mathscr{M}(G)$. By choosing perfect subsets Q of G carefully enough, we can extend "most" linear functionals L on $\mathscr{M}(Q)$ to multiplicative linear functionals M on $\mathscr{M}(G)$ such that $M \in X^-$.

To describe the needed restrictions on L, we introduce two sets of complex numbers for every nondiscrete locally compact Abelian group G. If every neighborhood of e in G contains an element of infinite order, let Γ_0 be the set of all complex numbers z such that $|z| = 1$ and let Γ_1 be the set of all complex numbers z such that $|z| \leqslant 1$. If there is a neighborhood of e in G containing only elements of finite order, then it can be proved that there is at least one integer $a > 1$ such that every neighborhood of e in G contains a compact subgroup isomorphic with the direct product of a countably infinite number of cyclic groups of order a. Select any such a, let Γ_0 be all of the a-th roots of unity, and let Γ_1 be the convex hull in the complex plane of Γ_0. Our second extension theorem can now be stated.

THEOREM B. *Let Q be any subset of G homeomorphic with Cantor's ternary set and having the property that every continuous function defined on Q with values in Γ_0 is arbitrarily uniformly approximable by continuous characters of G. Let L be any linear functional on $\mathscr{M}(Q)$ such that $L(\lambda) \in \Gamma_1$ if $\lambda \in \mathscr{M}(Q)$, $\lambda \geqslant 0$, and $\lambda(G) \leqslant 1$, and such that $L(\varepsilon_x) \in \Gamma_0$ if $x \in Q$. Then there is a multiplicative linear functional $M \in X^-$ such that $M(\lambda) = L(\lambda)$ for all $\lambda \in \mathscr{M}(Q)$.*

Theorem B is nonvacuous: every nonvoid open subset of G contains a set Q with the properties specified in Theorem B. Note the curious fact that if Γ_0 is finite, then every continuous function on Q with values in Γ_0 is the restriction to Q of a continuous character on G.

Theorem B is an analogue of an interesting result of Yu. A. Šreĭder [23]. In $\mathscr{M}(R)$, let $\mathscr{A}_\sigma$ denote the linear subspace of all measures absolutely continuous with respect to Lebesgue's singular measure σ on the Cantor set. Then there is a multiplicative linear functional N on $\mathscr{M}(R)$ such that M is in the closure of the Fourier-Stieltjes transforms $\mu \to \int_{-\infty}^{\infty} \exp(itx)\,d\mu(x)$ and $M(\lambda) = \gamma \cdot \lambda(R)$ for all $\lambda \in \mathscr{A}_\sigma$, where γ is a complex number such that $0 < |\gamma| < 1$.

Theorems A and B at once raise a question. What commutative Banach algebras contain nontrivial closed linear subspaces having the property that all linear functionals [satisfying perhaps some simple necessary conditions] on these subspaces can be extended to be multiplicative linear functionals on the whole algebra? This question has not been systematically explored. We give one example in an algebra very different from $\mathscr{M}(G)$. Let G be as above any nondiscrete locally compact Abelian group, and let X be its dual group, topologized as usual so as to be a locally compact Hausdorff space. Let $\mathfrak{C}(X)$ denote the algebra of all bounded, continuous, complex-valued functions on X. The functions $\hat{\mu}$, where $\mu \in \mathscr{M}(G)$ and $\hat{\mu}(\chi) = \int_G \chi(x)\,d\lambda(x)$ for $\chi \in X$, are of course in $\mathfrak{C}(X)$. For simplicity, we suppose that every neighborhood of e in G contains an element of infinite order. Let Q be any subset of G as specified in Theorem B, and let $\mathfrak{S}$ be the uniformly closed subspace of $\mathfrak{C}(X)$ generated by all of the functions $\hat{\mu}$ for $\mu \in \mathscr{M}_c(Q)$. Let L be any linear functional on $\mathfrak{S}$ of norm not exceeding 1 in the uniform norm for $\mathfrak{S} \subset \mathfrak{C}(X)$. Thus we have $|L(\mu)| \leqslant \sup\{|\hat{\mu}(\chi)| : \chi \in X\} \leqslant \|\mu\|$ for all $\mu \in \mathscr{M}_c(Q)$. By defining $L(\varepsilon_x)$ to be any complex number of absolute value 1 for each $x \in Q$, we extend L in an obvious way over $M(Q)$. It is easy to see that L satisfies the hypotheses of Theorem B. Thus there is a point M in $X^- \subset \mathfrak{S}$ such that $M(\mu) = L(\hat{\mu})$ for all $\mu \in \mathscr{M}_c(Q)$. Let βX be the Stone-Čech compactification of X [note that X is a normal space]. The identity mapping ι of X onto itself admits a continuous extension ι_0 that maps βX onto X^-: this is a characteristic property of βX. Let $p \in \beta X$ be any point in $\iota_0^{-1}(M)$. Then the evaluation $f(p)$ is a multiplicative linear functional on $\mathfrak{C}(X)$ that agrees with L on the functions $\hat{\mu}$ $(\mu \in \mathscr{M}_c(Q))$ and hence agrees with L on $\mathfrak{S}$, since $\hat{\mathscr{M}}_c(Q)$ is dense in $\mathfrak{S}$. Using X with its discrete topology, denoted by X_d, we can similarly extend L to be a multiplicative linear functional on the algebra $\mathfrak{C}(X_d)$, consisting of all bounded complex-valued functions on X.

The general problem of finding the closed linear subspaces of commutative Banach algebras on which linear functionals of norm $\leqslant 1$ are actually multiplicative seems to be completely open, even for the algebra of all continuous complex-valued functions on a compact Hausdorff space. A triviality may be mentioned. If Y is a compact Hausdorff space of cardinal less than $2^{\aleph_0}$, then $\mathfrak{C}(Y)$ contains no closed linear subspace different from $\{0\}$ with this property.

To conclude this essay, we mention some important open questions in the theory of $\mathscr{M}(G)$.

1. Is there a concrete representation, more specific than Šreĭder's generalized characters [24], for *all* multiplicative linear functionals on $\mathscr{M}(R)$? A similar query applies to $\mathscr{M}(G)$ for arbitrary locally compact Abelian G, but a concrete construction of the multiplicative linear functionals on $\mathscr{M}(R)$ would itself be of great interest. Theorems A and B show that the space of multiplicative linear functionals on $\mathscr{M}(R)$ is extremely complicated.

2. For locally compact Abelian or non-Abelian G, what are the closed ideals of $\mathscr{M}(G)$?

3. For locally compact non-Abelian G, what are the simple algebras that are homomorphic images of $\mathscr{M}(G)$?

4. For locally compact non-Abelian G, what are the idempotents of $\mathscr{M}(G)$?

Question 1 and questions 3 and 4 for *compact* groups probably can be answered with presently available techniques. Question 2 [even for Abelian groups] and questions 3 and 4 for noncompact groups appear to be very far from a solution. Such statements as the preceding sentence are of course dangerous: some ingenious person may any day prove the writer wrong. Such a dénouement would please him greatly.

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Linear differential equations in Banach algebras

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1. We denote by B a complex B-algebra with unit element e. Let ζ be a complex variable, and $f(\zeta)$ a function whose values are in B. Then

$$w'(\zeta) = f(\zeta)w(\zeta) \tag{1}$$

may be regarded as a generalization of the classical system of first order linear differential equations. It reduces to such a system if B is the algebra of n by n matrices.

If $f(\zeta)$ is holomorphic in a simply-connected domain Δ, and if w_0 is a given element of B, then there exists a unique solution $w(\zeta; \zeta_0, w_0)$ of (1) which is holomorphic in Δ and which reduces to w_0 when $\zeta \to \zeta_0$. This solution is a regular element of algebra if and only if w_0 has this property. We have

$$w(\zeta; \zeta_0, w_0) = w(\zeta; \zeta_0, e)w_0. \tag{2}$$

2. Suppose next that Δ is a sector with $\zeta = 0$ as vertex and that $f(\zeta)$ is not holomorphic at $\zeta = 0$. The behavior of the solution as $\zeta \to 0$ depends upon the integrability properties of $\|f(\zeta)\|$. If this function is integrable down to 0, the solutions have finite limits as $\zeta \to 0$ and the initial value problem may be set also at the singular point $\zeta = 0$.

If the integral diverges, several cases arise. Thus, if $\zeta = 0$ is a simple pole of $f(\zeta)$, then a priori estimates of the form

$$a^{-1}|\zeta|^a \leqslant \|w(\zeta)\| \leqslant a|\zeta|^{-a} \tag{3}$$

hold and we may speak of a regular singular point in analogy with the classical case. If the pole is of order k, $k > 1$, we have the analogue of an irregular singular point of rank $k-1$ and $\|w(\zeta)\|$ now lies between two exponential functions of $|\zeta|^{-1}$.

3. In the regular singular case, suppose that

$$f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^{n-1}, \quad c_n \in B, \ |\zeta| < \varrho. \tag{4}$$