Let $\|f\|_1$ be the norm in $\Lambda_{p_1,r}$ and $\|f\|_2$ the norm in $\Lambda_{p_2,r}$ where $1 \leqslant r \leqslant \infty$. Then, if $p_1 \neq p_2$ we have, up to equivalence,

$$\|f\|_{\Lambda_{p,r_1}} \leqslant N(\beta_1, \beta_2, s, f) \leqslant \|f\|_{\Lambda_{p,r_2}}$$

where $1/p = (1-s)/p_1 + s/p_2$, $r_1 = p\max(1/p_i\beta_i)$; $r_2 = p\min(1/p_i\beta_i)$. This combined with Theorem 3 gives the following result:

THEOREM 4. *Let $A$ be a linear operator on functions which is continuous from $\Lambda_{p_i,1}$ to $\Lambda_{q_i,\infty}$, $i = 1, 2$, $q_1 \neq q_2$, $p_1 \neq p_2$. Then $A$ maps $\Lambda_{p,r}$ continuously into $\Lambda_{q,P}$ where $1/p = (1-s)/p_1 + s/p_2$, $0 < s < 1$, $1/q = (1-s)/q_1 + s/q_2$* and $P > r$.

STUDIA MATHEMATICA; SERIA SPECJALNA, Z. I. (1963)

# The fundamental principle and some of its applications

by

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Let $R(C)$ denote real (complex) Euclidean space of dimension $n$ with coordinates $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_n)$. Let $W$ be a reflexive space of functions or distributions on $R$ such that differentation and translation are continuous on $W$; denote by $W'$ the dual of $W$. We assume that $W'$ is a convolution algebra and that the Fourier transform $\hat{W}'$ of $W'$ is a space of entire functions on $C$ and that the topology of $\hat{W}'$ can be described as follows:

There exists a family $A$ of continuous functions $a(z) > 0$ such that the sets

$$N_a = \{F \in W : |F(z)| \leqslant a(z)\}$$

form a fundamental system of neighborhoods of zero. In what follows we assume certain other "natural" conditions on $A$.

Let $D_1, \ldots, D_r$ denote partial differential operators on $R$; we want to find a description for $W(D_1, \ldots, D_r)$ which is the intersection of the kernels of $D_1, \ldots, D_r$ acting on $W$, that is, $W(D_1, \ldots, D_r)$ is the set of $f \in W$ for which $D_j f = 0$ for $j = 1, 2, \ldots, r$. Denote by $(D_1, \ldots, D_r)W'$ the ideal generated by the $D_j$ in $W'$. We can show that $(D_1, \ldots, D_r)W'$ is closed in $W'$. Thus, the dual of $W(D_1, \ldots, D_r)$ is $W'/(D_1, \ldots, D_r)W'$.

Denote by $P_j$ the Fourier transform of $D_j$ so $P_j$ is a polynomial on $C$; denote by $V$ the complex affine variety of common zeros of the $P_j$. The Fourier transform of $W'/(D_1, \ldots, D_r)W'$ is $\hat{W}'/(P_1, \ldots, P_r)W'$. The *fundamental principle* gives an analytic description of this quotient space; by means of this we shall obtain a complete description of $W(D_1, \ldots, D_r)$.

FUNDAMENTAL PRINCIPLE. *There exists a finite sequence of complex affine subvarieties $V_k$ (not necessarily distinct) of $V$. For each $k$ we can find a constant coefficient differential operator $\partial_k$ with the following properties:*

*The mapping*

$$F \to \textit{set of restrictions of } \partial_k F \textit{ to } V_k$$

*defines a topological isomorphism of* $\hat{W}'/(P_1, \ldots, P_r)\hat{W}'$ *onto the direct sum of the spaces* $\hat{W}'(V_k)$ *where* $\hat{W}'(V_k)$ *is the space of all entire functions* $G$ *on* $V_k$ *which satisfy*

$$G(z) = O(a(z)), \quad z \in V_k,\ a \in A,$$

*and the topology of* $\hat{W}'(V_k)$ *is defined by the sets*

$$N_a = \{G \in \hat{W}'(V_k) : |G(z)| \leqslant a(z) \text{ for all } z \in V_k\}.$$

From the fundamental principle it follows that each $f \in W(D_1, \ldots, D_r)$ can be represented as a sum of integrals of exponential polynomials in $W(D_1, \ldots, D_r)$. In addition, we can give a complete treatment of questions of hypoellipticity, hyperbolicity, uniqueness (a la Taeklind), existence for a Cauchy-like problem, for the space $W(D_1, \ldots, D_r)$.



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# Tensorial measures and homology on a compact differentiable manifold*

by

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Let $V^{(r)}$ be a differentiable manifold of the dimension $r$. The differential structure on $V^{(r)}$ be of class $C_L^1$ (this means that transformations with Lipschitz-continuous first derivatives change the local systems of admissible coordinates). Let $\varepsilon$ be an admissible map of $V^{(r)}$ ($r$-cell of $V^{(r)}$, with a local system of admissible coordinates on it). An object $\mu$ called *tensorial measure* is introduced. It is defined by 1°) a set of real valued measure functions $\mu_{j_1 \ldots j_k}^{i_1 \ldots i_n}(B)$ $(i_1, \ldots, i_n = 1, \ldots, r; j_1 \ldots j_k = 1, \ldots, r)$ (components of $\mu$ in $\varepsilon$) defined on the reduced $\sigma$-ring of all the Borel set contained with their closures in the support of $\varepsilon$; 2°) the following transformation rule for two maps $\varepsilon$ and $\bar{\varepsilon}$ with overlapping supports:

$$\mu_{j_1 \ldots j_k}^{i_1 \ldots i_n}(B) = \int\limits_B \frac{\bar{A}^m}{|\bar{A}|^{p+m}} a_{j_1}^{s_1} \ldots a_{j_k}^{s_k} \bar{a}_{h_1}^{i_1} \ldots \bar{a}_{h_n}^{i_n} d\bar{\mu}_{s_1 \ldots s_k}^{h_1 \ldots h_n}$$

($m = 0, 1$, $p =$ real number, $a_j^i = \partial \bar{x}^i / \partial x^j$, $\bar{a}_j^i = \partial x^i / \partial \bar{x}^j$, $A = \det\{a_j^i\}$, $\bar{A} = \det\{\bar{a}_j^i\}$, $x^i$ local coordinates in $\varepsilon$, $\bar{x}^i$ local coordinates in $\bar{\varepsilon}$). $n$ is the first rank of $\mu$, $k$ the second rank; $\mu$ is called of the *first* (*second*) *kind* if $m = 0$ ($m = 1$), $p$ is the weight of $\mu$. Every tensorial measure is uniquely decomposed as a sum of an absolutely continuous tensorial measure $\mu_0$ (every component of $\mu_0$ in $\varepsilon$ is absolutely continuous with respect to the measure $x(\tau B) =$ Lebesgue measure of the image of $B$ in the unit sphere of the Euclidean space by the homeomorphism $\tau$ that introduces on $\varepsilon$ the coordinate system) and a singular tensorial measure $\mathring{\mu}$ (every component of $\mathring{\mu}$ in $\varepsilon$ is singular with respect to $x(\tau B)$). The linear space $\mathfrak{M}_0$ of the abs. cont. tens. measure (for given $n, k, m$) is isomorphic to the space of tensors $f$ with locally integrable components (respect to $x$) of first rank $n$, second rank $k$, of the kind $m$, and weight $p+1$. This isomorphism is denoted by $f \leftrightarrow \int f \equiv \mu_0$.

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