Intermediate spaces and interpolation[*]

by

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E. Gagliardo and J. L. Lions and S. Krein have given methods to construct "intermediate" spaces between two Banach spaces. In the present note alternative approaches to the construction of intermediate spaces are given.

Let B be a complex Banach space in which two additional norms $\|x\|_1$ and $\|x\|_2$ are defined. A norm is a subadditive homogeneous function on B which vanishes only at zero. We assume that $\|x\|_i \leqslant \|x\|$, $i = 1, 2$, and $\|x\|_1 + \|x\|_2 \geqslant C\|x\|$, $c > 0$, where $\|x\|$ is the norm of x as an element of B. Then there exists a norm $\|x\|_0$ which also only vanishes at zero and such that $\|x\|_0 \leqslant \|x\|_i$, $i = 1, 2$.

1. Consider now functions $F(z)$ with values in B which are continuous and bounded in the strip $0 \leqslant R(z) \leqslant 1$, $z = s + it$ and analytic in $0 < R(z) < 1$. Define in this class $\mathfrak{F}$ of functions the norm $\|F\| = \max\left(\sup_t \|F(it)\|_1\right), \left(\sup_t \|F(1+it)\|_2\right)$ and given s, $0 \leqslant s \leqslant 1$, and $x \in B$, define

$$\|x\|_{1+s} = \inf\left(\lim_{n \to \infty} \|F_n\|\right)$$

where the infimum on the right is taken over all Cauchy sequences $F_n(z)$, i. e. such that $\lim_{n,m \to \infty} \|F_n - F_m\| = 0$, with the property that $\lim \|F_n(s) - x\|_0 = 0$. The quantity $\|x\|_{1+s}$ is a norm. Further if B is a Banach algebra and the norms $\|x\|_i$ are submultiplicative, the same is true of $\|x\|_{1+s}$.

The norms $\|x\|_{1+s}$, $0 \leqslant s \leqslant 1$, are compatible with $\|x\|_0$ that is if a sequence of elements $x_n \in B$ is Cauchy with respect to both norms $\|\ \|_{1+s}$ and $\|\ \|_0$ and tends to zero with respect to $\|\cdot\|_0$, it also tends to zero with respect to $\|\ \|_{1+s}$. As a consequence of this the completions B_{1+s} of B

with respect to $\| \ \|_{1+s}$ can be simultaneously isometrically imbedded in the completion B_0 of B with respect to $\| \ \|_0$.

THEOREM 1. *Let $F(z)$ be a function with values in B_0 which is continuous and bounded with respect to $\| \ \|_0$ in $0 \leqslant R(z) \leqslant 1$ and analytic in $0 < R(z) < 1$. Suppose that $F(it) \epsilon B$, and is continuous and bounded with respect to the norm $\| \ \|_1$ and $F(1+it) \epsilon B_2$ and is continuous and bounded with respect to the norm $\| \ \|_2$. Then for $0 \leqslant s \leqslant 1$, $F(s+it) \epsilon B_{1+s}$ and*

$$\|F(s+it)\|_{1+s} \leqslant [\sup_t \|F(it)\|_1]^{1-s} [\sup_t \|F(1+it)\|_2]^s.$$

THEOREM 2. *Let $\bar{B}$ be a second space with two additional norms $\|x\|_i$. Let $\|x\|_{1+s}$, $\bar{B}_0$, $\bar{B}_{1+s}$ be defined as in the case of the space B. Let A_z be a function defined on $0 \leqslant R(z) \leqslant 1$, $z = s+it$ whose values are bounded operators from B_0 to $\bar{B}_0$ and which is continuous and bounded in the operator norm in $0 \leqslant R(z) \leqslant 1$ and analytic in $0 < R(z) < 1$. Then if A_{it} maps B_1 into $\bar{B}_1$, A_{1+it} maps B_2 into $\bar{B}_2$ and*

$$\|A_{it}x\|_1 \leqslant M_1\|x\|_1, \qquad \|A_{1+it}x\|_2 \leqslant M_2\|x\|_2,$$

then A_{s+it} maps B_{1+s} into $\bar{B}_{1+s}$ and

$$\|A_{s+it}x\|_{1+s} \leqslant M_1^{1-s}M_2^s\|x\|_{1+s}.$$

This theorem can be interpreted as an abstract version of the extension of the theorem of M. Riesz given by E. Stein.

Examples. By suitable choice of B and the norms $\|x\|_i$ one can obtain various classes of "intermediate" spaces.

a) Let L_Φ be the Orlicz space associated with a convex strictly increasing function $\Phi(u)$. We assume that $\Phi(u)/u$ tends to zero and infinity as u tends to zero and infinity respectively. Let L_Φ^0 be the closure in L_Φ of the class of simple functions. Then if $B_1 = L_{\Phi_1}^0$ and $B_2 = L_{\Phi_2}^0$, the intermediate space B_{1+s} coincides with L_Φ^0 where Φ is defined by

$$\Phi(u^{1-s}v^s) = \Phi_1(u) = \Phi_2(v).$$

b) Let L_a^p be the class of Bessel potentials of order a of functions in $L^p(E_n)$ (for $a = k$, a positive integer, L_a^p coincides with the space $W_p^{(k)}$ of Sobolev). Then if $B_1 = L_{a_1}^{p_1}$ and $B_2 = L_{a_2}^{p_2}$, $1 < p_i < \infty$, $-\infty < a_i < \infty$, then B_{1+s} coincides with L_a^p where $1/p = 1/p_1(1-s)+1/p_2 s$ and $a = (1-s)a_1+sa_2$.

c) Let $\mathrm{lip}\,a$, $0 < a < 1$, be the class of bounded functions in E_n, which satisfy a Lipschitz condition of order a. Then if $B_1 = \mathrm{lip}\,a_1$ and $B_2 = \mathrm{lip}\,a_2$ then $B_{1+s} = \mathrm{lip}\,[(1-s)a_1+sa_2]$.

2. By introducing other norms in the space $\mathfrak{F}$ of analytic functions with values in B, it is possible to obtain a greater variety of intermediate spaces. We mention only one example. Consider the functions $F(z)$ which have a representation of the form

$$F(z) = \int_{-\infty}^{+\infty} \varphi(u)e^{zu}du, \qquad 0 \leqslant R(z) \leqslant 1,$$

where $\varphi(u)$ is a continuous bounded function with values in B such that $\int_{-\infty}^{+\infty} \|\varphi(u)\| e^{su}du$ is finite for $0 \leqslant s \leqslant 1$. Given β_1, β_2, $0 \leqslant \beta_1 \leqslant 1$, $0 \leqslant \beta_2 \leqslant 1$, we define

$$N(F) = N(\beta_1, \beta_2, F) = \max\left[\int_{-\infty}^{+\infty} \|\varphi(u)\|_1^{1/\beta_1}\,du\right]^{\beta_1}; \quad \left[\int_{-\infty}^{+\infty} \|\varphi(u)e^u\|_2^{1/\beta_2}du\right]^{\beta_2}$$

and

$$N(\beta_1, \beta_2, s, x) = \inf[\lim_{n\to\infty} N(F_n)]$$

where the infimum is taken over all sequences F_n such that $N(F_n - F_m) \to 0$ and $\|F_n(s) - x\|_0 \to 0$.

The intermediate norms just defined have the property of decreasing (up to equivalence) when β_1 or β_2 decrease. This has as a consequence the following

THEOREM 3. *Let $\bar{B}$ be a Banach space with two additional norms $\|x\|_i$, $i = 1, 2$, and A a bounded operator from B into $\bar{B}$ such that*

$$\|Ax\|_i \leqslant M_i\|x\|_i, \qquad i = 1, 2.$$

Then, if $\bar{\beta}_i \leqslant \beta_i$,

$$N(\bar{\beta}_1, \bar{\beta}_2, s, Ax) \leqslant CM_1^{1-s}M_2^sN(\beta_1, \beta_2, s, x)$$

where C depends on $\beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2$ and s.

This theorem can be interpreted as an abstract formulation and an improvement of the theorem on interpolation of Marcinkiewicz. For with suitable choice of the norms $\|x\|_i$ one obtains a sharpened version of that theorem and an extension of it to upper half of the square in the diagram of exponents. Let $\Lambda_{p,r}$ be the Lorentz space of functions f whose non-increasing rearrangement $f^*(u)$ in $0 < u < \infty$ has the property that

$$\left[\int_0^\infty [f^*(u)u^{1/p}]^r \frac{du}{u}\right]^{1/r} < \infty.$$

Let $\|f\|_1$ be the norm in $\varLambda_{p_1,r}$ and $\|f\|_2$ the norm in $\varLambda_{p_2,r}$ where $1 \leqslant r \leqslant \infty$. Then, if $p_1 \neq p_2$ we have, up to equivalence,

$$\|f\|_{\varLambda_{p,r_1}} \leqslant N(\beta_1, \beta_2, s, f) \leqslant \|f\|_{\varLambda_{p,r_2}}$$

where $1/p = (1-s)/p_1 + s/p_2$, $r_1 = p\max(1/p_i\beta_i)$; $r_2 = p\min(1/p_i\beta_i)$. This combined with Theorem 3 gives the following result:

THEOREM 4. *Let A be a linear operator on functions which is continuous from $\varLambda_{p_i,1}$ to $\varLambda_{q_i,\infty}$, $i = 1, 2$, $q_1 \neq q_2$, $p_1 \neq p_2$. Then A maps $\varLambda_{p,r}$ continuously into $\varLambda_{q,P}$ where $1/p = (1-s)/p_1 + s/p_2$, $0 < s < 1$, $1/q = (1-s)/q_1 + s/q_2$ and $P > r$.*

The fundamental principle and some of its applications

by

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Let $R(C)$ denote real (complex) Euclidean space of dimension n with coordinates $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_n)$. Let W be a reflexive space of functions or distributions on R such that differentation and translation are continuous on W; denote by W' the dual of W. We assume that W' is a convolution algebra and that the Fourier transform $\hat{W}'$ of W' is a space of entire functions on C and that the topology of $\hat{W}'$ can be described as follows:

There exists a family A of continuous functions $a(z) > 0$ such that the sets

$$N_a = \{F \epsilon W : |F(z)| \leqslant a(z)\}$$

form a fundamental system of neighborhoods of zero. In what follows we assume certain other "natural" conditions on A.

Let $D_1, \ldots, D_r$ denote partial differential operators on R; we want to find a description for $W(D_1, \ldots, D_r)$ which is the intersection of the kernels of $D_1, \ldots, D_r$ acting on W, that is, $W(D_1, \ldots, D_r)$ is the set of $f \epsilon W$ for which $D_j f = 0$ for $j = 1, 2, \ldots, r$. Denote by $(D_1, \ldots, D_r)W'$ the ideal generated by the D_j in W'. We can show that $(D_1, \ldots, D_r)W'$ is closed in W'. Thus, the dual of $W(D_1, \ldots, D_r)$ is $W'/(D_1, \ldots, D_r)W'$.

Denote by P_j the Fourier transform of D_j so P_j is a polynomial on C; denote by V the complex affine variety of common zeros of the P_j. The Fourier transform of $W'/(D_1, \ldots, D_r)W'$ is $\hat{W}'/(P_1, \ldots, P_r)\hat{W}'$. The *fundamental principle* gives an analytic description of this quotient space; by means of this we shall obtain a complete description of $W(D_1, \ldots, D_r)$.

FUNDAMENTAL PRINCIPLE. *There exists a finite sequence of complex affine subvarieties V_k (not necessarily distinct) of V. For each k we can find a constant coefficient differential operator ∂_k with the following properties:*
The mapping

$$F \to \text{set of restrictions of } \partial_k F \text{ to } V_k$$