onto the whole of X so as to remain γ-linear. Some sufficient conditions for extensibility are known.

THEOREM 13. *Let $\langle X_0, \|\ \|, \|\ \|^*\rangle$ be a γ-reflexive subspace of a γ-normal space $\langle X, \|\ \|, \|\ \|^*\rangle$. Then every γ-linear functional on X_0 possesses a γ-linear extension on X.*

A universal space. Two two-norm spaces, $\langle X, \|\ \|, \|\ \|^*\rangle$ and $\langle Y, \|\ \|, \|\ \|^*\rangle$ are called *γ-equivalent* if there exists a distributive operation T from X onto Y which establishes an isometry of $\langle X, \|\ \|\rangle$ and $\langle Y, \|\ \|\rangle$ and, at the same time, T is a homomorphism between $\langle X, \|\ \|^*\rangle$ and $\langle Y, \|\ \|^*\rangle$.

Let us consider the following example: suppose we are given a linear space Z with a sequence $[\ ]_i$ of seminorms such that $[x]_i = 0$ for $i = 1, 2, \ldots$ implies $x = 0$. Let $Z_\pi = \{x: \sup [x]_i < \infty\}$, $\|x\| = \sup_i [x]_i$, $\|x\|^* = \sum_{i=1}^{\infty} 2^{-i}[x]_i$ for $x \in Z_\pi$. Then $\langle Z_\pi, \|\ \|, \|\ \|^*\rangle$ is a γ-normal space.

In particular, let C denote the space of continuous functions $x = x(t)$ on the half-line $0 \leqslant t < \infty$ with $[x]_i = \sup\{|x(t)|: 0 \leqslant t \leqslant i\}$. Then γ-convergence in the space $\langle C_\pi, \|\ \|, \|\ \|^*\rangle$ means uniform boundedness plus uniform convergence on compact subsets of $[0, \infty)$.

THEOREM 14. *Every γ-normal two-norm space is γ-equivalent to a subspace of a certain space $\langle Z_\pi, \|\ \|, \|\ \|^*\rangle$.*

The space $\langle X, \|\ \|, \|\ \|^*\rangle$ is called *γ-separable* if there exists a countable set dense for the convergence γ.

THEOREM 15. *Every γ-separable space is γ-equivalent to a subspace of the space $\langle C_\pi, \|\ \|, \|\ \|^*\rangle$.*

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The group of invertible elements of a commutative Banach algebra

by

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Let f be continuous, complex-valued on a compact subset D of the complex plane C. Then f has the form $f = ae^g$ where a is *rational*, and g continuous on D. This classical theorem we generalize in a Banach algebra manner (see 1, below). Reformulated as in 4 (below) it represents another step along the path begun by Shilov [3] of finding algebraic invariants of a commutative Banach algebra A (over C, with unit) depending only on the space Δ of complex linear-algebra homomorphisms. In a sense, Shilov shows that the cohomology group $H^0(\Delta, Z)$ is isomorphic to the subring of A generated by its idempotents; and we show that $H^1(\Delta, Z)$ is isomorphic to G/G_0 (see 4).

Notations: C, A, Δ have always the meaning as above. $C^* = C - \{0\}$. $\mathscr{C}(X, Y)$ is the space of continuous functions. If $F \subset \mathscr{C}(X, C)$ then $\operatorname{ex} F = \{\exp(2\pi i f): f \in F\}$. If $W \subset C^n$ then $\mathrm{Hol}(W, Y)$ are the holomorphic Y-valued functions on W, $Y = C$ or C^*. $\{f \neq 0\}$ is the set where $f \neq 0$. For $b \in A$ and $\delta \in \Delta$, $b_\Delta(\delta) = \delta(b)$.

1. LEMMA. *Let $f \in \mathscr{C}(\Delta, C^*)$. Then there exists an $a \in A$, and a $g \in \mathscr{C}(\Delta, C)$ such that $f = a_\Delta e^g$. If $f = b_\Delta e^h$ is another such representation, then $b = ae^c$ for some $c \in A$.*

We shall deduce this from the following mere combination of two theorems of H. Cartan's. For our notation we refer closely to [2].

2. PROPOSITION. *Let $P_1, \ldots, P_N$ be polynomials in n complex variables, and form*

$$W = \{|P_1| < 1, \ldots, |P_N| < 1\}.$$

Then there is a natural isomorphism of the multiplicative groups.

3. $\mathscr{C}(W, C^*)/\operatorname{ex}\mathscr{C}(W, C) \leftrightarrow \mathrm{Hol}(W, C^*)/\operatorname{ex}\mathrm{Hol}(W, C)$.

We sketch the proof. For the Stein manifold W we have the exact sequence of sheaves ([2], 27(11)) $0 \to Z \to C_\omega \overset{\mathrm{ex}}{\to} C_\omega^* \to 0$, and the exact sequence ([2], 35 (2.10.1)), of chronology groups of W,

$$0 \to H^0(Z) \to H^0(C_\omega) \to H^0(C^*_\omega) \to H^1(Z) \to H^1(C_\omega) \to \ldots$$

By Cartan's theorem ([2], 119), the group written last is 0, so that $H'(Z) = H^0(C^*_\omega)/\mathrm{ex}\, H^0(C_\omega)$, and this quotient-group is ([2], 24, 29 (2.62)) the one on the right side of 3.

But we also have ([2], 26 (9)) $0 \to Z \to C^*_c \to C^*_c \to 0$, and so by analogous reasoning, using [2], 37 (2.11.1) (noting that C_c is fine) we obtain Brushlinsky's theorem (generalized): $H^1(Z) = \mathscr{C}(W, C^*)/\mathrm{ex}\,\mathscr{C}(W, C)$. A carefull tracing of the isomorphism (3) shows that for $\varphi \in \mathscr{C}(W, C^*)$ there is an $\alpha \in \mathrm{Hol}(W, C)$ and a $\psi \in \mathscr{C}(W, C)$ such that $\varphi = \alpha e \psi$.

Proof of 1. By Stone-Weierstrass, $f = f_1 e^g$ where $f_1 = a_{1\varDelta}\bar{a}_{2\varDelta} + \ldots + a_{k-1,\varDelta}\bar{a}_{k,\varDelta}$, and $g_1 \in \mathscr{C}(\varDelta, C)$. Define $\mu_k(\zeta) = (\zeta(a_1), \ldots, \zeta(a_k))$. Then $\mu_k(\varDelta) = \sigma(a_1, \ldots, a_r; A) \subset C^k$ is the *joint spectrum* of these elements relative to A. Evidently $x = z_1\bar{z}_2 + \ldots + z_{k-1}\bar{z}_k$ never vanishes on it. One can find $a_{k+1}, \ldots, a_n \in A$ such that $x \neq 0$ on $\sigma = \sigma(a_1, \ldots, a_n; A_n)$ where A_n is the subalgebra generated by $a_1, \ldots, a_n$ [1, 2, 3], understanding $z_1, \ldots, z_k$ now to be the first k coordinate-functions in C^n. From Shilov's observation ([1], 206), there exist polynomials $P_1, \ldots, P_N$ such that $\sigma \subset W = \bigcap \{|P_j| < 1\} \subset \{x \neq 0\}$. Thus (2.2) $x = ae^\psi$ where $\alpha \in \mathrm{Hol}(W, C^*)$, $\psi \in (W, C)$. By the theorem of Oka-Weil [4] there is a polynomial P such that $\max\{|P(\lambda) - \alpha(\lambda)|;\ \lambda \in \sigma\} < \varepsilon$. Take ε so small that $P \neq 0$ on σ and also $\alpha = Pe^p$ where $p \in \mathscr{C}(\sigma, C)$. Then $x = Pe^g$ where $g \in \mathscr{C}(\sigma, C)$. Let $a = P(a_1, \ldots, a_n) \in A$. Then $f_1 = a_\varDelta e^{g_2}$ where $g_2 \in \mathscr{C}(\varDelta, C)$, yielding half of the lemma. Suppose now that $a_\varDelta e^g = b_\varDelta e^h$ on $\varDelta$, and, by [1], 8.1, $ab^{-1} = e^c$ for some $c \in A$.

4. Theorem. *Let G be the group of invertible elements of A. Then there is a subgroup $\varGamma$ of G such that $G = G_0 + \varGamma$ where $G_0 = \{e^a : a \in A\}$ is the component of 1 in G, and $G/G_0 \longleftrightarrow \varGamma \longleftrightarrow H^1(\varDelta, Z)$.*

Sending a into $a_\varDelta$ induces a homomorphism H of G/G_0 into $\overset{\vartriangle}{G}/\overset{\vartriangle}{G}_0$ (where $\overset{\vartriangle}{G} = \mathscr{C}(\varDelta, C^*)$). The lemma shows that H is "onto" and 1:1. G/G_0 has no elements of finite order, so $\varGamma$ exists. Clearly $G_0 = \{e^a\}$.

Lemma 1 can be extended to commutative F algebras, by the use ([5], 2.4).

I should like to add that when I announced my reduction of the problem to the set σ, Professor H. L. Royden independently took up the matter and also arrived at [1]. Royden also used Cartan's Théorème B.

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