

## The two-norm spaces

by

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This is a report on work done jointly with Dr. Semadeni.

Suppose we are given a linear space  $X$  in which two norms  $\| \cdot \|$  and  $\| \cdot \|_*$ , are defined; let the norm  $\| \cdot \|_*$  be coarser than  $\| \cdot \|$  (this means that  $\|x_n\| \rightarrow 0$  implies  $\|x_n\|_* \rightarrow 0$ ). The triplet  $\langle X, \| \cdot \|, \| \cdot \|_* \rangle$  will be called *two-norm space*. We introduce a notion of convergence:  $x_n \xrightarrow{\gamma} x_0$  means that  $\sup_n \|x_n\| < \infty$  and  $\lim_{n \rightarrow \infty} \|x_n - x_0\|_* = 0$ . A functional  $\xi(x)$  defined on  $X$  will be called  *$\gamma$ -linear* if it is distributive and  $x_n \xrightarrow{\gamma} x_0$  implies  $\xi(x_n) \rightarrow \xi(x_0)$ . The totality of the  $\gamma$ -linear functionals will be denoted by  $\mathcal{E}_\gamma$ . Let us write further

$\langle \mathcal{E}, \| \cdot \| \rangle =$  the space conjugate to  $\langle X, \| \cdot \| \rangle$ ,

$\langle \mathcal{E}^*, \| \cdot \|_* \rangle =$  the space conjugate to  $\langle X, \| \cdot \|_* \rangle$ ;

thus

$$\|\xi\| = \sup\{|\xi(x)| : \|x\| \leq 1\},$$

$$\|\xi\|_* = \sup\{|\xi(x)| : \|x\|_* \leq 1\}.$$

Obviously  $\mathcal{E}^* \subset \mathcal{E}_\gamma \subset \mathcal{E}$ .

I need the following definitions:

The space  $\langle X, \| \cdot \|, \| \cdot \|_* \rangle$  will be called  *$\gamma$ -normal* if  $\lim_{n \rightarrow \infty} \|x_n - x_0\|_* = 0$  implies  $\|x_0\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$ . The space will be called  *$\gamma$ -complete* if the following condition is satisfied:

if  $(x_{p_n} - a_{q_n}) \xrightarrow{\gamma} 0$  as  $p_n, q_n \rightarrow \infty$ , then there exists  $x_0$  such that  $x_n \xrightarrow{\gamma} x_0$ .

**The structure of the space  $\mathcal{E}_\gamma$ .** A trivial case of two-norm spaces occurs when the norms  $\| \cdot \|$  and  $\| \cdot \|_*$  are equivalent. Then  $\langle X, \| \cdot \|, \| \cdot \|_* \rangle$  reduces to a normed space and  $\mathcal{E}^* = \mathcal{E}_\gamma = \mathcal{E}$ . A partial converse is true:

**THEOREM 1.** *Let the space  $\langle X, \| \cdot \|, \| \cdot \|_* \rangle$  be  $\gamma$ -normal and let  $\mathcal{E}^* = \mathcal{E}_\gamma$ . Then the norms  $\| \cdot \|$  and  $\| \cdot \|_*$  are equivalent.*

There exist non-trivial examples of two-norm spaces for which  $\mathcal{E}_\gamma = \mathcal{E}$ . Such spaces will be called *saturated*. A special case of saturated spaces is described by

**THEOREM 2.** Let  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  be a two-norm space and let the space  $\langle X, \|\cdot\| \rangle$  be reflexive. Then the space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  is saturated.

We are able to establish a lot of necessary and sufficient conditions for a two-norm space to be saturated, for instance:

**THEOREM 3.** Each of the following conditions is necessary and sufficient for  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  to be saturated:

- (a)  $\mathcal{E}^*$  is dense in  $\langle \mathcal{E}, \|\cdot\| \rangle$ ,  
 (b) for every  $\xi \in \mathcal{E}$  and  $\varepsilon > 0$  there exists a constant  $K$  such that

$$\xi(x) \leq \varepsilon + K\|x\|^* \quad \text{for} \quad \|x\| \leq 1.$$

Theorem 2 combined with the condition (a) of Theorem 3 admits a partial converse:

**THEOREM 4.** A Banach space  $\langle X, \|\cdot\| \rangle$  is reflexive if and only if for every norm  $\|\cdot\|^*$ , coarser than  $\|\cdot\|$ , the space  $\mathcal{E}^*$  is dense in  $\langle \mathcal{E}, \|\cdot\| \rangle$ .

A very useful tool for establishing in particular cases the general form of  $\gamma$ -linear functionals is given by

**THEOREM 5.** Let  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  be  $\gamma$ -normal. Then  $\mathcal{E}_\gamma$  is identical with the closure in the space  $\langle \mathcal{E}, \|\cdot\| \rangle$  of the set  $\mathcal{E}^*$ ; in other words the general form of  $\gamma$ -linear functionals is

$$\xi(x) = \lim_{n \rightarrow \infty} \xi_n(x) \quad \text{with} \quad \xi_n \in \mathcal{E}^*, \quad \lim_{n \rightarrow \infty} \|\xi_n - \xi\| = 0.$$

It is easily proved that in non-trivial cases the convergence  $\gamma$  cannot be produced by a metric. So we may ask whether a topology can reach this goal. The answer is positive.

**THEOREM 6** (of Wiweger). Let the space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  be  $\gamma$ -normal. Then there exists a convex linear topology  $\mu$  such that

- (i)  $\gamma$ -convergence is equivalent to the sequential  $\mu$ -convergence,  
 (ii)  $\mathcal{E}_\gamma$  is identical with the set of distributive functionals continuous for the topology  $\mu$ .

In view of this theorem it could be expected that the  $\gamma$ -linear functionals possess the extension property. This is, however, not the case.

One of the fundamental principles of functional analysis is contained in the theorem of Banach stating that the limit of a convergent sequence of continuous linear functionals on a Banach space is continuous. Is an analogous statement true for two-norm spaces? The answer is negative: we know examples of  $\gamma$ -complete  $\gamma$ -normal spaces in which such a theorem is not true. So we may ask for the sufficient conditions. Let me write one of them:

( $\Sigma$ ) For every  $x_0 \in \mathcal{S} = \{x: \|x\| \leq 1\}$  and  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $x \in \mathcal{S}$ ,  $\|x\|^* < \delta$ , implies  $x = z_1 - z_2$  with  $z_1, z_2 \in \mathcal{S}$ ,  $\|z_1 - x_0\|^* < \varepsilon$ ,  $\|z_2 - x_0\|^* < \varepsilon$ .

**THEOREM 7.** Let the space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  be  $\gamma$ -normal and  $\gamma$ -complete, and let it possess the property ( $\Sigma$ ). Then the limit of every convergent sequence of  $\gamma$ -linear functionals is  $\gamma$ -linear.

This theorem and some similar ones play an important role as applied to Toeplitz methods of summability. Concerning more details I refer to the papers of Orlicz.

**Duality theory.** Let us confine ourselves in this section to  $\gamma$ -normal spaces. In the space  $\mathcal{E}^*$  the norm  $\|\cdot\|^*$  is finer than  $\|\cdot\|$ ; so we shall call the space  $\langle \mathcal{E}^*, \|\cdot\|^*, \|\cdot\| \rangle$   $\gamma$ -conjugate to  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$ . The space  $\langle \mathcal{E}^*, \|\cdot\|^*, \|\cdot\| \rangle$  is always  $\gamma$ -normal and  $\gamma$ -complete.

Now let us consider the second  $\gamma$ -conjugate space of  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$ . Let us denote it by  $\langle \mathcal{X}^{(\gamma)}, \|\cdot\|, \|\cdot\|^* \rangle$ ; then  $\mathcal{X}^{(\gamma)}$  is the space conjugate to  $\langle \mathcal{E}^*, \|\cdot\|^* \rangle$  or, what amounts to the same, it is conjugate to  $\langle \mathcal{E}_\gamma, \|\cdot\|^* \rangle$ .

Now let us consider the functional  $\mathfrak{y}(\xi) = \xi(x)$  defined for  $\xi \in \mathcal{E}^*$ . The canonical embedding  $x \rightarrow \mathfrak{y}_x$  maps the space  $X$  into  $\mathcal{X}^{(\gamma)}$ ; moreover we have:

**THEOREM 8.** Let the space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  be  $\gamma$ -normal. Then the canonical embedding maps  $X$  into  $\mathcal{X}^{(\gamma)}$  with preservation of both norms, i. e.

$$\|\mathfrak{y}_x\| = \|x\|, \quad \|\mathfrak{y}_x\|^* = \|x\|^*.$$

The space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  will be called  $\gamma$ -reflexive if the canonical map of  $X$  is the whole of  $\mathcal{X}^{(\gamma)}$ .

**THEOREM 9.** A  $\gamma$ -normal space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  is  $\gamma$ -reflexive if and only if the ball  $S = \{x: \|x\| \leq 1\}$  is compact for the topology  $\sigma(X, \mathcal{E}^*)$ .

A lot of theorems analogous to the case of Banach space holds:

**THEOREM 10.** Any  $\gamma$ -closed subspace of a  $\gamma$ -reflexive space is  $\gamma$ -reflexive.

(A subspace  $Y \subset X$  is called  $\gamma$ -closed if  $y_n \in Y$ ,  $y_n \xrightarrow{\gamma} y_0$ , implies  $y_0 \in Y$ .)

**THEOREM 11.** The  $\gamma$ -conjugate space of a  $\gamma$ -reflexive space is  $\gamma$ -reflexive. If the  $\gamma$ -conjugate space is  $\gamma$ -reflexive, so is the primitive space, provided it be  $\gamma$ -complete.

We can also give a characterization of reflexivity in terms of  $\gamma$ -reflexivity.

**THEOREM 12.** A Banach space  $\langle X, \|\cdot\| \rangle$  is reflexive if there exists a norm  $\|\cdot\|^*$  coarser than  $\|\cdot\|$  such that the space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  is saturated and  $\gamma$ -reflexive.

**Extension of  $\gamma$ -linear functionals.** The  $\gamma$ -linear functionals do not possess in general the extension property. We know examples of  $\gamma$ -normal,  $\gamma$ -complete two-norm spaces  $X$ , for which there exist  $\gamma$ -closed subspaces  $Y$  and  $\gamma$ -linear functionals on  $Y$  which cannot be extended

onto the whole of  $X$  so as to remain  $\gamma$ -linear. Some sufficient conditions for extensibility are known.

**THEOREM 13.** *Let  $\langle X_0, \|\cdot\|, \|\cdot\|^* \rangle$  be a  $\gamma$ -reflexive subspace of a  $\gamma$ -normal space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$ . Then every  $\gamma$ -linear functional on  $X_0$  possesses a  $\gamma$ -linear extension on  $X$ .*

**A universal space.** Two two-norm spaces,  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  and  $\langle Y, \|\cdot\|, \|\cdot\|^* \rangle$  are called  $\gamma$ -equivalent if there exists a distributive operation  $T$  from  $X$  onto  $Y$  which establishes an isometry of  $\langle X, \|\cdot\| \rangle$  and  $\langle Y, \|\cdot\| \rangle$  and, at the same time,  $T$  is a homomorphism between  $\langle X, \|\cdot\|^* \rangle$  and  $\langle Y, \|\cdot\|^* \rangle$ .

Let us consider the following example: suppose we are given a linear space  $Z$  with a sequence  $[x]_i$  of seminorms such that  $[x]_i = 0$  for  $i = 1, 2, \dots$  implies  $x = 0$ . Let  $Z_\pi = \{x: \sup [x]_i < \infty\}$ ,  $\|x\| = \sup [x]_i$ ,  $\|x\|^* = \sum_{i=1}^{\infty} 2^{-i} [x]_i$  for  $x \in Z_\pi$ . Then  $\langle Z_\pi, \|\cdot\|, \|\cdot\|^* \rangle$  is a  $\gamma$ -normal space.

In particular, let  $C$  denote the space of continuous functions  $x = x(t)$  on the half-line  $0 \leq t < \infty$  with  $[x]_i = \sup\{|x(t)|: 0 \leq t \leq i\}$ . Then  $\gamma$ -convergence in the space  $\langle C_\pi, \|\cdot\|, \|\cdot\|^* \rangle$  means uniform boundedness plus uniform convergence on compact subsets of  $[0, \infty)$ .

**THEOREM 14.** *Every  $\gamma$ -normal two-norm space is  $\gamma$ -equivalent to a subspace of a certain space  $\langle Z_\pi, \|\cdot\|, \|\cdot\|^* \rangle$ .*

The space  $\langle X, \|\cdot\|, \|\cdot\|^* \rangle$  is called  $\gamma$ -separable if there exists a countable set dense for the convergence  $\gamma$ .

**THEOREM 15.** *Every  $\gamma$ -separable space is  $\gamma$ -equivalent to a subspace of the space  $\langle C_\pi, \|\cdot\|, \|\cdot\|^* \rangle$ .*

## The group of invertible elements of a commutative Banach algebra

by

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Let  $f$  be continuous, complex-valued on a compact subset  $D$  of the complex plane  $C$ . Then  $f$  has the form  $f = ae^g$  where  $a$  is rational, and  $g$  continuous on  $D$ . This classical theorem we generalize in a Banach algebra manner (see 1, below). Reformulated as in 4 (below) it represents another step along the path begun by Shilov [3] of finding algebraic invariants of a commutative Banach algebra  $A$  (over  $C$ , with unit) depending only on the space  $\Delta$  of complex linear-algebra homomorphisms. In a sense, Shilov shows that the cohomology group  $H^0(\Delta, Z)$  is isomorphic to the subring of  $A$  generated by its idempotents; and we show that  $H^1(\Delta, Z)$  is isomorphic to  $G/G_0$  (see 4).

Notations:  $C, A, \Delta$  have always the meaning as above.  $C^* = C - \{0\}$ .  $\mathcal{C}(X, Y)$  is the space of continuous functions. If  $F \subset \mathcal{C}(X, C)$  then  $\text{ex } F = \{\exp(2\pi if): f \in F\}$ . If  $W \subset C^n$  then  $\text{Hol}(W, Y)$  are the holomorphic  $Y$ -valued functions on  $W$ ,  $Y = C$  or  $C^*$ .  $\{f \neq 0\}$  is the set where  $f \neq 0$ . For  $b \in A$  and  $\delta \in \Delta$ ,  $b_\Delta(\delta) = \delta(b)$ .

**1. LEMMA.** *Let  $f \in \mathcal{C}(\Delta, C^*)$ . Then there exists an  $a \in A$ , and a  $g \in \mathcal{C}(\Delta, C)$  such that  $f = a_\Delta e^g$ . If  $f = b_\Delta e^h$  is another such representation, then  $b = ae^c$  for some  $c \in A$ .*

We shall deduce this from the following mere combination of two theorems of H. Cartan's. For our notation we refer closely to [2].

**2. PROPOSITION.** *Let  $P_1, \dots, P_N$  be polynomials in  $n$  complex variables, and form*

$$W = \{|P_1| < 1, \dots, |P_N| < 1\}.$$

*Then there is a natural isomorphism of the multiplicative groups.*

**3.**  $\mathcal{C}(W, C^*) / \text{ex } \mathcal{C}(W, C) \leftrightarrow \text{Hol}(W, C^*) / \text{ex } \text{Hol}(W, C)$ .

We sketch the proof. For the Stein manifold  $W$  we have the exact sequence of sheaves ([2], 27(11))  $0 \rightarrow Z \rightarrow C_\omega \xrightarrow{\text{ex}} C_\omega^* \rightarrow 0$ , and the exact