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On Fourier transforms of rapidly increasing distributions

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L. Ehrenpreis [2], [3], J. M. Gelfand and G. E. Silov [4], [5], and B. Malgrange [6] have introduced Fourier transforms of rapidly increasing distributions as functionals on a space of entire functions. Their definition is based on the Parseval equation and agrees with L. Schwartz's concept of distributions.

The present paper is an attempt to discuss this problem from the standpoint of the sequential theory of distributions, as developed by J. Mikusiński and R. Sikorski in [7]. For simplicity's sake we restrict ourselves to distributions of one variable. The paper deals mainly with distributions of finite order. The case of all distributions involves a slight modification of our basic definitions, which is given in section 5.

We recall briefly the notions and principal results of [7]. A sequence $\{f_n(x)\}$ of continuous functions on the real line R is an *F-sequence* (or a *fundamental sequence*), if there exist an integer $k \geqslant 0$ and a sequence $\{F_n(x)\}$ of functions such that $F_n^{(k)}(x) = f_n(x)$ and $\{F_n(x)\}$ converges almost uniformly, i. e. uniformly in every finite interval. Following [7], we shall write $F_n(x) \rightrightarrows F(x)$ (resp. $F_n(x) \rightrightarrows$) if $\{F_n(x)\}$ converges almost uniformly to $F(x)$ (resp. to some function). Two F-sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are *equivalent*, if $f_1(x), g_1(x), f_2(x), g_2(x), \ldots$ is an F-sequence. *Distributions* are classes of equivalent F-sequences. The distribution determined by the F-sequence $\{f_n(x)\}$ is denoted by $f(x) = [f_n(x)]$. A continuous function $f(x)$ may be identified with a distribution determined by the F-sequence $\{f(x)\}$, all of whose terms are equal to $f(x)$. The algebraic operations on distributions, such as addition, subtraction, multiplication with a function and translation are defined in a natural way by means of operations on their representing sequences.

Each distribution $f(x)$ may be represented by a sequence $\{f_n(x)\}$ of indefinitely differentiable functions. Then, for every integer $m \geqslant 0$, $\{f_n^{(m)}(x)\}$ is an F-sequence and the m-th derivative of $f(x)$ is defined as $f^{(m)}(x) = [f_n^{(m)}(x)]$. Thus each distribution has a derivative of any order. On the other hand, distributions are derivatives, of some orders, of continuous functions.

A sequence $\{f_n(x)\}$ of distributions is said to *converge* to $f(x)$, if there exist an integer $k \geqslant 0$, a continuous function $F(x)$ and a sequence $\{F_n(x)\}$ of continuous functions, such that $F_n^{(k)}(x) = f_n(x)$, $F^{(k)}(x) = f(x)$ and $F_n(x) \rightrightarrows F(x)$.

A distribution admitting a representation $f(x) = [f_n(x)]$, where all $f_n(x)$ vanish outside a finite interval, is said to be of *compact carrier*. Then, for an arbitrary distribution $g(x) = [g_n(x)]$,

$$f_n(x) * g_n(x) = \int_{-\infty}^{\infty} f_n(x-t) g_n(t)\, dt$$

is an F-sequence and defines the convolution product $f(x)*g(x)$ of the distributions $f(x)$ and $g(x)$.

Let now $\{f_n(x)\}$ be an F-sequence of integrable functions on R. If there are integers $k, l \geqslant 0$ and a sequence $\{F_n(x)\}$ of continuous functions, such that $F_n^{(k)}(x) = f_n(x)$, $F_n(x) \rightrightarrows$ and $|F_n(x)| < M(1+|x|^l)$, where M is a constant, then $f(x) = [f_n(x)]$ is a *tempered distribution*. In this case

$$\varphi_n(s) = \int_{-\infty}^{\infty} f_n(x) e^{-2\pi i x s}\, dx \tag{1}$$

is an F-sequence and represents a tempered distribution $\varphi(s)$ — the Fourier transform $\mathscr{F}(f(x))$ of $f(x)$. However, Fourier transforms of rapidly increasing functions, such as e^x, e^{x^2} etc. cannot be defined in this way, since they are not distributions. In order to extend the Fourier transform to all distributions we define other "generalized functions", which we call *ultra-distributions* (following Sebastião e Silva [9])(¹).

Throughout the paper we take for granted the theory of Fourier transforms in the space L^2 of square integrable functions on R. The limit in the L^2 norm is called limit in mean. We also use Hadamard's "finite part" of an improper integral as given in [8] or the equivalent "regularisation of functions" developed in [5]. If, for example, $f(x)$ is a $k-1$ times continuously differentiable function on R and $f(x) = O(|x|^{k-2})$ as $|x| \to \infty$, then we put

$$Fp \int_{-\infty}^{\infty} \frac{f(x)}{x^k}\, dx = \lim_{\varepsilon\to 0+} \Big\{ \int_{\varepsilon}^{\infty} \frac{r(x)}{x^k}\, dx - \frac{r(0)}{k-1}\, \frac{1}{\varepsilon^{k-1}} - \frac{r'(0)}{1!(k-2)}\, \frac{1}{\varepsilon^{k-1}} - \ldots - \frac{r^{(k-2)}(0)}{(k-2)!1}\, \frac{1}{\varepsilon} - \frac{r^{(k-1)}(0)}{(k-1)!} \ln\varepsilon \Big\} \tag{2}$$

(¹) In Sebastião e Silva's paper ultra-distributions are Fourier transforms of distributions of "exponential type".

$$= \int_0^{\infty} \frac{1}{x^k} \Big\{ r(x) - r(0) - \frac{x}{1!} r'(0) - \ldots - \frac{x^{k-2}}{(k-2)!} r^{(k-2)}(0) - \frac{x^{k-1}}{(k-1)!} r^{(k-1)}(0) H(1-x) \Big\}\, dx,$$

where

$$r(x) = f(x) + (-1)^k f(-x)$$

and $H(x)$ is Heaviside's unique function:

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x > 0. \end{cases}$$

Using the complex plane one can express (2) in the following form:

$$Fp \int_{-\infty}^{\infty} \frac{f(x)}{x^k}\, dx = \lim_{\varepsilon\to 0+} \frac{1}{2} \Big\{ \int_{-\infty}^{\infty} \frac{f(x)}{(x+i\varepsilon)^k}\, dx + \int_{-\infty}^{\infty} \frac{f(x)}{(x-i\varepsilon)^k}\, dx \Big\}. \tag{3}$$

For the m-th derivative of a function $f(x)$ we shall use both symbols $f^{(m)}(x)$ and $D^m f(x)$.

Let now $z = \xi + i\eta$ be a complex variable with ξ and η as real and imaginary parts respectively, and let $\lambda > 0$. An entire function $f(z)$ is of exponential type $\leqslant \lambda$, if, for every $\varepsilon > 0$,

$$|f(z)| < M_\varepsilon e^{(\lambda+\varepsilon)|z|};$$

M_ε is a constant.

1. Φ-sequences of functions. The starting point in our approach is the family of sequences formed from complex-valued, indefinitely differentiable functions, which are slowly increasing together with all their derivatives, i. e. the functions and their derivatives increase at infinity like polynomials. We proceed in a way similar to that given in [7].

Let us write

$$K_\lambda(z) = \frac{1}{\pi}\, \frac{\sin 2\pi\lambda z}{z}$$

where $\lambda \geqslant 0$. From Fourier's single-integral formula (see [10], p. 25) it follows that $K_\lambda(\xi)$ tends to the δ-Dirac distribution $\delta(\xi)$, as $\lambda \to \infty$.

If the function $\varphi(x)$ is $k-1$ times continuously differentiable and $\varphi(x) = O(|x|^{k-1})$ as $|x| \to \infty$, then

$$\Phi_\lambda(z) = Fp \int_{-\infty}^{\infty} K_\lambda(z-x) \frac{\varphi(x)}{x^k}\, dx \tag{4}$$

is an entire function of exponential type $\leqslant \lambda$, square integrable on R. Furthermore we have

PROPOSITION 1. *Let k and l be integers $\geqslant 0$, and $0<\mu\leqslant\lambda$. If $\varphi(x)$ is a $k+l-1$ times continuously differentiable function such that $\varphi(x)=O(|x|^{k-1})$ as $|x|\to\infty$, and if $\Phi_\lambda(z)$ is defined by (4), then*

$$Fp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{\Phi_\lambda(x)}{x^l}\,dx=Fp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{\varphi(x)}{x^{k+l}}\,dx+P(D)K_\mu(z),\tag{5}$$

where $P(D)$ is a polynomial in D of degree $\leqslant l-1$, whose coefficients may depend on λ [2].

In particular, for $l=0$ equation (5) *takes the form*

$$\Phi_\mu(z)=\int_{-\infty}^{\infty}K_\mu(z-x)\,\Phi_\lambda(x)\,dx.\tag{6}$$

Proof. By (4) we get for the left-hand side of (5)

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)}{x^l}\left\{Fp\int_{-\infty}^{\infty}\frac{K_\lambda(x-y)\varphi(y)}{y^k}\,dy\right\}dx$$

$$=Fp\int_{-\infty}^{\infty}\frac{\varphi(y)}{y^k}\left\{Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)K_\lambda(x-y)}{x^l}\,dx\right\}dy.$$

The change of the order of integration is valid; it may be verified by use of definition (2). To complete the proof of the proposition it suffices thus to show that, for any integer $l\geqslant 0$,

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)K_\lambda(x-y)}{x^l}\,dx=\frac{K_\mu(z-y)}{y^l}-\frac{1}{y^l}\sum_{j=0}^{l-1}A_{l,j}(y)K_\mu^{(j)}(z),\tag{7}$$

where $A_{l,0}(z), A_{l,1}(z), \ldots, A_{l,l-1}(z)$ are entire functions satisfying for $|\xi|\to\infty$ conditions

$$A_{l,j}(\xi)=O(|\xi|^{l-1}),\qquad j=0,1,\ldots,l-1.\tag{8}$$

We use induction. Applying formula (3) and the theory of residues one obtains for $l=0$ and $l=1$ respectively

$$\int_{-\infty}^{\infty}K_\mu(z-x)K_\lambda(x-y)\,dx=K_\mu(z-y)\tag{9}$$

[2] A polynomial of degree -1 vanishes identically.

and

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)K_\lambda(x-y)}{x}\,dx=\frac{K_\mu(z-y)}{y}-\frac{K_\mu(z)\cos 2\pi\lambda y}{y}.\tag{10}$$

Suppose now equation (7) holds for some $l\geqslant 1$. Its right-hand side can be written in the form

$$\frac{1}{y^l}\left\{K_\mu(z-y)-\sum_{j=0}^{l-1}\frac{(-y)^j}{j!}K_\mu^{(j)}(z)\right\}+\frac{1}{y^l}\sum_{j=0}^{l-1}\left\{\frac{(-y)^j}{j!}-A_{l,j}(y)\right\}K_\mu^{(j)}(z)\tag{11}$$

and it has a limit as $y\to 0$ (which is an entire function in z). The first part of (11) tends to $(-1)^lK_\mu^{(l)}(z)/l!$. On substituting in the second part $z=1/4\mu$ one gets a polynomial of degree l in μ, whose coefficients in $\mu,\mu^2,\ldots,\mu^l$ are proportional to y^{-1} times the terms in curly brackets. Hence we infer (see [7], lemma on p. 8) that

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)K_\lambda(x)}{x^l}\,dx=\sum_{j=0}^{l}B_{l,j}K_\mu^{(j)}(z),\tag{12}$$

where

$$B_{l,j}=\lim_{y\to 0}\frac{1}{y^l}\left\{\frac{(-y)^j}{j!}-A_{l,j}(y)\right\},\quad j=0,1,\ldots,l-1,\quad\text{and}\quad B_{l,l}=\frac{(-1)^l}{l!}.$$

We now multiply equation (7) with $K_\lambda(y-z)/y$ and take the Fp's of integrals with respect to y. This leads to equation

$$Fp\int_{-\infty}^{\infty}\frac{K_\lambda(y-t)}{y}\left\{Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)K_\lambda(x-y)}{x^l}\,dx\right\}dy\tag{13}$$

$$=Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-y)K_\lambda(y-t)}{y^{l+1}}\,dy-\sum_{j=0}^{l-1}K_\mu^{(j)}(z)\,Fp\int_{-\infty}^{\infty}\frac{K_\lambda(t-y)A_{l,j}(y)}{y^{l+1}}\,dy.$$

By change of the order of integration and in view of (7), (10) and (12), the left-hand side of (13) may be transformed into

$$\frac{K_\mu(z-t)}{t^{l+1}}-\frac{1}{t^{l+1}}\left\{\sum_{j=0}^{l-1}A_{l,j}(t)K_\mu^{(j)}(z)+t^l\cos 2\pi\lambda t\sum_{j=0}^{l}B_{l,j}K_\mu^{(j)}(z)\right\}.$$

Consequently

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-y)K_\lambda(y-t)}{y^{l+1}}\,dy=\frac{K_\mu(z-t)}{t^{l+1}}-\frac{1}{t^{l+1}}\sum_{j=0}^{l}A_{l+1,j}(t)K_\mu^{(j)}(z),$$

where the functions

$$A_{l+1,j}(t) = A_{l,j}(t)+B_{l,j}t^l\cos 2\pi\lambda t - Fp\int_{-\infty}^{\infty}\frac{K_\lambda(t-y)A_{l,j}(y)}{y^{l+1}}\,dy,$$

$j = 0, 1, \ldots, l-1$, and

$$A_{l+1,l}(t) = B_{l,l}t^l\cos 2\pi\lambda t$$

are $O(|t|^l)$ as $|t|\to\infty$ (on R). This means that equation (7), with the functions $A_{l,j}(y)$, $j = 0, 1, \ldots, l-1$, satisfying conditions (8), holds for $l+1$, and thus our assertion is proved.

PROPOSITION 2. *If k, l, μ, λ and $\varphi(x)$ are as in proposition* 1, *and if*

$$\Phi_\lambda(z) = Fp\int_{-\infty}^{\infty}K_\lambda(z-x)\frac{\varphi(x)}{x^k}\,dx+Q(D)K_\lambda(z),$$

where $Q(D)$ is a polynomial of degree $\leqslant k-1$, then equation (5) *holds with $P(D)$ a polynomial of degree $\leqslant k+l-1$.*

Proof. On account of proposition 1 it remains to prove that, for any integers m, $l\geqslant 0$,

$$Fp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{K_\lambda^{(m)}(x)}{x^l}\,dx = \sum_{j=0}^{l+m}C_j^{m,l}K_\mu^{(j)}(z), \tag{14}$$

where $C_j^{m,l}$, $j = 0, 1, \ldots, l+m$, are constants.

For $m = 0$ and any l, equation (14) coincides with equation (12), when $C_j^{0,l} = B_{l,j}$. Suppose now (14) holds for some m and every l. Then

$$Fp\int_{-\infty}^{\infty}K'_\mu(z-x)\frac{K_\lambda^{(m)}(x)}{x^l}\,dx = \sum_{j=0}^{l+m}C_j^{m,l}K_\mu^{(j+1)}(z). \tag{15}$$

But the integral on the left-hand side of (15) may be written in the following form:

$$Fp\int_{-\infty}^{\infty}K'_\mu(z-x)\frac{K_\lambda^{(m)}(x)}{x^l}\,dx$$

$$= Fp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{K_\lambda^{(m+1)}(x)}{x^l}\,dx - lFp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{K_\lambda^{(m)}(x)}{x^{l+1}}\,dx.$$

Hence equation (14) holds for $m+1$ and every l, with

$$C_0^{m+1,l} = lC_0^{m,l+1},$$

$$C_j^{m+1,l} = C_{j-1}^{m,l}+lC_j^{m,l+1}, \quad j = 1, 2, \ldots, l+m+1.$$

Thus, by induction, it holds for every integer m, and the proposition is proved.

We also need a more general result, namely

PROPOSITION 3. *Let $k, l, \varphi(x)$ be as in proposition* 1, *and $a(z)$ an entire function of exponential type $\leqslant\nu$, such that on R,*

$$a(\xi) = O(|\xi|^{l-1}), \quad \textit{as} \quad |\xi|\to\infty.$$

If $\mu\leqslant\lambda$ and

$$\Phi_{\lambda+\nu}(z) = Fp\int_{-\infty}^{\infty}K_{\lambda+\nu}(z-x)\frac{\varphi(x)}{x^k}\,dx+Q(D)K_{\lambda+\nu}(z),$$

then

$$Fp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{a(x)\Phi_{\lambda+\nu}(x)}{x^l}\,dx = Fp\int_{-\infty}^{\infty}K_\mu(z-x)\frac{a(x)\varphi(x)}{x^{k+l}}\,dx+P(D)K_\mu(z),$$

where $Q(D)$ and $P(D)$ are polynomials of degree $\leqslant k-1$ and $\leqslant k+l-1$, respectively.

Proof. Using the same argument as in propositions 1 and 2 we need only to prove that

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)\,a(x)K_{\lambda+\nu}(x-y)}{x^l}\,dx \tag{17}$$

$$= \frac{K_\mu(z-y)\,a(y)}{y^l} - \frac{1}{y^l}\sum_{j=0}^{l-1}A_{l,j}^a(y)K_\mu^{(j)}(z),$$

where $A_{l,0}^a(z), A_{l,1}^a(z), \ldots, A_{l,l-1}^a(z)$ are entire functions such that, for $|\xi|\to\infty$,

$$A_{l,j}^a(\xi) = O(|\xi|^{l-1}), \quad j = 0, 1, \ldots, l-1.$$

In fact, if $a(0) = a'(0) = \ldots = a^{(l-1)}(0) = 0$, then by methods of the theory of residues it is easy to verify that

$$\int_{-\infty}^{\infty}\frac{K_\mu(z-x)\,a(x)K_{\lambda+\nu}(x-y)}{x^l}\,dx = \frac{K_\mu(z-y)\,a(y)}{y^l}. \tag{18}$$

In the general case consider

$$a^*(z) = a(z)-a(0)-\ldots-\frac{z^{l-1}}{(l-1)!}a^{(l-1)}(0).$$

We have $a^*(0) = a^{*\prime}(0) = \ldots = a^{*(l-1)}(0) = 0$ and, on R,

$$a^*(\xi) = O(|\xi|^{l-1}) \quad \text{as} \quad |\xi|\to\infty.$$

Therefore, by (18),

$$\int_{-\infty}^{\infty}\frac{K_\mu(z-x)\,a^*(x)K_{\lambda+\nu}(x-y)}{x^l}\,dx = \frac{K_\mu(z-y)\,a^*(y)}{y^l}.$$

It follows that

$$Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)\,a(x)K_{\lambda+\nu}(x-y)}{x^l}\,dx$$

$$= \frac{K_\mu(z-y)\,a(y)}{y^l} - \sum_{j=0}^{l-1}\frac{a^{(j)}(0)}{j!}\left\{\frac{K_\mu(z-y)}{y^{l-j}} - Fp\int_{-\infty}^{\infty}\frac{K_\mu(z-x)K_{\lambda+\nu}(x-y)}{x^{l-j}}\,dx\right\}$$

and, by (7), the terms in curly brackets are of the form stated in equation (17).

Let now $\{P_n(D)\}$ be a sequence of polynomials of degree $\leqslant k-1$,

$$P_n(D) = a_{n,0}+a_{n,1}D+\ldots+a_{n,k-1}D^{k-1}.$$

We have

PROPOSITION 4. *If the sequence $\{P_n(D)K_\lambda(z)\}$ converges for k different values $\lambda=\lambda_j>0$ and $z=1/4\lambda_j$, $j=1,2,\ldots,k$, then the limits*

$$\lim_{n\to\infty} a_{n,p} = a_p, \quad p = 0,1,\ldots,k-1,$$

exist and thus the limit of the sequence in question is of the form $P(D)K_\lambda(z)$, where $P(D) = a_0+a_1D+\ldots+a_{k-1}D^{k-1}$.

Proof. The substitution $z=1/4\lambda$ in $P_n(D)K_\lambda(z)$ leads to polynomials in λ of degree $\leqslant k$, whose free terms are zero and the coefficients in $\lambda,\lambda^2,\ldots,\lambda^k$ are proportional to $a_{n,0},\ldots,a_{n,k-1}$. For the proof it suffices thus to apply the lemma on p. 8 in [7].

We shall now consider sequences of functions which, unless otherwise stated, are indefinitely differentiable and slowly increasing together with all their derivatives.

A sequence $\{\varphi_n(s)\}$ is said to be a *Φ-sequence*, if there exist an integer $k\geqslant 0$ and a sequence of polynomials $P_n(D)$ of degree $\leqslant k-1$, such that

(19) $$\varphi_n(s) = O(|s|^{k-1}) \quad \text{as} \quad |s|\to\infty$$

and, for every $\lambda>0$,

(20) $$\Phi_n(\lambda,z) = Fp\int_{-\infty}^{\infty}K_\lambda(z-s)\frac{\varphi_n(s)}{s^k}\,ds + P_n(D)K_\lambda(z)$$

converges uniformly in every strip $-N<\eta<N$ $(z=\xi+i\eta)$.

For brevity we shall write $\Phi_n(\lambda,z)\Rightarrow\Phi(\lambda,z)$ (resp. $\Phi_n(\lambda,z)\Rightarrow$) if $\{\Phi_n(\lambda,z)\}$ converges to $\Phi(\lambda,z)$ (resp. to some function) uniformly in every strip $-N<\eta<N$.

Remark 1. From proposition 2 it follows that the integer k in (19) and (20) may be replaced by any greater one. In fact, if $\{\Phi_n(\lambda,z)\}$ converges uniformly in every strip $-N<\eta<N$, then so does

(21) $$\Phi_n^*(\lambda,z) = Fp\int_{-\infty}^{\infty}\frac{K_\lambda(z-s)\Phi_n(\lambda,s)}{s}\,ds.$$

Moreover, since, for every $\lambda>0$, the functions $\Phi_n(\lambda,\xi)$ are uniformly bounded on R,

(22) $$|\Phi_n^*(\lambda,\xi)| \leqslant \frac{M_\lambda}{|\xi|},$$

where M_λ is a constant, and so $\{\Phi_n^*(\lambda,\xi)\}$ converges in mean.

Remark. 2. According to a theorem of S. Bernstein (see [1], chapter II), if $f(z)$ is an entire function of exponential type $\leqslant\lambda$ and $|f(\xi)|\leqslant M$, then $|f'(\xi)|\leqslant M\lambda$. Therefore $\Phi_n(\lambda,z)\Rightarrow$ implies $\Phi_n'(\lambda,z)\Rightarrow$.

The limit $\Phi(\lambda,z)$ of (20) is an entire function of exponential type $\leqslant\lambda$, tending to 0 as z tends to ∞ on R. By remark 1 it satisfies condition (22), if k is sufficiently large.

If $\{\varphi_n(s)\}$ and $\{\psi_n(s)\}$ are Φ-sequences and a,b complex numbers, then $\{a\varphi_n(s)+b\psi_n(s)\}$ is a Φ-sequence; if a,b are real numbers and $a\neq 0$, then also $\{\varphi_n(ax+b)\}$ is a Φ-sequence.

If $\{\varphi_n(s)\}$ is a Φ-sequence, then, for any integer $m\geqslant 0$, $\{s^m\varphi_n(s)\}$ is a Φ-sequence.

More general, if $\{\varphi_n(s)\}$ is a Φ-sequence and $\alpha(z)$ an entire function of exponential type, slowly increasing on R, then $\{\alpha(s)\varphi_n(s)\}$ is a Φ-sequence This follows from proposition 3.

Each uniformly convergent sequence $\{\varphi_n(s)\}$ of indefinitely differentiable functions, bounded on R, is a Φ-sequence (with $k=1$).

If $\varphi_n(s)\rightrightarrows$ and $|\varphi_n(s)|<M(1+|s|^l)$ for some integer $l\geqslant 0$ and a constant M, then $\{\varphi_n(s)\}$ is a Φ-sequence. In fact, we can decompose the sequence as follows: $\{\varphi_n(s)\} = \{\tilde\varphi_n(s)\}+\{\tilde{\tilde\varphi}_n(s)\}$, where $\tilde\varphi_n(s)$, $\tilde{\tilde\varphi}_n(s)$ are indefinitely differentiable functions, such that $\tilde\varphi_n(s)=0$ for $|s|>2$, $\tilde{\tilde\varphi}_n(s)=0$ for $|s|<1$ and $\tilde\varphi_n(s)\rightrightarrows$, $\tilde{\tilde\varphi}_n(s)\rightrightarrows$. Since the sequences $\{\tilde\varphi_n(s)\}$ and $\{\tilde{\tilde\varphi}_n(s)s^{-l-1}\}$ converge uniformly on R, they are Φ-sequences, and from what we have said before it follows that $\{\varphi_n(s)\}$ is also a Φ-sequence.

We prove

PROPOSITION 5. *If $\{\varphi_n(s)\}$ is a Φ-sequence and if for some k, the derivatives $\varphi_n'(s)$ satisfy condition (19), then $\{\varphi_n'(s)\}$ is also a Φ-sequence.*

Proof. We may always choose k sufficiently large so as to be the corresponding integer for the Φ-sequence $\{\varphi_n(s)\}$ in (19) and (20). Then $\Phi_n(\lambda, z) \rightrightarrows$ and, by remark 2, also $\Phi_n'(\lambda, z) \rightrightarrows$ for every $\lambda > 0$. We prove that, for $k \geqslant 1$,

$$(23)\qquad k\Phi_n^*(\lambda, z)+\Phi_n'(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n'(s)}{s^k}\,ds+\tilde{P}_n(D)K_\lambda(z),$$

where $\Phi_n^*(\lambda, z)$ are defined by (21) and $\tilde{P}_n(D)$ are polynomials of degree $\leqslant k-1$. In fact

$$kFp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n(s)}{s^{k+1}}\,ds+Fp\int_{-\infty}^{\infty} K_\lambda'(z-s)\frac{\varphi_n(s)}{s^k}\,ds = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n'(s)}{s^k}\,ds.$$

Furthermore, by (20),

$$Fp\int_{-\infty}^{\infty} K_\lambda'(z-s)\frac{\varphi_n(s)}{s^k}\,ds = \Phi_n'(\lambda, z)-DP_n(D)K_\lambda(z),$$

and, by proposition 2,

$$kFp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n(s)}{s^{k+1}}\,ds = k\Phi_n^*(\lambda, z)-Q_n(D)K_\lambda(z),$$

where the polynomials $DP_n(D)$ and $Q_n(D)$ are of degree $\leqslant k$. But from (16) and (12) it follows that

$$\tilde{P}_n(D) = DP_n(D)+Q_n(D)$$

is a polynomial of degree $\leqslant k-1$. Hence equation (23) is valid and this gives the required result.

A Φ-sequence does not in general converge in the distributional sense, i. e. it is not an F-sequence. For illustration we give the following

Example. The functions

$$(24)\qquad \varphi_n(s) = K_n(s+i) = \frac{\sin 2\pi n(s+i)}{\pi(s+i)}$$

satisfy condition (19) with $k=0$. Moreover, since for $n>\lambda$,

$$K_\lambda(z+i) = \int_{-\infty}^{\infty} K_\lambda(z-s)K_n(s+i)\,ds,$$

it is a Φ-sequence. However, for $\psi_k(s) = \mathcal{F}[(\sin\pi x/\pi x)^{2k}]$, $k = 1, 2, \ldots$,

$$\Big|\int_{-\infty}^{\infty} \varphi_n(s)\psi_k(s)\,ds\Big| \geqslant M_k n,$$

where $M_k > 0$, and therefore $\{\varphi_n(s)\}$ is not an F-sequence.

2. Ultra-distributions and operations on them. We say that two Φ-sequences $\{\varphi_n(s)\}$ and $\{\psi_n(s)\}$ are *equivalent*, if there exist an integer $k \geqslant 0$ and sequences of polynomials $P_n(D)$ and $Q_n(D)$ of degree $\leqslant k-1$ such that

$$\varphi_n(s) = O(|s|^{k-1}),\quad \psi_n(s) = O(|s|^{k-1})\quad \text{as}\quad |s|\to\infty,\ n = 1, 2, \ldots,$$

and for every $\lambda > 0$,

$$\Phi_n(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n(s)}{s^k}\,ds+P_n(D)K_\lambda(z)$$

and

$$\Psi_n(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\psi_n(s)}{s^k}\,ds+Q_n(D)K_\lambda(z)$$

converge to the same limit, uniformly in every strip $-N<\eta<N$. We denote this relation between Φ-sequences by $\sim$, i. e. we write $\{\varphi_n s)\}\sim\{\psi_n(s)\}$.

By proposition 2, the integer $k \geqslant 0$ can be replaced by any greater integer.

It is easy to show that the above definition of equivalent Φ-sequences can be expressed briefly as follows: Two Φ-sequences $\{\varphi_n(s)\}$ and $\{\varphi_n(s)\}$ are equivalent, if $\varphi_1(s), \psi_1(s), \varphi_2(s), \psi_2(s), \ldots$, is a Φ-sequence.

Proposition 6. *If $\{\varphi_n(s)\}$ is a Φ-sequence, k the corresponding integer in* (19) *and* (20), *and $\Phi(\lambda, z)$ the limit of* (20), *then*

$$\{\varphi_n(s)\} \sim \{s^k\Phi(n, s)\}.$$

Proof. We have

$$\varphi_n(s) = O(|s|^k),\quad s^k\Phi(n, s) = O(|s|^k),\quad \text{as}\quad |s|\to\infty$$

for $n = 1, 2, \ldots$, and by proposition 2, the functions $\Phi_n^*(\lambda, z)$ defined by (21) are of the form

$$\Phi_n^*(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n(s)}{s^{k+1}}\,ds+P_n^*(D)K_\lambda(z),$$

where $P_n^*(D)$ are polynomials of degree $\leqslant k$.

On the other hand, by proposition 2, for every $n \geqslant \lambda$,

$$\Phi_m^*(\lambda, z) = Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\Phi_m(n, s)}{s} ds$$

and hence, as $m \to \infty$,

$$\Phi^*(\lambda, z) = Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\Phi(n, s)}{s} ds.$$

Since $\Phi_n^*(\lambda, z) \Rightarrow \Phi^*(\lambda, z)$, the proposition is proved.

The relation $\sim$ is reflexive, symmetric and transitive. The abstraction classes of Φ-sequences with regard to this relation will be called *ultra-distributions*. We denote the ultra-distribution represented by the Φ-sequence $\{\varphi_n(s)\}$ by $\varphi(s) = [\varphi_n(s)]$ (3).

We shall show later on (see section 4) that the ultra-distribution determined by the sequence (24) may be regarded as a translated δ-Dirac distribution with the singularity at the imaginary point $s = -i/2\pi$, i. e.

$$\delta\left(s+\frac{i}{2\pi}\right) = \left[\frac{\sin 2\pi n(s+i)}{\pi(s+i)}\right].$$

It was proved in section 1 that a sequence of indefinitely differentiable functions $\varphi_n(s)$ such that

(25) $$\varphi_n(s) \rightrightarrows \quad \text{and} \quad |\varphi_n(s)| < M(1+|s|^l)$$

for some integer $l \geqslant 0$ and a constant M, is a Φ-sequence. Let $\varphi(s)$ be the limit of $\{\varphi_n(s)\}$. All Φ-sequences satisfying conditions (25) and having $\varphi(s)$ as limit are equivalent. On the other hand, sequences of that kind having different limits determine different ultra-distributions. This one-to-one correspondence between ultra-distributions represented by sequences with the property (25) and continuous, slowly increasing functions (the limits of those sequences) enables us to identify both notions. Ultra-distributions may therefore be regarded as a generalization of continuous, slowly increasing functions. We shall show later on that this generalization includes also all tempered distributions.

Let now $\varphi(s) = [\varphi_n(s)]$ and $\psi(s) = [\psi_n(s)]$ be two ultra-distributions, a, b real numbers ($a \neq 0$), and $\alpha(z)$ an entire function of exponential type, slowly increasing on R. We define

(26) $$\varphi(s) \pm \psi(s) = [\varphi_n(s) \pm \psi_n(s)],$$

(27) $$\varphi(as+b) = [\varphi_n(as+b)],$$

(28) $$\alpha(s)\varphi(s) = [\alpha(s)\varphi_n(s)].$$

(3) Similarly as in the case of distributions, the "variable" s in $\varphi(s)$ is purely symbolic and, in general, one cannot substitute for it any number.

The sequences in square brackets are Φ-sequences. To prove the consistency we need only to show that in each case the new ultra-distribution does not depend on the choice of the Φ-sequence representing $\varphi(s)$ or $\psi(s)$. In other words, if $\{\tilde{\varphi}_n(s)\} \sim \{\varphi_n(s)\}$ and $\{\tilde{\psi}_n(s)\} \sim \{\psi_n(s)\}$, then $\{\tilde{\varphi}_n(s) \pm \tilde{\psi}_n(s)\} \sim \{\varphi_n(s) \pm \psi_n(s)\}$, $\{\tilde{\varphi}_n(as+b)\} \sim \{\varphi_n(as+b)\}$ and $\{\alpha(s)\tilde{\varphi}_n(s)\} \sim \{\alpha(s)\varphi_n(s)\}$. We restrict ourselves to the third item.

Suppose that

$$\left.\begin{aligned} \varphi_n(s) &= O(|s|^{k-1}), \\ \tilde{\varphi}_n(s) &= O(|s|^{k-1}), \end{aligned}\right. \quad \text{as} \quad |s| \to \infty$$

and that there are polynomials $P_n(D)$ and $\tilde{P}_n(D)$ of degree $\leqslant k-1$ such that for every $\lambda > 0$,

$$\left.\begin{aligned} \Phi_n(\lambda, z) &= Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\varphi_n(s)}{s^k} ds + P_n(D)K_\lambda(z) \\ \tilde{\Phi}_n(\lambda, z) &= Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\tilde{\varphi}_n(s)}{s^k} ds + P_n(D)K_\lambda(z) \end{aligned}\right\} \Rightarrow \Phi(\lambda, z).$$

If

$$\alpha(s) = O(|s|^{l-1}) \quad \text{as} \quad |s| \to \infty,$$

we infer that for every $\lambda > 0$,

$$\left.\begin{aligned} &Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\alpha(s)\Phi_n(\lambda, s)}{s^l} ds \\ &Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\alpha(s)\tilde{\Phi}_n(\lambda, s)}{s^l} ds \end{aligned}\right\} \Rightarrow Fp \int_{-\infty}^{\infty} K_\lambda(z-s) \frac{\alpha(s)\Phi(\lambda, s)}{s^l} ds.$$

Hence

$$\{\alpha(s)\tilde{\varphi}_n(s)\} \sim \{\alpha(s)\varphi_n(s)\}$$

by proposition 3.

From propositions 5 and 6 it follows, in view of remark 2, that every ultra-distribution admits a representation $\varphi(s) = [\varphi_n(s)]$, such that $\{\varphi_n'(s)\}$ is a Φ-sequence. We define the derivative of $\varphi(s)$ by $\varphi'(s) = [\varphi_n'(s)]$. The consistency of this definition can be easily verified by use of equation (21). We have thus

THEOREM 1. *Each ultra-distribution has a derivative of an arbitrary order.*

If an ultra-distribution is a function having a continuous (and slowly increasing) m-th derivative, then its m-th derivative in the above sense coincides with its ordinary m-th derivative.

Ultra-distributions are not in general derivatives of some orders of continuous (slowly increasing) functions. There is a one-to-one correspondence between derivatives (in the sense of ultra-distributions) of some orders of continuous, slowly increasing functions and tempered distributions. We shall identify both notions. In this way ultra-distributions are a generalization of tempered distributions.

3. Sequences and series of ultra-distributions. For every Φ-sequence $\{\varphi_n(s)\}$ and every integer $k \geqslant 0$ sufficiently large, the sequence of functions $\Phi_n(\lambda, z)$ defined by (20) converges uniformly in every strip $-N < \eta < N$. The limit $\Phi(\lambda, z)$ depends on the Φ-sequence representing a given ultra-distribution $\varphi(s)$, say, but by proposition 4, it is determined up to a term of the form $P(D)K_\lambda(z)$, where $P(D)$ is a polynomial of degree $\leqslant k-1$. Conversely, by proposition 6, $\Phi(\lambda, z)$ determines $\varphi(s)$ entirely. Each $\Phi(\lambda, z)$ is said to be a *regular* function corresponding to the integer k and the ultra-distribution $\varphi(s)$.

Two ultra-distributions $\varphi(s)$ and $\psi(s)$ are equal if and only if for every $k \geqslant 0$, the corresponding regular functions $\Phi(\lambda, z)$ and $\Psi(\lambda, z)$, if they exist, differ by a term $P(D)K_\lambda(z)$, where $P(D)$ is a polynomial of degree $\leqslant k-1$.

We say that a sequence $\{\varphi_n(s)\}$ of ultra-distributions *converges* to $\varphi(s)$ and we write $\varphi_n(s) \rightarrow\!\!\rightarrow \varphi(s)$, if there exist an integer $k \geqslant 0$ and regular functions $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ corresponding to k and $\varphi_n(s)$, $\varphi(s)$, respectively, such that for every $\lambda > 0$, $\Phi_n(\lambda, z) \Rightarrow \Phi(\lambda, z)$.

The limit $\varphi(s)$ of a convergent sequence of ultra-distributions $\varphi_n(s)$ is unique.

In fact, if $\varphi_n(s) \rightarrow\!\!\rightarrow \varphi(s)$ and $\varphi_n(s) \rightarrow\!\!\rightarrow \psi(s)$, then there are integers $k, l \geqslant 0$ and regular functions $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ and $\Psi_n(\lambda, z)$, $\Psi(\lambda, z)$ corresponding to $k, \varphi_n(s)$, $\varphi(s)$ and $l, \varphi_n(s)$, $\psi(s)$ respectively, such that for every $\lambda > 0$,

$$\Phi_n(\lambda, z) \Rightarrow \Phi(\lambda, z) \quad \text{and} \quad \Psi_n(\lambda, z) \Rightarrow \Psi(\lambda, z).$$

We may assume that $k \leqslant l$. Then

$$\tilde{\Phi}_n(\lambda, z) = Fp \int_{-\infty}^{\infty} K_\lambda(z-s)\, \frac{\Phi_n(\lambda, s)}{s^{l-k}}\, ds \Rightarrow \tilde{\Phi}(\lambda, z),$$

where $\tilde{\Phi}_n(\lambda, z)$, $\tilde{\Phi}(\lambda, z)$ are regular functions corresponding to l and $\varphi_n(s)$, $\varphi(s)$ respectively. Moreover, by propositions 2 and 4,

$$\tilde{\Phi}_n(\lambda, z) - \Psi_n(\lambda, z) = P_n(D)K_\lambda(z) \Rightarrow P(D)K_\lambda(z),$$

where $P_n(D)$ and $P(D)$ are polynomials of degree $\leqslant l-1$. Consequently $\tilde{\Phi}(\lambda, z)$ and $\Psi(\lambda, z)$ correspond to the same ultra-distribution, i. e. $\varphi(s) = \psi(s)$.

If $\varphi(s)$ is an indefinitely differentiable function on R, such that

$$|\varphi(s)| = O(|s|^{k-1}) \quad \text{as} \quad |s| \to \infty,$$

then

$$\Phi(\lambda, z) = Fp \int_{-\infty}^{\infty} K_\lambda(z-s)\, \frac{\varphi(s)}{s^k}\, ds$$

is a regular function corresponding to k and $\varphi(s)$. Hence we get

PROPOSITION 7. *A sequence $\{\varphi_n(s)\}$ of indefinitely differentiable and slowly increasing functions converges to an ultra-distribution $\varphi(s)$ if and only if $\{\varphi_n(s)\}$ is a Φ-sequence, and $\varphi(s) = [\varphi_n(s)]$.*

If $\{\varphi_n(s)\}$, $\{\psi_n(s)\}$ are sequences of ultra-distributions, a, b real numbers $(a \neq 0)$, and $\alpha(z)$ is an entire function of exponential type, slowly increasing on R, then

$$\varphi_n(s) \rightarrow\!\!\rightarrow \varphi(s) \quad \text{and} \quad \psi_n(s) \rightarrow\!\!\rightarrow \psi(s)$$

implies

$$\varphi_n(s) + \psi_n(s) \rightarrow\!\!\rightarrow \varphi(s) + \psi(s),$$

$$\varphi_n(as+b) \rightarrow\!\!\rightarrow \varphi(as+b),$$

$$\alpha(s)\varphi_n(s) \rightarrow\!\!\rightarrow \alpha(s)\varphi(s).$$

Similarly as for distributions we have the useful

THEOREM 2. *Each convergent sequence of ultra-distributions may be differentiated term by term.*

In other words, if $\{\varphi_n(s)\}$ is a sequence of ultra-distributions and $\varphi_n(s) \rightarrow\!\!\rightarrow \varphi(s)$, then, for every integer $m \geqslant 0$, $\varphi_n^{(m)}(s) \rightarrow\!\!\rightarrow \varphi^{(m)}(s)$.

Proof. There exist an integer $k \geqslant 0$ and regular functions $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ corresponding to k and $\varphi_n(s)$, $\varphi(s)$ respectively, such that for every $\lambda > 0$, $\Phi_n(\lambda, z) \Rightarrow \Phi(\lambda, z)$. Putting

$$\Phi_n^*(\lambda, z) = Fp \int_{-\infty}^{\infty} K_\lambda(z-s)\, \frac{\Phi_n(\lambda, s)}{s}\, ds$$

we get

$$k\Phi_n^*(\lambda, z) + \Phi_n'(\lambda, z) \Rightarrow k\Phi^*(\lambda, z) + \Phi'(\lambda, z). \tag{29}$$

But from the proof of proposition 5 it follows that $k\Phi_n^*(\lambda, z) + \Phi_n'(\lambda, z)$ are regular functions corresponding to k and $\varphi_n'(s)$. Similarly $k\Phi^*(\lambda, z) + \Phi'(\lambda, z)$ is a regular function corresponding to k and $\varphi(s)$. Therefore (29) implies $\varphi_n'(s) \rightarrow\!\!\rightarrow \varphi(s)$ and this argument can be repeated any number of times.

A sequence $\{\varphi_n(s)\}$ of tempered distributions is said to *converge*, if there exist integers $k, l \geqslant 0$, a constant M, and continuous functions $\Phi_n(s)$ on R, such that

(30) $$\Phi_n^{(k)}(s) = \varphi_n(s), \quad \Phi_n(s) \rightrightarrows \quad \text{and} \quad |\Phi_n(s)| \leqslant M(1+|s|^l).$$

This convergence is stronger than in the sense of ultra-distributions, i. e. if a sequence of tempered distributions converges in the above sense then it converges as a sequence of ultra-distributions, but not conversely.

A series $\sum_{n=1}^{\infty} \varphi_n(s)$ of ultra-distributions converges if the sequence of partial sums

$$\psi_n(s) = \sum_{j=1}^{n} \varphi_j(s)$$

converges. Its limit $\varphi(s)$ is called the *sum of the series* and we write

$$\varphi(s) = \sum_{n=1}^{\infty} \varphi_n(s).$$

From theorem 2 one gets immediately

THEOREM 2′. *Each convergent series of ultra-distributions may be differentiated terms by term, i. e.*

$$\Big(\sum_{n=1}^{\infty} \varphi_n(s)\Big)' = \sum_{n=1}^{\infty} \varphi_n'(s).$$

We show that

(31) $$\delta\left(s+\frac{i}{2\pi}\right) = \sum_{n=0}^{\infty} \left(\frac{i}{2\pi}\right)^n \frac{\delta^{(n)}(s)}{n!}.$$

In fact, regular functions corresponding to $k=0$ and $\delta^{(n)}(s)$ are $K_\lambda^{(n)}(z)$. But

$$K_\lambda\left(z+\frac{i}{2\pi}\right) = \sum_{n=0}^{\infty} \left(\frac{i}{2\pi}\right)^n \frac{K_\lambda^{(n)}(z)}{n!},$$

and the convergence is uniform in every strip $-N < \eta < N$. Since $K_\lambda(s+i/2\pi)$ is a regular function corresponding to $k=0$ and $\delta(s+i/2\pi)$, this proves (31).

4. Fourier transforms of distributions of finite order. Let $f(x)$ be a continuous, rapidly decreasing function, that means, for every integer $m \geqslant 0$, $f(x) = O(|x|^{-m})$ as $|x| \to \infty$. Then the Fourier transform

$$\varphi(s) = \mathscr{F}\big(f(x)\big) = \int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)\,dx$$

is indefinitely differentiable and bounded on R together with all its derivatives.

Now, for a positive integer k, consider

(32) $$F(x) = \frac{1}{2}\left\{\int_{-\infty}^{x} \frac{(x-y)^{k-1}}{(k-1)!} f(y)\,dy - \int_{x}^{\infty} \frac{(x-y)^{k-1}}{(k-1)!} f(y)\,dy\right\};$$

it is a continuous, slowly increasing function, such that $F^{(k)}(x) = f(x)$. We assert that

(33) $$\mathscr{F}\big(F(x)U_\lambda(x)\big) = \frac{1}{(2\pi i)^k} Fp \int_{-\infty}^{\infty} K_\lambda(s-t)\frac{\varphi(t)}{t^k}\,dt$$

where

$$U_\lambda(x) = \begin{cases} 1 & \text{for} \quad |x| \leqslant \lambda, \\ 0 & \text{for} \quad |x| > \lambda. \end{cases}$$

In order to prove it we write for $\varepsilon > 0$,

$$F_+(\varepsilon, x) = \int_0^{\infty} \frac{y^{k-1}}{(k-1)!} e^{-2\pi\varepsilon y} f(x-y)\,dy$$

and

$$F_-(\varepsilon, x) = -\int_{-\infty}^{0} \frac{y^{k-1}}{(k-1)!} e^{2\pi\varepsilon y} f(x-y)\,dy.$$

On account of (32),

(34) $$F(x) = \tfrac{1}{2}\{F_+(0+, x) + F_-(0+, x)\}.$$

Moreover, $F_+(\varepsilon, x)$ is a convolution product of $f(x)$ with

$$\frac{x^{k-1}}{(k-1)!} e^{-2\pi\varepsilon x} H(x),$$

where $H(x)$ is Heaviside's unique function. Therefore,

$$\mathscr{F}\big(F_+(\varepsilon, x)\big) = \mathscr{F}\big(f(x)\big)\mathscr{F}\left(\frac{x^{k-1}}{(k-1)!} e^{-2\pi\varepsilon x} H(x)\right) = \frac{\varphi(s)}{(2\pi i)^k (s-i\varepsilon)^k}$$

and since $\mathscr{F}\big(U_\lambda(x)\big) = K_\lambda(s)$, we infer that

$$\mathscr{F}\big(F_+(\varepsilon, x)U_\lambda(x)\big) = \frac{1}{(2\pi i)^k} \int_{-\infty}^{\infty} \frac{K_\lambda(s-t)\varphi(t)}{(t-i\varepsilon)^k}\,dt.$$

Similarly

$$\mathscr{F}\big(F_{-}(\varepsilon, x)U_\lambda(x)\big) = \frac{1}{(2\pi i)^k}\int_{-\infty}^{\infty}\frac{K_\lambda(s-t)\varphi(t)}{(t+i\varepsilon)^k}\,dt.$$

Hence, letting ε tend to zero and applying (3) and (34) we obtain equation (33).

We now prove

Proposition 8. *Assume that $f_n(x)$ are continuous, rapidly decreasing functions and $\varphi_n(s)$ their Fourier transforms. Then $\{f_n(x)\}$ is an F-sequence if and only if $\{\varphi_n(s)\}$ is a Φ-sequence.*

Proof. If $\{f_n(x)\}$ is an F-sequence, then there exist a positive integer k and a sequence of polynomials $P_n(x)$ of degree $\leqslant k-1$, such that

$$(35)\quad F_n(x) = \frac{1}{2}\left\{\int_{-\infty}^{x}\frac{(x-y)^{k-1}}{(k-1)!}f_n(y)\,dy - \int_{x}^{\infty}\frac{(x-y)^{k-1}}{(k-1)!}f_n(y)\,dy\right\} + P_n(x) \rightrightarrows .$$

Since

$$P_n\left(\frac{D}{2\pi i}\right)K_\lambda(s) = \mathscr{F}\big(P_n(x)\,U_\lambda(x)\big),$$

it follows from (33) and (35) that for every $\lambda > 0$,

$$(36)\qquad \frac{1}{(2\pi i)^k}\,Fp\int_{-\infty}^{\infty}K_\lambda(z-t)\frac{\varphi_n(t)}{t^k}\,dt + P_n\left(\frac{D}{2\pi i}\right)K_\lambda(z) \Rightarrow ,$$

i. e. $\{\varphi_n(s)\}$ is a Φ-sequence.

Conversely, for sufficiently large k, the sequence (36) converges in mean on R. Hence, for every $\lambda > 0$, $\{F_n(x)U_\lambda(x)\}$ converges in mean. Consequently

$$\tilde{F}_n(x) = \int_0^x F_n(y)\,dy \rightrightarrows \quad\text{and}\quad \tilde{F}_n^{(k+1)}(x) = f_n(x),$$

and thus $\{f_n(x)\}$ is an F-sequence.

It is easy to see that in proposition 8 equivalent F-sequences correspond to equivalent Φ-sequences. There is therefore a one-to-one correspondence between distributions of finite order and ultra-distributions. We call $\varphi(s) = [\varphi_n(s)]$ the Fourier transform of $f(x) = [f_n(x)]$ and we write $\varphi(s) = \mathscr{F}\big(f(x)\big)$ or $f(x) = \mathscr{F}^{-1}\big(\varphi(s)\big)$. We have thus

Theorem 3. *Each distribution (of finite order) has a Fourier transform, which is an ultra-distribution. Conversely, each ultra-distribution is a Fourier transform of a distribution (of finite order).*

The Fourier transform in question is a linear operation and coincides for tempered distributions with that mentioned in the introduction

(equation (1)). In particular, for integrable and square integrable functions it is the usual Fourier transform.

Also the following properties are valid:

$$(37)\qquad \mathscr{F}\big(f'(x)\big) = 2\pi i s\,\varphi(s),$$

$$(38)\qquad \mathscr{F}\big(-2\pi i x f(x)\big) = \varphi'(s),$$

$$(39)\qquad \mathscr{F}\big(f(x)*g(x)\big) = \mathscr{F}\big(f(x)\big)\mathscr{F}\big(g(x)\big),$$

where $\varphi(s) = \mathscr{F}\big(f(x)\big)$, $g(x)$ is a distribution with compact carrier, and the product on the right-hand side of (39) is well defined, since $\mathscr{F}\big(g(x)\big)$ is an entire function of exponential type, slowly increasing on R.

If $\varphi(s) = \mathscr{F}\big(f(x)\big)$, then for k sufficiently large, each regular function $\Phi(\lambda, z)$ corresponding to k and $\varphi(s)$ is a Fourier transform of $F(x)U_\lambda(x)$, where $F(x)$ is a continous function such that $F^{(k)}(x) = f(x)$.

Theorem 4. *Suppose that $f_n(x)$ and $f(x)$ are distributions (of finite order) with $\varphi_n(s)$ and $\varphi(s)$ as Fourier transforms respectively. Then*

$$f_n(x) \to f(x) \ \textit{if and only if}\ \varphi_n(s) \twoheadrightarrow \varphi(s).$$

In other words, the Fourier transforms $\mathscr{F}$ and $\mathscr{F}^{-1}$ are continous operations.

Proof. If $f_n(x) \to f(x)$, then there exist an integer $k \geqslant 0$ and continuous functions $F_n(x)$, $F(x)$ such that

$$(40)\qquad F_n^{(k)}(x) = f_n(x),\qquad F^{(k)}(x) = f(x)\qquad\text{and}\qquad F_n(x)\rightrightarrows F(x).$$

Write

$$\Phi_n(\lambda, s) = \mathscr{F}\big(F_n(x)U_\lambda(x)\big)\quad\text{and}\quad \Phi(\lambda, s) = \mathscr{F}\big(F(x)U_\lambda(x)\big).$$

From (40) it follows that for every $\lambda > 0$, $\Phi_n(\lambda, z) \Rightarrow \Phi(\lambda, z)$. But $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ are regular functions corresponding to k and $\varphi_n(s)$, $\varphi(s)$ respectively. Consequently $\varphi_n(s) \twoheadrightarrow \varphi(s)$.

On the other hand, assume that $\varphi_n(s) \twoheadrightarrow \varphi(s)$. There exist an integer $k \geqslant 0$ and regular functions $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ corresponding to k and $\varphi_n(s)$, $\varphi(s)$ respectively such that $\Phi_n(\lambda, z) \Rightarrow \Phi(\lambda, z)$. Now for sufficiently large k,

$$\mathscr{F}^{-1}\big(\Phi_n(\lambda, s)\big) = F_n(x)U_\lambda(x),\qquad \mathscr{F}^{-1}\big(\Phi(\lambda, s)\big) = F(x)U_\lambda(x),$$

where $F_n(x)$, $F(x)$ are locally square integrable functions such that $F_n^{(k)}(x) = f_n(x)$, $F^{(k)}(x) = f(x)$ and $\{F_n(x)U_\lambda(x)\}$ converges to $F(x)U_\lambda(x)$ in mean. It follows that

$$\tilde{F}_n(x) = \int_0^x F_n(y)\,dy \rightrightarrows \int_0^x F(y)\,dy = \tilde{F}(x),$$

where $\tilde{F}_n^{(k+1)}(x) = f_n(x)$, $\tilde{F}^{(k+1)}(x) = f(x)$, and thus $f_n(x) \to f(x)$.

Comparing proposition 7 with theorem 4 we conclude:

COROLLARY. *If $\{f_n(x)\}$ is an F-sequence representing the distribution $f(x)$, then $\mathcal{F}(f_n(x)) \to\to \mathcal{F}(f(x))$.*

Similarly, if $\{\varphi_n(s)\}$ is a Φ-sequence representing the ultra-distribution $\varphi(s)$, then $\mathcal{F}^{-1}(\varphi_n(s)) \to \mathcal{F}^{-1}(\varphi(s))$.

Let $\{f_n(x)\}$ be an F-sequence representing the distribution $f(x)$, and $a(x)$ an indefinitely differentiable function, which vanishes outside a finite interval. Then the "scalar products"

$$\int_{-\infty}^{\infty} f_n(x)\,a(x)\,dx \tag{41}$$

converge. The limit does not depend on the choice of the F-sequence in (41). It is said to be the *scalar product* of the distribution $f(x)$ with $a(x)$ and we denote it by

$$\int_{-\infty}^{\infty} f(x)\,a(x)\,dx.$$

Now, the Fourier transform $a(s)$ of $a(x)$ is extendable to an entire function of exponential type, rapidly decreasing on R. The scalar product of the ultra-distribution $\varphi(s) = \mathcal{F}(f(x))$ with $a(s)$ is defined so as to preserve the Parseval equation, i. e.

$$\int_{-\infty}^{\infty} \varphi(s)\,a(s)\,ds = \int_{-\infty}^{\infty} f(x)\,a(-x)\,dx. \tag{42}$$

Equation (42) is the essential point in the theory of Fourier transforms of distributions given in [2], [3], [4], [5] and [6].

As an example consider the ultra-distribution

$$\delta\left(s-\frac{b}{2\pi}\right) = [K_n(s-b)],$$

where b is a complex number. For every function $a(s)$ having the stated properties

$$\int_{-\infty}^{\infty} \delta\left(s-\frac{b}{2\pi}\right) a(s)\,ds = a\left(\frac{b}{2\pi}\right). \tag{43}$$

If $b = -i\beta$ (β is real), we get

$$K_n(s+i\beta) = \mathcal{F}(f_n(x)), \quad \text{where} \quad f_n(x) = \begin{cases} e^{\beta x} & \text{for} \quad |x| < n, \\ 0 & \text{for} \quad |x| > n, \end{cases}$$

and therefore

$$\delta\left(s+\frac{i\beta}{2\pi}\right) = \mathcal{F}(e^{\beta x}).$$

5. The case of distributions of infinite order. The following modifications of our definitions are required to enable us to define Fourier transforms of all distributions, in particular of distributions of infinite order.

A sequence $\{\varphi_n(s)\}$ of indefinitely differentiable, slowly increasing functions is called a *Φ-sequence*, if to every $\lambda > 0$, there exist an integer $k \geqslant 0$ and a sequence of polynomials $P_n(D)$ of degree $\leqslant k-1$ such that

$$\varphi_n(s) = O(|s|^{k-1}) \quad \text{as} \quad |s| \to \infty \tag{44}$$

and

$$\Phi_n(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n(s)}{s^k}\,ds + P_n(D)K_\lambda(z) \tag{45}$$

converges uniformly in every strip $-N < \eta < N$ [4].

Two Φ-sequences $\{\varphi_n(s)\}$ and $\{\psi_n(s)\}$ are said to be *equivalent*, if to every λ there exist an integer $k \geqslant 0$ and sequences of polynomials $P_n(D)$, $Q_n(D)$ of degree $\leqslant k-1$, such that

$$\varphi_n(s) = O(|s|^{k-1}), \quad \psi_n(s) = O(|s|^{k-1}) \quad \text{as} \quad |s| \to \infty$$

and

$$\Phi_n(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\varphi_n(s)}{s^k}\,ds + P_n(D)K_\lambda(z),$$

$$\Psi_n(\lambda, z) = Fp\int_{-\infty}^{\infty} K_\lambda(z-s)\frac{\psi_n(s)}{s^k}\,ds + Q_n(D)K_\lambda(z)$$

converge to the same limit, uniformly in every strip $-N < \eta < N$.

Classes of equivalent Φ-sequences are called *ultra-distributions*.

The new definitions differ essentially from those used before, since now the integer k depends, in general, on λ. If in particular one can choose to all $\lambda > 0$ a common k, then the ultra-distribution obtained can be identified with an ultra-distribution in the earlier sense.

Operations on the new ultra-distributions, such as addition, subtraction, linear substitution, multiplication by a function, can be defined in the usual way.

The property of Φ-sequences expressed in proposition 6 is now lacking. Yet one can represent every ultra-distribution by a sequence of entire functions of exponential type, bounded on the real line. If $\varphi(s) = [\varphi_n(s)]$ is such a representation, then $\{\varphi_n'(s)\}$ is a Φ-sequence defining the derivative $\varphi'(s)$.

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[4] Condition (44) may be satisfied for some k, independent of λ; then it is satisfied for every $k' \geqslant k$.

Given an ultra-distribution $\varphi(s)$. For every λ and k the limit $\Phi(\lambda, z)$ of (45), if it exists, is said to be a *regular function* corresponding to λ, k and $\varphi(s)$.

A sequence $\{\varphi_n(s)\}$ of ultra-distributions converges to $\varphi(s)$, if to every $\lambda > 0$ there exist an integer k and regular functions $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ corresponding to λ, k and $\varphi_n(s)$, $\varphi(s)$ respectively, such that $\Phi_n(\lambda, z)$ converges to $\Phi(\lambda, z)$ uniformly in every strip $-N < \eta < N$.

All properties of section 3 remain valid for the new convergence.

The theory of ultra-distributions extended in this way is equivalent with the theory of functionals on a space of entire function defined by authors mentioned in the introduction.

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Properties of the orthonormal Franklin system

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1. Introduction. The purpose of this paper is to present some properties of the orthonormal Franklin set and to indicate similarity of this system to another bases of the Banach space $C\langle 0,1\rangle$ of continuous functions on $\langle 0,1\rangle$. It was proved [6] a long time ago that the Franklin set forms a Schauder basis for $C\langle 0,1\rangle$. Not many more properties of these functions are known to be published. The other bases have been investigated in [6], [7], [1], [2] and [3]. Some applications of the author's results are given in [4]. Using the results of this paper one can get the same kind of applications for the Franklin functions. Some of these results were announced on the Conference of Functional Analysis held in Warszawa-Jabłonna in September 1960. Theorem 1 has a very simple proof. This theorem together with the Banach-Steinhaus theorem for sequences of linear operators gives a very simple proof of the Franklin result. The proofs of that result presented in [6] and [7] (p. 122-125) are very tedious.

2. Preliminaries and notation. The Haar functions are defined as follows:

$$\chi_1(t) \equiv 1 \quad \text{in} \quad \langle 0,1\rangle,$$

$$\chi_{2^n+1}(1) = -\sqrt{2^n},$$

$$(1) \qquad \chi_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{in} \quad \left\langle \dfrac{k-1}{2^n}, \dfrac{2k-1}{2^{n+1}} \right), \\ -\sqrt{2^n} & \text{in} \quad \left\langle \dfrac{2k-1}{2^{n+1}}, \dfrac{k}{2^n} \right), \\ 0 \text{ elswhere in } \langle 0,1\rangle, \end{cases}$$

where $n = 0, 1, \ldots;\ k = 1, 2, \ldots, 2^n$.

We shall define the Schauder functions using the Haar functions as follows:

$$(2) \qquad \varphi_0(t) \equiv 1, \quad \varphi_n(t) = \int_0^t \chi_n(\tau)\,d\tau, \quad t \in \langle 0,1\rangle,\ n = 1, 2, \ldots$$